**CUBO** *a Mathematical Journal*
*Vol. 6, Nº 1, (281-292). March 2004.*

# Interactions between partial differential equations and generalized analytic functions

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**1.** Interactions between partial differential equations (=PDE) and Complex Analysis (=CA) have a long history. Originally they were connected with holomorphic functions because real and imaginary parts of such functions satisfy the Cauchy-Riemann system and the Laplace equation as well. Present interactions concern more general differential equations including non-linear ones. In addition, higher-dimensional versions of complex methods can also be developed.

**2.** Two classical connexions between CA and PDE concern

- the Dirichlet boundary value problem and
- initial value problems of Cauchy-Kovalevskaya type.

As it is well-known, the Dirichlet boundary value problem for the Laplace equation in a simply connected bounded domain in the plane with a sufficiently smooth boundary can be solved as follows: In view of Riemann's Mapping Theorem there exists a conformal mapping which transforms the given domain into the unit disk. Since conformal mappings transform solutions of the Laplace equation again into solutions of the Laplace equation, it remains to solve the Laplace equation for the unit disk. This, however, can be done by the Poisson Integral Formula which can easily be obtained from the Cauchy Integral Formula for holomorphic functions. In view of the Cauchy-Riemann system, the imaginary part of a holomorphic function with a given real part is uniquely determined up to an arbitrary (real) constant. Thus a holomorphic function is uniquely determined by the boundary values of its real part and the imaginary part at one point (= *Dirichlet problem for holomorphic functions*).

Initial value problems of Cauchy-Kovalevskaya type have the form

$$\frac{\partial u}{\partial t} = \mathcal{F}\left(t, x, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right) \tag{1}$$

$$u(0, x) = \varphi(x) \tag{2}$$

where the desired (real-, complex- or vector-valued) $u = u(t, x)$ depends on the time $t$ and a spacelike variable $x = (x_1, ..., x_n)$. Then the classical Cauchy-Kovalevskaya Theorem states that the solution exists and is a power series in its variables provided both the right-hand side $\mathcal{F}$ of the differential equation and the initial function $\varphi$ have this property as well. Moreover, in view of the classical Holmgren Theorem the solution is unique not only among all power series in $x$ but also among all continuously differentiable functions.

The connexion of the (classical) Cauchy-Kovalevskaya Theorem with CA is based on the fact that a power series in real variables can be extended to a power series in complex variables which automatically define holomorphic functions. In order to generalize the classical Cauchy-Kovalevskaya Theorem, one has, however, more thoroughly to analyse the relations between that theorem and Complex Analysis. This is done by modern functional-analytic proofs of the Cauchy-Kovalevskaya Theorem such as W. Walter's [15] one. Such proofs reveal that the Cauchy-Kovalevskaya Theorem is based on the following property of the derivative of a holomorphic function:

*If $f$ is holomorphic and bounded in a domain $\Omega$, then its complex derivative $f'$ at $z$ can be estimated by*

$$|f'(z)| \leq \frac{\sup_\Omega |f|}{\operatorname{dist}(z, \partial\Omega)} \tag{3}$$

*where* $\operatorname{dist}(z, \partial\Omega)$ *means the distance of $z$ from the boundary $\partial\Omega$ of $\Omega$.*

This estimate can easily be seen by applying Cauchy's Integral Formula for $f'$ to a disk with radius $\delta < \operatorname{dist}(z, \partial\Omega)$ centred at $z$.

The present article will show that the described interactions between CA and PDE remain valid when Laplace equation and Cauchy-Riemann system are replaced by more general differential equations.

**3.** Classical Complex Function Theory shows that a (continuously differentiable) solution $u, v$ of the Cauchy-Riemann system is a holomorphic function, i.e., it is complex differentiable everywhere. If one wants to solve more general PDE by complex methods, one has to replace the ordinary complex differentiation $d/dz$ by two partial complex differentiations. In order to come to a natural definition of such partial complex differentiations, we consider a complex-valued function $f$ defined in a domain $\Omega$ of the $z = x + yi$-plane which is continuously differentiable with respect to the real variables $x$ and $y$. Then the linearization $\tilde{f}$ of $f$ at a point $z_0 = x_0 + iy_0$ is given by

$$\tilde{f}(z_0) = f(z_0) + c_1(x - x_0) + c_2(y - y_0) \tag{4}$$

where $c_1$ and $c_2$ are the partial derivatives of $f$ with respect to the real variables $x$ and $y$ resp. at $z_0 = x_0 + iy_0$, i.e.,

$$c_1 = \frac{\partial f}{\partial x}(z_0), \quad c_2 = \frac{\partial f}{\partial y}(z_0).$$

Clearly, since $z - z_0 = (x - x_0) + i(y - y_0)$ and $\overline{z - z_0} = (x - x_0) - i(y - y_0)$, the increments $x - x_0$ and $y - y_0$ can be expressed by $z - z_0$ and $\overline{z - z_0}$. Doing this, formula (4) can be rewritten in the form

$$\tilde{f}(z_0) = f(z_0) + d_1(z - z_0) + d_2\overline{(z - z_0)}.$$

Since $d_1$ and $d_2$ are the coefficients of the complex increments $z - z_0$ and $\overline{z - z_0}$, these coefficients are called the *partial complex derivatives* of $f$ with respect to $z$ and $\overline{z}$ at $z_0$. Expressing $d_1$ and $d_2$ by $c_1$ and $c_2$, one sees that the partial complex differentiations $\partial/\partial z$ and $\partial/\partial\overline{z}$ are connected with the real differentiations $\partial/\partial x$ and $\partial/\partial y$ by the relations

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big). \tag{5}$$

These operators $\partial/\partial z$ and $\partial/\partial\overline{z}$ can be applied to any function $f$ which has (continuous) derivatives with respect to the real variables $x$ and $y$, whereas the ordinary complex differentiation $d/dz$ is defined only for holomorphic functions, i.e., for functions $f = u + iv$ whose real part $u$ and imaginary part $v$ satisfy the Cauchy-Riemann system. Using the Cauchy-Riemann system, an easy calculation shows that for holomorphic functions $f$ the following relations hold:

$$\frac{\partial f}{\partial \overline{z}} = 0 \tag{6}$$

$$\frac{\partial f}{\partial z} = \frac{df}{dz} = f'.$$

Equation (6) is nothing but a complex version of the Cauchy-Riemann system.

Taking into account the definition (5) of the partial complex differentiations, and combining the Green-Gauss Formulae

$$\iint\limits_{\Omega} \frac{\partial f}{\partial x} dxdy = \int\limits_{\partial\Omega} f dy, \qquad \iint\limits_{\Omega} \frac{\partial f}{\partial y} dxdy = -\int\limits_{\partial\Omega} f dx,$$

one gets the following *complex versions of the Green-Gauss Formulae*:

$$\iint\limits_{\Omega} \frac{\partial f}{\partial \overline{z}} f dxdy = \frac{1}{2i}\int\limits_{\partial\Omega} f dz \tag{7}$$

$$\iint\limits_{\Omega} \frac{\partial f}{\partial z} f dxdy = -\frac{1}{2i}\int\limits_{\partial\Omega} f d\overline{z}.$$

Formula (7) contains, for instance, also the Cauchy Integral Theorem. Indeed, if $f$ is holomorphic in $\Omega$, then the integrand of the domain integral vanishes identically and thus the boundary integral equals zero.

Also a generalization of Cauchy's Integral Formula can easily be obtained from formula (7). For this purpose consider a fixed point $\zeta$ in $\Omega$. If $f$ is continuously differentiable in $\overline{\Omega}$, then

$$g(z) = \frac{f(z)}{z-\zeta}$$

has an isolated singularity in $\Omega$. Applying the complex Green-Gauss Formula (7) to $g$ in $\Omega_\varepsilon = \left\{z \in \Omega : |z-\zeta| \geq \varepsilon\right\}$, it follows

$$\iint\limits_{\Omega_\varepsilon} \frac{1}{z-\zeta} \cdot \frac{\partial f}{\partial \bar{z}} dx dy = \frac{1}{2i} \int\limits_{\partial\Omega} \frac{f(z)}{z-\zeta} dz - \frac{1}{2i} \int\limits_{|z-\zeta|=\varepsilon} \frac{f(z)}{z-\zeta} dz. \tag{8}$$

Note that

$$\begin{aligned} \int\limits_{|z-\zeta|=\varepsilon} \frac{f(z)}{z-\zeta} dz &= \int\limits_{|z-\zeta|=\varepsilon} \frac{f(z)-f(\zeta)}{z-\zeta} dz + f(\zeta) \int\limits_{|z-\zeta|=\varepsilon} \frac{1}{z-\zeta} dz \\ &= \int\limits_{|z-\zeta|=\varepsilon} \frac{f(z)-f(\zeta)}{z-\zeta} dz + f(\zeta) \cdot 2\pi i. \end{aligned} \tag{9}$$

The absolute value of the integral on the right hand side of (9) can be estimated by

$$\sup_{|z-\zeta|=\varepsilon} |f(z) - f(\zeta)| \cdot \frac{1}{\varepsilon} \cdot 2\pi\varepsilon.$$

Note that $f$ is continuous at $\zeta$, in particular. Thus the integral on the right hand side of (9) tends to zero as $\varepsilon \to 0$. Carrying out the limiting process $\varepsilon \to 0$ in (8), we get thus the following *Cauchy-Pompeiu Integral Formula*

$$f(\zeta) = \frac{1}{2\pi i} \int\limits_{\partial\Omega} \frac{f(z)}{z-\zeta} dz - \frac{1}{\pi} \iint\limits_{\Omega} \frac{1}{z-\zeta} \cdot \frac{\partial f}{\partial \bar{z}} dx dy \tag{10}$$

which is true for each (complex-valued) function $f$ which has continuous first order derivatives with respect to the (real) variables $x$ and $y$. If, especially, $f$ is holomorphic in $\Omega$, then this formula (10) passes into the well-known Cauchy Integral Formula for holomorphic functions.

4. A decisive breakthrough in the interactions of PDE and CA can be reached by the combination of complex methods with distributional methods for PDE. In order to explain the basic idea of distributional solutions of a PDE, consider limear differential operators $\mathcal{L}$ of divergence type. A differential operator $\mathcal{L}$ of order $k$ in $n$ real variables

$x_1, ..., x_n$ is said to be of ***divergence type*** if there exists an associated operator $\mathcal{L}^*$ of the same order $k$ and $n$ differential operators $P_j$ of order $k-1$ such that

$$v\mathcal{L}u + (-1)^{k+1}u\mathcal{L}^*v = \sum_j \frac{\partial P_j}{\partial x_j}.$$

Then the Gauss Integral Formula yields the following *Green Integral Formula* for differential operators of divergence type

$$\int_\Omega \Big(v\mathcal{L}u + (-1)^{k+1}u\mathcal{L}^*v\Big)dx = \int_{\partial\Omega} \sum_j P_j N_j d\mu \tag{11}$$

where $N = (N_1, ..., N_n)$ is the outer unit normal and $d\mu$ is the measure element of the boundary $\partial\Omega$ of $\Omega$. For instance, if $\mathcal{L}$ is the (self-adjoint) Laplace operator $\Delta$, then $P_j = v\dfrac{\partial u}{\partial x_j} - u\dfrac{\partial v}{\partial x_j}$, and the Green Integral Formula passes into

$$\int_\Omega (v\Delta u - u\Delta v)\, dx = \int_{\partial\Omega} \Big(v\frac{\partial u}{\partial N} - u\frac{\partial v}{\partial N}\Big)d\mu.$$

In the case of the Laplace operator we see that the $P_j$ are equal to zero on the boundary if $v$ is a function which vanishes identically in a neighbourhood of the boundary. An analogous property is supposed for general differential operators of divergence type.

Now let $u$ be a solution of $\mathcal{L}u = 0$ where $\mathcal{L}$ is again an arbitrary differential operator of divergence type. For $v$ we choose a $k$ times differentiable function which vanishes identically in a neighbourhood of the boundary. Then the integrand of the boundary integral in (11) vanishes identically, and (11) leads to

$$\int_\Omega u\mathcal{L}^*\varphi dx = 0. \tag{12}$$

Thus for a solution $u$ of $\mathcal{L}u = 0$ the relation (12) is necessarily satisfied if $\varphi$ is any ($k$ times continuously differentiable) function vanishing identically in a neighbourhood of the boundary.

Conversely, assume that (12) is satisfied for each function $\varphi$ which vanishes identically in a neighbourhood of the boundary. Then formula (11) shows that

$$\int_\Omega \varphi\mathcal{L}u dx = 0$$

for each $\varphi$. This, however, is only possible if $\mathcal{L}u = 0$, otherwise one gets a contradiction in view of the Fundamental Lemma of Variational Calculus because $\varphi$ is arbitrary. Therefore, a $k$ times continuously differentiable function $u$ turns out to be a solution of the equation $\mathcal{L}u = 0$ if only (12) is satisfied for each $\varphi$. Since (12) is a derivation-free characterization of solutions of the Laplace equation, the functions $\varphi$ are called *test functions*.

Relation (12) is not only a characterization of solutions of a PDE, but also it can be used in order to generalize the concept of a solution. An integrable function $u$ is called a *distributional solution* of the differential equation $\mathcal{L}u = 0$ in case (7) holds for each choice of $\varphi$.

I. N. Vekua's theory of generalized analytic functions [14] is based on the use of *partial complex derivatives in the distributional sense.* Here one starts from the complex Green-Gauss formula (7). Suppose $w$ is a solution of the inhomogeneous Cauchy-Riemann equation

$$\frac{\partial w}{\partial \bar{z}} = h \tag{13}$$

where the right hand side is a given complex-valued function. Suppose, further, that $\varphi$ is a complex-valued test function. Applying (7) to $f = w\varphi$, it follows

$$\iint_{\Omega} \left( w \frac{\partial \varphi}{\partial \bar{z}} + h\varphi \right) dx dy = 0 \tag{14}$$

for each choice of the test function $\varphi$.

Again, relation (14) is not only a derivation-free characterization of classical solutions of the inhomogeneous Cauchy-Riemann equation (7), but also it can be used in order to define distributional solutions of (13): An integrable function $w$ is called a distributional solution of (13) if (14) is satisfied for each test function $\varphi$.

Using this concept of distributional solutions, one can show that the weakly singular integral

$$(T_{\Omega} h)[\zeta] = -\frac{1}{\pi} \iint_{\Omega} \frac{h(z)}{z - \zeta} dx dy$$

defines a special distributional solution of (13). The general solution of (13) is given by $T_{\Omega} h + \Phi$ where $\Phi$ is a distributional solution of the homogeneous Cauchy-Riemann equation $\partial w / \partial \bar{z} = 0$, i.e.,

$$\iint_{\Omega} \Phi \frac{\partial \varphi}{\partial \bar{z}} dx dy = 0 \tag{15}$$

for each choice of $\varphi$. Relation (15) implies, however, that such a $\Phi$ is necessarily an ordinary holomorphic function in the classical sense (*complex version of the Weyl Lemma*). Consequently, the general (distributional) solution of the inhomogeneous equation (13) has the form

$$w = T_{\Omega} h + \Phi \tag{16}$$

where $\Phi$ is an arbitrary holomorphic function. This relation (16) makes it possible to reduce problems for generalized analytic functions (such as boundary value problems) to analogous problems for holomorphic functions.

Notice, further, that the partial complex differentiation $\partial/\partial z$ in the distributional sense can be defined by a relation which is analogous to (14). Using this definition, one can show that the distributional derivative of $T_\Omega h$ with respect to $z$ is equal to the strongly singular integral

$$\frac{\partial T_\Omega h}{\partial z}[\zeta] = (\Pi_\Omega h)[\zeta] = -\frac{1}{\pi}\iint\limits_{\Omega} \frac{h(z)}{(z-\zeta)^2}dxdy$$

which is to be understood in the sense of Cauchy's principal value.

**5.** Next we shall outline how the above techniques can be used to investigate real systems of PDE using methods of CA. To begin with, consider a linear first order system

$$a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y} + b_{11}\frac{\partial v}{\partial x} + b_{12}\frac{\partial v}{\partial y} + a_1 u + b_1 v = f_1$$

$$a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y} + b_{21}\frac{\partial v}{\partial x} + b_{22}\frac{\partial v}{\partial y} + a_2 u + b_2 v = f_2$$

for two desired real-valued functions $u(x,y)$ and $v(x,y)$ in the plane whose coefficients may depends on $x$ and $y$. Provided the system is elliptic ( i.e., the coefficients of the first order derivatives satisfy a certain relation, cf. I. N. Vekua [14]), such a system can be reduced to a system of the form

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = c_1 u + c_2 v + d_1 \tag{17}$$

$$\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} = c_3 u + c_4 v + d_2 \tag{18}$$

(this can be done by a transformation of coordinates and by introducing new desired functions). Setting $u + iv = w$ and $x + iy = z$, one has

$$\frac{\partial w}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\Big) + \frac{i}{2}\Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big) \tag{19}$$

and hence the system (17), (18) can be written as one complex equation

$$\frac{\partial w}{\partial \overline{z}} = A(z)w + B(z)\overline{w} + D(z). \tag{20}$$

In the homogeneous case ($D \equiv 0$) this equation (20) is known as Vekua's equation, its solutions are called *generalized analytic functions.* Such functions can also be interpreted as *pseudo-analytic* functions in L. Bers' sense [2].

In order to illustrate the power of the above introduced $T_\Omega$-operator, we shall prove the following statement:

*The zeros of a generalized analytic function $w(z)$ are isolated unless the function vanishes identically.*

Indeed, define $g$ by

$$g = \begin{cases} A + B\frac{\overline{w}}{w} \text{ if } w \neq 0 \\ 0 \text{ otherwise.} \end{cases}$$

Then $\Phi = w \cdot \exp(-T_\Omega g)$ is holomorphic because

$$\begin{aligned} \frac{\partial \Phi}{\partial \overline{z}} &= \frac{\partial w}{\partial \overline{z}} \cdot \exp(-T_\Omega g) + w \cdot \exp(-T_\Omega g) \cdot (-g) \\ &= \exp(-T_\Omega g)\Big( Aw + B\overline{w} + w\Big( -A - B\frac{\overline{w}}{w}\Big)\Big) = 0. \end{aligned}$$

Thus the generalized analytic function $w$ can be factorized by a holomorphic function $\Phi$ and a factor different from zero:

$$w = \Phi \cdot \exp(T_\Omega g).$$

Since the zeros of a (not identically vanishing) holomorphic functions are isolated, the same is true for generalized analytic functions. The first proof of this statement is due to T. Carleman [5], whereas the above short proof was given by I. N. Vekua [13].

**6.** Complex methods are also applicable to non-linear systems of PDE. Consider the real implicit system of two (real) equations

$$H_j\Big(x, y, u, v, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\Big) = 0, \quad j = 1, 2, \tag{21}$$

for two desired real-valued functions $u(x, y)$ and $v(x, y)$. Introduce complex variables $z = z + yi$ and $w = u + vi$. Then one has

$$\frac{\partial w}{\partial z} = p_1 + p_2 i \quad \text{and} \quad \frac{\partial w}{\partial \overline{z}} = q_1 + q_2 i \tag{22}$$

where

$$p_1 = \frac{1}{2}\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big), \qquad p_2 = \frac{1}{2}\Big(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\Big), \tag{23}$$

$$q_1 = \frac{1}{2}\Big(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\Big), \qquad q_2 = \frac{1}{2}\Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big) \tag{24}$$

(see formula (19) and an analogous equation for $\partial w / \partial z$). In view of (23), (24) one can express the first order derivatives of $u$ and $v$ by $p_1, p_2, q_1, q_2$. Substitute these expressions into (21). Now assume that the system (21) can be solved for $q_1$ and $q_2$. In view of (22), $q_1 + q_2 i$ equals $\partial w / \partial \overline{z}$ and so the system (21) can be rewritten in the form

$$\frac{\partial w}{\partial \overline{z}} = \mathcal{F}\Big(z, w, \frac{\partial w}{\partial z}\Big). \tag{25}$$

This differential equation (25) can be considered as complex normal form of the real system (21).

Note that (25) can be interpreted as an inhomogeneous Cauchy-Riemann equation of the form (13) whose right hand side depends on the desired function $w = w(z)$ and its derivative $\partial w/\partial z$. By virtue of (16), to each solution $w$ of (25) there exists a holomorphic function $\Phi$ such that $w$ satisfies the integro-differential equation

$$w = \Phi + T_\Omega \mathcal{F}(\cdot, w, \partial w/\partial z).$$

This relation, however, leads to the following fixed-point method for solving boundary value problems

$$\mathcal{B}w = g \quad \text{on} \quad \partial\Omega \tag{26}$$

for solutions of (25):

Define the image $W$ of a given $w$ by

$$W = \Phi_{(w)} + T_\Omega \mathcal{F}(\cdot, w, \partial w/\partial z) \tag{27}$$

where $\Phi_{(w)}$ is a holomorphic function (depending on the choice of $w$) such that $W$ satisfies the given boundary condition (26), i.e., $\mathcal{B}W = g$ on $\partial\Omega$. Then

$$\partial W/\partial z = \Phi'_{(w)} + \Pi_\Omega \mathcal{F}(\cdot, w, \partial w/\partial z)$$
$$\text{and} \qquad \partial W/\partial \bar{z} = \mathcal{F}(\cdot, w, \partial w/\partial z).$$

Consequently, a fixed point of the operator (27) turns out to be a solution of the boundary value problem (26) for the non-linear partial complex differential equation (25).

A special boundary value problem for (25) is again the Dirichlet boundary value problem (cf. Section 2): one prescribes the real part of $w$ on the whole boundary and the imaginary part of $w$ at one point. In [10] the corresponding fixed point is constructed using the contraction-mapping principle.

7. The contraction-mapping principle can also be applied to initial value problems of type (1), (2). The classical approach to this initial value problem is to look for a power-series representation of the desired solution (provided both the right hand side of (1) and the initial function (2) are given power series in their variables). The initial coefficients of the desired power series follow from the initial condition, whereas the differential equation (1) leads to a recursion formula for the coefficients. The convergence of the formally constructed power series, finally, can be proved by a comparison method.

A functional-analytic approach to the initial value problem (1), (2) was initiated by M. Nagumo [8] who rewrote this problem as an integro-differential equation

$$u(t, x) = \varphi(x) + \int_0^t \mathcal{F}(\tau, x, u, \partial u/\partial x_j)\, d\tau. \tag{28}$$

This is analogous to the rewriting of the initial value problems for ordinary differential equations as integral equation. Here, however, the integrand depends on the derivatives of the desired solution. It can be shown that the corresponding operator (defined by the right hand side of (28)) possesses a fixed point in a suitably chosen function space whose elements permit an estimate of type (3) where if necessary the supremum norm is to be replaced by another norm. This leads to the concept of *associated spaces* for initial value problems of type (1), (2):

*Suppose a differential equation $\mathcal{G}u = 0$ (with respect to the spacelike variables $x_1, ..., x_n$) defines a function space whose elements satisfy an estimate of type (3). Suppose, further, that for each fixed $t$ the right hand side $\mathcal{F}$ of (1) transforms this function space into itself (i.e., the operators $\mathcal{F}$ and $\mathcal{G}$ form a so-called associated pair). Provided the initial function $\varphi$ satisfies the side condition $\mathcal{G}\varphi = 0$, the initial value problem (1), (2) has a uniquely determined solution $u = u(t, x)$ satisfying the side condition $\mathcal{G}u = 0$ for each $t$.*

In the case of the classical Cauchy-Kovalevskaya problem the side condition $\mathcal{G}u = 0$ is given by the Cauchy-Riemann equation $\partial w/\partial \bar{z} = 0$, and the right hand side transforms holomorphic functions into themselves.

Generalized analytic functions, too, can serve as initial functions (see [11]). Then one has to consider right hand sides $\mathcal{F}$ transforming generalized analytic functions into themselves (the construction of associated pairs can be reduced to the inhomogeneous Cauchy-Riemann equation, see [11]). The necessary estimate of type (3) can easily be obtained from mapping properties of the $T_\Omega$- and $\Pi_\Omega$-operators:

Suppose $w$ is a solution of equation (20) with $D \equiv 0$. Define

$$\Phi = w - T_\Omega(Aw + B\overline{w}). \tag{29}$$

Then

$$\frac{\partial \Phi}{\partial \bar{z}} = \frac{\partial w}{\partial \bar{z}} - (Aw + B\overline{w}) = 0,$$

i.e., $\Phi$ is a holomorphic function. Since $T_\Omega$ is a bounded operator, (29) implies

$$\|\Phi\| \leq \text{const} \cdot \|w\|\,.$$

Differentiating (29) with respect to $z$, one gets

$$\frac{\partial w}{\partial z} = \Phi' + \Pi_\Omega(Aw + B\overline{w}).$$

Taking into account the boundedness of $\Pi_\Omega$, an estimate of type (3) for $\Phi'$ implies that an analogous estimate holds for $\partial w/\partial z$ (a suitable space for $w$ is the space of Hölder-continuously differentiable functions). Details may be found in [11].

**8.** In conclusion we hint to some related methods, further generalizations, and open problems.

Integral operators for the inversion of partial differential operators can be generated by fundamental solutions (cf. H. Begehr and R. P. Gilbert [1]). Generally speaking, each (elliptic) differential equation has its own fundamental solution. An advantage of the Cauchy kernel $1/(z-\zeta)$ (which is a fundamental solution of the Cauchy-Riemann equation) is the fact that it can be used to solve boundary value problems for general systems of the form (21). Guo-Chun Wen and H. Begehr [16] apply complex methods also for solving boundary value problems for elliptic equations and systems of second order. Related complex methods such as Bergman's Integral Operators can be found in R. P. Gilbert's book [7].

Initial value problems with generalized analytic vectors as initial data are investigated by A. Crodel in his Thesis [6]. A general theory of generalized analytic vectors was founded by B. Bojarski [3]. Generalized analytic vectors satisfy systems of the form

$$\frac{\partial w}{\partial \bar{z}} - Q(z)\frac{\partial w}{\partial z} = A(z)w + B(z)\overline{w}$$

where $w = (w_1, ..., w_n)$ is a desired vector with complex-valued components $w_j$ and $A(z)$, $B(z)$ and $Q(z)$ are matrices. Such systems are complex normal forms of uniformly elliptic linear system for $2n$ desired real-valued functions.

At present initial value problems with still more general initial functions are under consideration (see [12]). In this connexion interior estimates of type (3) are to be proved for general elliptic differential equations. Further, associated pairs are to be constructed.

In higher dimensions complex methods can be used not only with respect to several complex variables but also in connexion with the Cauchy-Riemann operator in Clifford Analysis (see F. Brackx, R. Delanghe and F. Sommen [4], see also E. Obolashvili [9]). Another promising field in higher dimensions is to investigate functions depending on several variables where each of these variables runs in a Clifford Algebra.

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