CUBO *a Mathematical Journal*
Vol. 6, Nº 1, (259-279). March 2004.

The complex Monge-Ampère equation and methods of pluripotential theory[1 2 3]

Slawomir Kolodziej
Jagiellonian University, Institute of Mathematics
Reymonta 4, 30-059 Kraków, Poland
kolodzie im.uj.edu.pl

ABSTRACT

One can prove fairly sharp results on the existence of weak solutions to the complex Monge-Ampère equation applying the methods based on the concept of the positive current. We survey those results both in a strictly pseudoconvex domain and on a compact Kähler manifold.

0. Introduction

A function u defined on an open subset Ω of $\mathbb{C}^n$ with values in $[-\infty, +\infty)$, and not identically $-\infty$ is called plurisubharmonic (shortly psh) if it is upper semicontinuous in Ω and subharmonic on any intersection of Ω with a complex line. When a psh function is of class C^2 then the complex Hessian $(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k})$ is positive semidefinite, that is

$$\sum_{j,k} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} w_j w_k \geq 0$$

for any vector $w \in \mathbb{C}^n$. The last inequality says that the differential form

$$dd^c u = 2i\partial\bar{\partial}u$$

defined by means of the operators

$$d = \partial + \bar{\partial}, \quad d^c = i(\bar{\partial} - \partial), \quad \partial = \sum_j \frac{\partial}{\partial z_j}, \quad \bar{\partial} = \sum_j \frac{\partial}{\partial \bar{z}_j},$$

[1] Mathematics Subject Classification: primary 32F07, secondary 32F05
[2] Key words and phrases: plurisubharmonic function, complex Monge-Ampère operator.
[3] Partially supported by KBN Grant No. 2 PO3A 010 19

is positive. Let us observe that a psh function is subharmonic and the positivity of the Hessian is the complex analogue of the condition defining smooth convex functions. So psh functions share not only all the properties of subharmonic functions but also their behavior often resembles that of the convex functions. However, in pluripotential theory we rather use methods which are not derived neither from theory of convex functions nor classical potential theory.

The plurisubharmonic functions have been investigated since 1942 when P. Lelong and K. Oka independently introduced them in their studies of some problems in several complex variables. An excellent historical account of the development of pluripotential theory (theory of plurisubharmonic functions) the reader will find in C.O. Kiselman's survey [KI2]. Plurisubharmonic functions constitute the natural class of solutions of the complex Monge-Ampère equation

$$\det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right) = f, \tag{0.1}$$

where u is unknown and f is a nonnegative function on a given domain in $\mathbb{C}^n$. In particular the Dirichlet problem for this equation is well posed if we require u to be psh. Recall that in the study of the real Monge-Ampère equation

$$\det\left(\frac{\partial^2 u}{\partial x_j \partial x_k}\right) = f, \tag{0.2}$$

we (usually) look for convex solutions. One way of attacking equation (1) is to apply the methods of fully nonlinear elliptic equations which give good results in solving (2) (see [CNS][GT]). This approach proved to be successful at least in the non degenerate case $f > 0$ (see CKNS]). In this article, however, we shall look at (1) from a different viewpoint. As it is explained below the equation (1) makes sense also for non smooth u with a positive Borel measure on the right hand side. Our focus will be on finding out which measures yield solutions in prescribed families of psh functions. It is easy to compute that for $u \in C^2$

$$const. \det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right) dV = (dd^c u)^n,$$

where the power on the right is taken with respect to the wedge product and dV denotes the Lebesgue measure. We shall see that the form on the right hand side is well defined also for locally bounded psh functions if we legalize differential forms with distribution coefficients (or currents). This is possible due to the fact that $dd^c u$ is positive. The generalized definition of $(dd^c u)^n$ is consistent with the classical one since if we take a sequence of smooth psh functions u_j decreasing to the given locally bounded psh function u then the measures $(dd^c u_j)^n$ converge to $(dd^c u)^n$ in the weak-star topology. The definition can be extended to some classes of unbounded functions as well.

The study of those generalized solutions of the complex Monge-Ampère equation, initiated by E. Bedford and B.A. Taylor in [BT1][BT2] (and prior to the results mentioned above) is intimately related to the study of certain properties of psh functions, especially the convergence properties of sequences of psh functions. We shall discuss the results from [BT1] and [BT2] after a brief review of the properties of positive forms and currents. Bedford and Taylor solved the Dirichlet problem for the complex Monge-Ampère equation in a strictly pseudoconvex domain for continuous data. Section 3 and Section 4 are devoted to the presentation of more general existence theorems with certain classes of Borel measures on the right hand side. Those results are taken from the papers of U. Cegrell [CE2][CE3] and the author [KO1][KO2][KO3][KO4][KO6]. We can give fairly sharp sufficient conditions on the measure to obtain continuous solutions. If we look for solutions in certain classes defined by Cegrell then it is possible to characterize measures for which the Dirichlet problem is solvable.

The Monge-Ampère equation is also intensively studied on Kähler manifolds since its solutions give Kähler metrics with prescribed Ricci curvature. If $(g_{j\bar{k}})$ is a Kähler metric and f is a positive function on the manifold then the Monge-Ampère equation has the form

$$\det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} + g_{j\bar{k}}\right) = f \det(g_{j\bar{k}}). \tag{0.3}$$

Solutions of a little bit more general equation of Monge-Ampère type, with the right hand side depending also on u, yield Kähler-Einstein metrics. The latter are, in a sense, canonical metrics on a given complex manifold. Their metric tensor is proportional to the Ricci curvature tensor. The works of E. Calabi [CA] , T. Aubin [AU1] and S.-T. Yau [YA] are milestones in the complex geometry. They solved the above equations for smooth non degenerate data. It is possible to generalize those results dropping the smoothness and non degeneracy assumptions ([KO3][KO7]). For example, if (suitably normalized) $f \geq 0$ belongs to $L^p(M)$ for $p > 1$ then there exists a continuous solution of (3). In the last two sections we discuss those results.

For more background on pluripotential theory we refer to [BE] [KI2] [KL]. Most of the proofs of the results presented below can be found in [KO8]. They are too long to be included in this survey.

1. Positive forms and currents

Let us denote by $C^\infty_{(p,p)}(\Omega)$ the set of all smooth differential forms of bidegree (p,p) defined in an open set $\Omega \subset \mathbb{C}^n$. Thus any form ω from $C^\infty_{(p,p)}(\Omega)$ is given by

$$\omega = i^p {\sum}'_{|J|=p,|K|=p} \omega_{JK}\, dz_J \wedge d\bar{z}_K,$$

where ω_{JK} are C^∞ functions in Ω, $dz_J = dz_{j_1} \wedge dz_{j_2} \wedge \ldots \wedge dz_{j_p}$, $d\bar{z}_J = d\bar{z}_{j_1} \wedge d\bar{z}_{j_2} \wedge \ldots \wedge d\bar{z}_{j_p}$, and $\sum'$ indicates that we sum up over multi indices $J = (j_1, \ldots, j_p), K = (k_1, \ldots, k_p)$ such that $j_1 < j_2 < \ldots < j_p$; $k_1 < k_2 < \ldots < k_p$.

A (p,p) form is *positive* if and only if its restriction to any complex analytic submanifold of dimension p in Ω is equal to the volume form of the submanifold multiplied by a nonnegative function. Equivalently, ω is positive ($\omega \geq 0$) if

$$\omega \wedge \alpha$$

is a nonnegative measure for any α which has a representation

$$\alpha = i^p \alpha_1 \wedge \bar{\alpha}_1 \wedge \alpha_2 \wedge \bar{\alpha}_2 \wedge \ldots \wedge \alpha_{n-p} \wedge \bar{\alpha}_{n-p} \tag{1.1}$$

where $\alpha_j \in C^\infty_{(1,0)}(\Omega)$. It is easy to check that a (1,1) form $\omega = \frac{i}{2} \sum \omega_{jk} dz_j \wedge d\bar{z}_k$ is positive if and only if (ω_{jk}) is a positive semidefinite Hermitian matrix. Hence for $u \in PSH \cap C^2(\Omega)$ the form

$$dd^c u = 2i\partial\bar{\partial}u$$

is positive. In particular

$$\beta = \frac{i}{2}\partial\bar{\partial}|z|^2 = \frac{i}{2}\sum_1^n dz_j \wedge d\bar{z}_j.$$

is positive in $\mathbb{C}^n$. The wedge product $\frac{1}{n!}\beta^n$ gives the standard volume form in $\mathbb{C}^n$. There is a theorem (see [LE1]) saying that a wedge product of a $(1,1)$ positive form and any other positive form is again positive. (It is essential that one of the forms is of bidegree $(1,1)$.) Thus for a collection of smooth psh functions $u_1, u_2, \ldots, u_k$ the form

$$dd^c u_1 \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_k \tag{1.2}$$

is positive. Since we do not want to restrict ourselves to smooth functions we are going to define the above differential form so that its coefficients could be identified with distributions. Here the notion of a *current* comes in handy.

Let $\mathcal{D}_{(p,q)}(\Omega)$ denote the space of smooth, compactly supported forms (shortly: test forms) in Ω of bidegree (p,q) equipped with Schwartz' topology. Any continuous linear functional on the space $\mathcal{D}_{(p,q)}(\Omega)$ is called a *current* of bidegree $(n-p, n-q)$ (equivalently: of bidimension (p,q)) in Ω. The collection of such currents will be denoted by $\mathcal{D}'_{(p,q)}(\Omega)$. When for $T \in \mathcal{D}'_{(p,p)}(\Omega)$ we have

$$(T, \alpha) \geq 0$$

for any test form α given in (1.1) we say that T is a *positive current*. An important property of a positive current is that its action can be extended to forms with continuous, compactly supported coefficients. Then the coefficients of the current can be identified with Radon measures. One can differentiate currents in the same way as distributions.

It is a well known fact that any (pluri-)subharmonic function u is (locally) the decreasing limit of a sequence of smooth (pluri-)subharmonic functions which are the convolutions of u with $\rho_j(z) := j^{2n}\rho(jz)$, where $\rho \in C_0^\infty(B)$ (B is the unit ball in $\mathbb{C}^n$) is a nonnegative, rotation invariant function with $\int \rho dV = 1$ (dV always denotes the Lebesgue measure). The sequence $u_j = u * \rho_j$ will be called *regularizing sequence* for u. Any reasonable definition of (1.2) for non smooth functions should be stable with respect to this type of regularization.

The following proposition was proved in [BT1].

Proposition 1.1 *For $u \in PSH \cap L^\infty_{loc}(\Omega)$ and a closed positive current T on Ω the currents uT and*

$$dd^c u \wedge T := dd^c(uT)$$

are well defined. Moreover, the latter current is closed and positive.

Proof The statement is local, so one can use a regularizing sequence u_j and assume that it is uniformly bounded. Since we know that distribution coefficients of T are complex measures it follows from Lebesgue's dominated convergence theorem that u_jT converges weakly to uT. Hence $dd^c(u_jT) \to dd^c(uT)$. For smooth functions u_j we have $dd^c(u_jT) = dd^c u_j \wedge T$ and so $dd^c u \wedge T$ is equal to the limit of positive closed currents $dd^c u_j \wedge T$. ∎

Applying this proposition repeatedly one can define (1.2) for locally bounded psh functions. We shall see that this definition can be extended further to cover some classes of unbounded functions but the following example, due to C. O. Kiselman [KI1], shows that we cannot apply the Monge-Ampère operator to all psh functions.

Example *The function*

$$u(z) = (-\log|z_1|)^{1/2}(|z'|^2 - 1)$$

for $z = (z_1, z') \in \mathbb{C} \times \mathbb{C}^{n-1}$ is psh in the ball $B(0, 1/2)$ but

$$\int_{B(0,r)\setminus L} (dd^c u)^n = \infty$$

for $L = \{z : z_1 = 0\}$ and $r \in (0, 1/2)$.

In 1969 S.S. Chern, H.I. Levine and L. Nirenberg [CLN] proved a very useful inequality which gives a bound on the total variation $||\cdot||$ of the measure $dd^c u_1 \wedge dd^c u_2 \wedge ... \wedge dd^c u_k$ in terms of $||\cdot||_\infty$ norms of u_j's.

Theorem 1.2 *If $K \subset\subset U \subset\subset \Omega \subset\subset \mathbb{C}^n$, T is a closed positive current and $u_j \in PSH \cap L^\infty(\Omega)$, $j = 1, 2, ..., k$ then for a constant C depending only on K, U, Ω the following inequality holds*

$$||dd^c u_1 \wedge dd^c u_2 \wedge ... \wedge dd^c u_k \wedge T||_K \le C||u_1||_{L^\infty(U)}||u_2||_{L^\infty(U)} \cdots ||u_k||_{L^\infty(U)}||T||_U,$$

Proof It is known that for a positive (p,p) current we have the following estimate for its total variation

$$||T||_K \le C_1 \int_K T \wedge \beta^{n-p},$$

where C_1 depends only on the dimension. Let us take a non negative test function ϕ in U which is equal to 1 on K and does not exceed 1 elsewhere in Ω. Applying Stokes' theorem (which is justified for those currents) and the above estimate we get for a $(n-j-1, n-j-1)$ current T:

$$||dd^c u_1 \wedge T||_K \le C_1 \int_U \phi dd^c u_1 \wedge T \wedge \beta^j = C_1 \int_U u_1 dd^c \phi \wedge T \wedge \beta^j$$
$$\le C||u_1||_U||T||_U,$$

where C depends on C_1 and the second order derivatives of ϕ. The statement follows by induction. ■

Demailly [DE] has strengthened this inequality by replacing one of the norms on the right hand side by L^1 norm.

2. Convergence of psh functions and capacity

It is well known (see e.g. [HO1][HO2]) that given a locally uniformly upper bounded sequence of subharmonic functions in some open connected subset of $\mathbb{R}^n$ which does not tend to $-\infty$ locally uniformly one can extract a subsequence which converges in L^1_{loc} to a subharmonic function. Furthermore, if subharmonic functions u_j tend to u as distributions then $u_j \to u$ in L^1_{loc}. One can strengthen the last statement when u_j are psh. Then the sequence converges in L^p_{loc} for $p \in [1, \infty)$.

However even if $u_j \to u$ in L^p_{loc} it does not imply that $(dd^c u_j)^n \to (dd^c u)^n$ as measures (see [LE2][CE1]). In [BT2] Bedford and Taylor have shown that the Monge-Ampère operator is continuous with respect to monotone sequences of psh functions. Later, Y. Xing [XI] found out that the convergence in capacity (defined below) entails the convergence of corresponding Monge-Ampère measures and that, in a way, this result is sharp. The capacity has been introduced in [BT2] and is nowadays called *the Bedford-Taylor capacity*:

$$cap(E, \Omega) = \sup\{\int_E (dd^c u)^n : u \in PSH(\Omega), -1 \leq u < 0\}$$

for a Borel subset E of Ω. It is a Choquet capacity and in bounded Ω it vanishes exactly on pluripolar sets. Recall that a set E in $\mathbb{C}^n$ is *pluripolar* if for any $z \in E$ there exists a neighbourhood V of z and $v \in PSH(V)$ such that $E \cap V \subset \{v = -\infty\}$.

Definition *A sequence u_j of functions defined in Ω is said to converge in capacity to u if for any $t > 0$ and $K \subset\subset \Omega$*

$$\lim_{j\to\infty} cap(K \cap \{|u - u_j| > t\}, \Omega) = 0.$$

The Monge-Ampère operator is continuous with respect to sequences of psh functions converging in this manner.

Theorem 2.1 (Convergence theorem)[XI] *Let $\{u_k^j\}_{j=1}^\infty$ be a locally uniformly bounded sequence of psh functions in Ω for $k = 1, 2, ..., n$; and let $u_k^j \to u_k \in PSH \cap L^\infty_{loc}(\Omega)$ in capacity as $j \to \infty$ for $k = 1, 2, ..., n$. Then*

$$dd^c u_1^j \wedge ... \wedge dd^c u_n^j \to dd^c u_1 \wedge ... \wedge dd^c u_n$$

in the weak topology of currents.

One can show that a decreasing, locally uniformly bounded, sequence of psh functions converges in capacity. The same is true for an increasing sequence converging almost everywhere to a psh function but this fact can be shown only with an application of Theorem 2.4 below which in turn requires the convergence theorem for increasing sequences (Theorem 2.3). Thus we neeed a proof of Theorem 2.3 which is independent of Theorem 2.1. For this proof one needs an important property of psh functions known as *quasicontinuity* and given in the following theorem.

Theorem 2.2 [BT2] *For any psh function u defined in Ω and any positive number ϵ one can find an open set $U \in \Omega$ with $cap(U, \Omega) < \epsilon$ and such that u restricted to $\Omega \setminus U$ is continuous.*

Theorem 2.3 (Convergence theorem for increasing sequences [BT2]**)** *The statement of Theorem 2.1 remains true if $u_k^j \uparrow u_k$ almost everywhere.*

A classical potential theory theorem, due to H. Cartan, says that any *negligible* set, that is a set of the form $\{u < u^*\}$, where u is a supremum over a family of subharmonic functions, is polar. In '60 P. Lelong conjectured that the corresponding statement should be true for psh functions, namely, that a negligible set for a family of psh functions is pluripolar. It is easy to see that the converse is true. Indeed, if $E \subset \{v = -\infty\}$ for $v \in PSH(\Omega)$ then E is negligible since $E \subset \{u < u^*\}$ for $u = \sup_{j \in \mathbb{N}} v/j$. The Lelong conjecture was proved by Bedford and Taylor.

Theorem 2.4 [BT2] *Negligible sets are pluripolar.*

The preceding theorems in this section are the essential ingredients in the proof of Theorem 2.4 as well as the solution to the Dirichlet problem for the Monge-Ampère equation which we shall discuss in the next section. This theorem has found many applications in complex analysis, especially in approximation theory and the theory of extremal functions (see e.g. [SIC]).

3. Bounded solutions of the Dirichlet problem for the Monge-Ampère equation

Let Ω be a strictly pseudoconvex domain (a sublevel set of a C^2 smooth strictly psh function whose gradient does not vanish on $\partial\Omega$), let φ be a continuous function on $\partial\Omega$ and let f be a non negative function in Ω. We consider the following Dirichlet problem

$$\begin{aligned} &u \in PSH \cap C(\overline{\Omega}) \\ &(dd^c u)^n = f\, dV \\ &\lim_{\zeta \to z} u(\zeta) = \varphi(z) \quad z \in \partial\Omega,\ \varphi \in C(\partial\Omega). \end{aligned} \tag{$*$}$$

Theorem 3.1 [BT1] *The Dirichlet problem* $(*)$ *has a unique solution when f is a continuous function on the closure of* Ω.

About the proof The proof is too long to give all the details but we shall highlight its main ideas. The uniqueness follows from the comparison principle.

Comparison principle *Let Ω be an open bounded subset of $\mathbb{C}^n$. For $u, v \in PSH \cap L^\infty(\Omega)$ satisfying $\underline{\lim}_{\zeta\to z}(u-v)(\zeta) \geq 0$ for any $z \in \partial\Omega$ we have*

$$\int_{\{u<v\}} (dd^c v)^n \leq \int_{\{u<v\}} (dd^c u)^n.$$

In particular the inequality $(dd^c u)^n \leq (dd^c v)^n$ implies $v \leq u$.

The comparison principle is also a main tool in the proofs of existence theorems which follow this one.

As for the existence part of Theorem 3.1, a difficult technical point is to handle $\det(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k})$ for non smooth u. A device that does the job is the Goffmann and Serin [GS] construction of a scalar measure associated to a vector valued measure via a given homogeneous superadditive functional. For our purpose we use the functional $\mathcal{F}$ defined on C - the cone of $n \times n$ nonnegative Hermitian matrices - by the formula

$$\mathcal{F}(A) = det^{1/n} A, \quad A \in C.$$

If μ is a C- valued measure on Ω then the scalar measure $\mathcal{F}\mu$ is given by

$$\mathcal{F}\mu(E) = \inf \sum_j \mathcal{F}(\mu(E_j)),$$

where the infimum is taken over all partitions $\{E_j\}$ of E into a finite number of disjoint Borel sets. Since for any psh function u its second order derivatives are measures one can define

$$\Phi(u) = 4(n!)^{1/n}\mathcal{F}(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}).$$

For smooth u we have $(dd^c u)^n = \Phi^n(u)\,dV$. From the corresponding properties of $\mathcal{F}\mu$ (see [GS]) one can infer the following properties of Φ.

Proposition 3.2

1) $\Phi(tu) = t\Phi(u)$ *for* $t > 0$ *and* $\Phi(u+v) \geq \Phi(u) + \Phi(v)$.

2) If ρ is a test function then $\Phi(u * \rho) \geq \Phi(u) * \rho$.

3) If a sequence of plurisubharmonic functions u_j tends weakly to u and $\Phi(u_j)$ is weakly convergent then $\Phi(u) \geq \lim \Phi(u_j)$.

4) For the regularizing sequence of u we have $\lim \Phi(u_j) = \Phi(u)$.

5) $\Phi(\max(u,v)) \geq \min(\Phi(u), \Phi(v))$.

In what follows *2)-4)* allow us to work with smooth psh functions and then to draw a conclusion passing to the limit. The last point is important since we shall define the solution of $(*)$ as the supremum over a family of psh functions. Finally, the superadditivity of Φ plays a crucial role in establishing the second order a priori estimates for the solution.

Let us now define the candidate for the solution

$$u = \sup_S v, \tag{3.1}$$

where

$$S = \{v \in PSH(\Omega) \cap C(\overline{\Omega}) : \Phi(v) \geq f^{\frac{1}{n}}\, dV,\ v_{|\partial\Omega} \leq \varphi\}.$$

To prove that u really solves the equation we start with the following proposition.

Proposition 3.3 *The function u is continuous and belongs to S. If $f^{\frac{1}{n}}$ and φ are Lipschitz then so is u.*

The proof of those facts uses an argument of Walsh [WA] which allows us to show that for $v \in S$ and small $|a|$ the function $v(a + \cdot)$ suitably modified close to the boundary also belongs to S.

In the next step of the proof of Theorem 3.1 we find second order a priori estimates for the solution in the case Ω is equal to a ball.

Proposition 3.4 *The function u in* (3.1) *has bounded second order derivatives if we assume that Ω is equal to the unit ball B, $f^{\frac{1}{n}} \in C^{1,1}(\bar{B})$ and φ is $C^{1,1}$.*

About the proof We seek for a good estimate of the expression

$$u(z+h) + u(z-h) - 2u(z),$$

when h is a small vector. Since $u(\cdot + h)$ is not defined everywhere in B and we need to work with functions from S, we shall replace $u(\cdot + h)$ by $u \circ T_h$, where T_h is a holomorphic automorphism of B with the property

$$T_h(z) = z + h + O(|h|^2).$$

This is where we need the assumption $\Omega = B$ just to have a rich group of automorphisms. Using the superadditivity of Φ one can show that for some uniform constant C the function

$$\frac{1}{2}(u \circ T_h + u \circ T_{-h}) - C|h|^2$$

belongs to S. From this the required estimate rather easily follows. In the course of the proof also the following neat formula for the Monge-Ampère measure of a composition of a psh function u with a holomorphic mapping T comes in handy

$$\Phi(u \circ T) = |\det T'|^{2/n} \Phi(u) \circ T,$$

where T' denotes the Jacobian of T. ■

Having Proposition 3.4 one can show the following theorem.

Theorem 3.5 *Suppose $0 \le f^{\frac{1}{n}} \in C^{1,1}(\bar{B})$ and $\varphi \in C^{1,1}(\partial B)$. Then the function u in (3.1) belongs to $C^{1,1}(B)$ and solves the Dirichlet problem (∗) in the unit ball.*

Let us now derive Theorem 3.1 in the case $\Omega = B$ from Theorem 3.5. One can fix two sequences of smooth functions f_j, φ_j which tend uniformly to f and φ respectively. From the comparison principle it follows that the solutions u_j of (∗), which correspond to the data f_j, φ_j converge uniformly to a psh function u. By the convergence theorem

$$(dd^c u_j)^n \to (dd^c u)^n,$$

so u is the desired solution.

The proof of Theorem 3.1 for general Ω easily follows from that special case. Indeed, by Proposition 3.3, it remains to prove that $(dd^c u)^n = f\,dV$. Let us fix a ball $B_0 \subset \Omega$ and denote by u_1 the solution of the Dirichlet problem $(dd^c u)^n = f\,dV$ in B_0, $u_1 = u$ on ∂B_0. Then v equal to u_1 in B_0 and equal to u elsewhere in Ω belongs to $\mathcal{S}$. Hence $v \le u$. Since, due to the comparison principle, $u_1 \ge u$ in B_0 we conclude that u_1 and u are equal in B_0 which shows that $(dd^c u)^n = f\,dV$ in Ω because the above is true for any ball in Ω.

One can prove the existence of solutions to (∗) under weaker assumptions on f. Suppose a measure μ satisfies the inequality

$$\mu(K) \le F(cap(K, \Omega)), \quad K \text{ compact},$$

with

$$F(x) = \frac{Ax}{h(x^{-1/n})}, \quad A > 0, \tag{3.2}$$

where $h : \mathbb{R}_+ \to (1, \infty)$ satisfies the conditions:

1) h is continuous and increasing,

2) $\int_1^\infty \frac{1}{x h^{1/n}(x)}\, dx < \infty$,

3) for some $a > 1$, $b > 1$ and $x_0 > 0$ we have $h(ax) \le bh(x)$ for $x > x_0$.

Then one can prove a priori L^∞ estimates for the solutions of

$$(dd^c u)^n = d\mu$$

with given continuous boundary data. Given F as above all such solutions are uniformly bounded (see [KO1][KO4]). Those estimates allow us to prove the following existence results.

Theorem 3.6 [KO1] [KO4] *Let us define the family of non negative Borel measures in Ω associated to a function h, satisfying the conditions above, and a positive constant A:*

$$\mathcal{F}(A, h) = \{\mu : \mu(K) \le F(cap(K, \Omega)) \quad \textit{for } F(x) = \frac{Ax}{h(x^{-1/n})}$$

$$\textit{and any compact } K \subset \Omega\}.$$

Then the Dirichlet problem (∗) has a unique solution for any $d\mu \in \mathcal{F}(A, h)$.

Theorem 3.7 [KO1][KO6] *Let*

$$f \in L^{\psi_h}(c_0) = \{g \in L^1(\Omega) : g \geq 0, \int_\Omega \psi_h(g)\, dV \leq c_0\},$$

where

$$\psi_h(t) = |t|(log(1+|t|))^n h(log(1+|t|)),$$

with h as above. Then the Dirichlet problem (∗) has a solution. Moreover, all such solutions for fixed h,c_0 are equicontinuous and uniformly bounded.

As a consequence of the last theorem we get the following statement.

Corollary 3.8 *For any $p > 1$ and any $f \in L^p(\Omega)$ the equation (∗) is solvable.*

One can take

$$\psi_h(t) = |t|(log(1+|t|))^n(1 + log(1 + log(1+|t|)))^m, m > n,$$

in Theorem 3.7. The following example, due to L. Persson [PE], shows that the assumptions in this theorem cannot be substantially weakened. If $\chi(t) = |t|(log(1 + |t|))^m, m < n$ then the Monge-Ampère equation admits unbounded solutions with pointwise singularities for some radially symmetric densities from L^χ. Indeed, one may verify that the function $f(z) = |z|^{-2n} \log^{-k} 2|z|^{-1}$ belongs to $L^\chi(B)$ for $k > m+1$ and the corresponding solution is equal $-\infty$ at 0 for $k < n+1$.

Let us now consider a slightly more general Dirichlet problem in a strictly pseudoconvex domain. Here we no longer require the solution be continuous.

$$\begin{aligned} &u \in PSH \cap L^\infty(\Omega), \\ &(dd^c u)^n = d\mu, \\ &\lim_{\zeta \to z} u(\zeta) = \varphi(z) \text{ for } z \in \partial\Omega. \end{aligned} \qquad (**)$$

A bounded psh function v is a subsolution to (∗∗) if

$$(dd^c v)^n \geq d\mu$$

and the boundary condition is met. The following theorem provides us with a large class of solutions.

Theorem 3.9 [KO2] *If there exists a subsolution for the Dirichlet problem (∗∗) then the problem is solvable.*

Applying Theorem 3.9 one can solve (∗∗) for many specific measures which are singular with respect to the Lebesgue measure.

Example *For Ω let us take the unit ball B. By B_r we denote the ball of radius r centered at the origin. Consider*

$$v(z) = \max(\log \frac{|z|}{r}, 0)$$

in B. Since $\log|z|$ is harmonic away from 0 on any complex plane containing the origin the Monge-Ampère measure of v is concentrated on $\partial B_r = \{z : \log\frac{|z|}{r} = 0\}$. This measure is also rotation invariant and thus it is proportional to the surface measure $d\sigma_r$ on ∂B_r. From Theorem 3.9 it follows that for any bounded measurable f the Dirichlet problem (**) with $d\mu = f\, d\sigma_r$ and $\varphi = 0$ is solvable.

One can prove an analogous statement for a surface measure of a smooth, strictly convex hypersurface and arbitrary φ but it requires a little bit more effort since the subsolution will not be explicitly given in general.

4. The exdended definition of the Monge-Ampère operator and the Dirichlet problem

As the Kiselman example from Section 1 shows there is no hope for a good definition of the Monge-Ampère operator on all unbounded psh functions. However, the Monge-Ampère operator can be defined on some classes of psh functions in such a way that $(dd^c u)^n$ is locally finite and that it is continuous with respect to monotone sequences of psh functions. In this section we shall follow U. Cegrell's work [CE2] [CE3].

Throughout the section Ω will denote a fixed hyperconvex domain in $\mathbb{C}^n$, $n > 1$. We call a domain in $\mathbb{C}^n$ hyperconvex if there exists nonzero $u \in PSH(\Omega) \cap C(\overline{\Omega})$ such that $u = 0$ on $\partial\Omega$. The set of such functions satisfying $\int_\Omega (dd^c u)^n < \infty$ we denote by $\mathcal{E}_0$. Observe that a polydisk in $\mathbb{C}^n$ is hyperconvex. A psh function in a polydisk which is continuous up to the boundary cannot assume arbitrary continuous boundary values. This follows from the fact that there are complex disks in the boundary and the boundary values on those disks are necessarily subharmonic. Therefore the Dirichlet problem in a hyperconvex domain may not be solvable for arbitrary continuous boundary data.

Definition *We say that a plurisubharmonic function u belongs to $\mathcal{E}_p$, $p \geq 1$ if there exists $u_j \in \mathcal{E}_0$ with $u_j \downarrow u, \sup_j \int_\Omega (-u_j)^p (dd^c u_j)^n < \infty$. A function from $\mathcal{E}_p$ belongs to $\mathcal{F}_p$ if $\sup_j \int_\Omega (dd^c u_j)^n < \infty$.*

The following Hölder-like estimate is a crucial technical tool in the proofs of the results of this section.

Theorem 4.1 [CP] *For $u, v \in \mathcal{E}_0$ and $p \geq 1$*

$$\int_\Omega (-u)^p (dd^c u)^j \wedge (dd^c v)^{n-j}$$
$$\leq C(j,p) \left(\int_\Omega (-u)^p (dd^c u)^n\right)^{(p+j)/(n+p)} \left(\int_\Omega (-v)^p (dd^c v)^n\right)^{(n-j)/(n+p)}$$

with $C(j,p) = 1$ if $p = 1$ and $C(j,p) = p(p+j)(n-j)/(p-1)$ otherwise.

The families of functions introduced above have following properties:

1) $\mathcal{E}_0 \subset \mathcal{F}_p \subset \mathcal{E}_p$, $\mathcal{F}_q \subset \mathcal{F}_p$ for $q > p$.
2) $\mathcal{E}_p$ and $\mathcal{F}_p$ are convex cones.
3) $\mathcal{E}_p$ and $\mathcal{F}_p$ are closed with respect to the operation of taking maximum of a finite number of functions.

One can define the Monge-Ampère operator on $\mathcal{E}_p$ applying the next statement.

Theorem 4.2 *Suppose $u \in PSH(\Omega)$ is the limit of a decreasing sequence $u_j \in \mathcal{E}_0$ such that $a = \sup_j \int_\Omega (-u_j)^p (dd^c u_j)^n < \infty$. Then $(dd^c u_j)^n$ is weakly convergent to a measure $d\mu$ which is independent of the choice of u_j satisfying the condition above. Thus one can define $(dd^c u)^n = d\mu$.*

It is possible to characterize those finite measures which give rise to a function from $\mathcal{F}_p$ as a solution of the Dirichlet problem.

Theorem 4.3 [CE2] *Let μ be a positive measure with finite total mass in Ω. Then there exists a unique $u \in \mathcal{F}_p$ solving*

$$(dd^c u)^n = d\mu$$

if and only if for some positive A the following inequality holds

$$\int (-v)^p \, d\mu \leq A\left(\int (-v)^p (dd^c v)^n\right)^{\frac{p}{n+p}},$$

for any $v \in \mathcal{E}_0$.

Let us observe that functions from $\mathcal{F}_p$ have essentially zero boundary values. Precisely, lim sup of such a function with the argument approaching the boundary is equal to zero. The last theorem can be generalized to cover other boundary data.

Let φ be a continuous function on $\partial\Omega$ such that there exists a solution u_φ of the homogeneous Monge-Ampère equation ($(dd^c u_\varphi)^n = 0$) which is continuous up to $\partial\Omega$ and equal to φ on $\partial\Omega$. For $p \geq 1$ the class $\mathcal{F}_p(\varphi)$ consists of those functions u for which there exists $v \in \mathcal{F}_p$ such that

$$u_\varphi \geq u \geq v + u_\varphi.$$

Theorem 4.4 [CE2] *Let μ be a positive measure with finite total mass in Ω and let φ be the function from the last paragraph. Then there exists a function $u \in \mathcal{F}_p(\varphi)$ solving*

$$(dd^c u)^n = d\mu$$

if and only if for some positive A the following inequality holds

$$\int (-v)^p \, d\mu \leq A\left(\int (-v)^p (dd^c v)^n\right)^{\frac{p}{n+p}},$$

for any $v \in \mathcal{E}_0$.

Let us now see how far can we go in extending the action of the Monge-Ampère operator. In [CE3] U. Cegrell defined a class $\mathcal{E} = \mathcal{E}(\Omega)$ of negative psh functions and proved that it is the largest family which satisfies the following two conditions:

1) If $u \in \mathcal{E}$ and v is a negative psh function then $\max(u, v) \in \mathcal{E}$.

2) If $u \in \mathcal{E}$ and $0 \geq u_j \in PSH \cap L^{\infty}_{loc}(\Omega)$ with $u_j \downarrow u$ then $(dd^c u_j)^n$ is weak* convergent.

The class $\mathcal{E}(\Omega)$ consists of those u - negative psh functions for which given $z \in \Omega$ one can find a neighbourhood of this point and a sequence $u_j \in \mathcal{E}_0(\Omega)$ with

$$\sup_j \int_\Omega (dd^c u_j)^n \leq \infty$$

and such that $u_j \downarrow u$ in U. For functions from $\mathcal{E}(\Omega)$ one can define their Monge-Ampère measure due to the following theorem.

Theorem 4.5 [CE3] *For $u \in \mathcal{E}(\Omega)$ and $u_j \in \mathcal{E}_0(\Omega)$ with $u_j \downarrow u$ the measures $(dd^c u_j)^n$ converge in the weak* topology. The limit is, by definition, equal to $(dd^c u)^n$.*

Similarly one can define

$$dd^c u_1 \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_k$$

for $u_1, u_2, \ldots, u_k \in \mathcal{E}(\Omega)$. It is possible to prove (see [CE2]) that the function in $\mathbb{C}^2$ given by

$$u(z_1, z_2) = -(-\log|z_2|)^a$$

belongs to $\mathcal{E}(B)$ (B is the unit ball) if and only if $a \in (0, 1/2)$.

To develop an interesting theory one needs to consider a smaller class $\mathcal{F}(\Omega) \subset \mathcal{E}(\Omega)$. A function $u \in \mathcal{E}(\Omega)$ belongs to $\mathcal{F}(\Omega)$ if there exists a sequence $u_j \in \mathcal{E}_0(\Omega)$ with $u_j \downarrow u$ in Ω and

$$\sup_j \int_\Omega (dd^c u_j)^n \leq \infty.$$

To have $u \in \mathcal{E}(\Omega)$ it was enough to find u_j convergent locally to u. In the class $\mathcal{F}$ one can perform integration by parts and prove the comparison principle. As for the solution of the Dirichlet problem one can reduce the solution to the case of a singular measure which is carried by a pluripolar set.

Theorem 4.6 [CE3] *Let μ be a positive measure in Ω with $\int_\Omega d\mu < \infty$. Then one can find $u \in \mathcal{E}_0(\Omega)$, $f \in L^1((dd^c u)^n)$ and a measure ν carried by a pluripolar set such that $\mu = f(dd^c u)^n + \nu$. Furthermore, if there exists $v \in \mathcal{F}(\Omega)$ with $(dd^c v)^n = d\nu$ then there exists $w \in \mathcal{F}(\Omega)$ with $(dd^c w)^n = d\mu$.*

The solution of the Dirichlet problem for measures carried by a pluripolar set is an interesting open problem.

5. The complex Monge-Ampère equation on a compact Kähler manifold

The geometers have good reasons to study the complex Monge-Ampère equation on Kähler manifolds. So far the satisfactory results have been obtained for compact Kähler manifolds. Let us consider a compact n-dimensional Kähler manifold M equipped with the fundamental form ω which is given in local coordinates by

$$\omega = \frac{i}{2}\sum_{k,j} g_{k\bar{j}} dz^k \wedge d\bar{z}^j. \tag{5.1}$$

For a Hermitian manifold the matrix $(g_{k\bar{j}})$ is positive definite and Hermitian symmetric at any point. A Hermitian manifold is Kähler if $d\omega = 0$. There is a theorem saying that locally in a neighbourhood of a given point in M there exists a psh function v such that

$$\omega = dd^c v.$$

This condition may also define Kähler manifolds among Hermitian ones. For the background in Kähler geometry we refer to [AU1][TI][YA].

The Monge-Ampère equation has the following form

$$(\omega + dd^c\varphi)^n = f\omega^n, \quad \omega + dd^c\varphi \geq 0, \tag{5.2}$$

where φ is the unknown function. The given non negative function $f \in L^1(M)$ is normalized by the condition

$$\int_M f\omega^n = \int_M \omega^n.$$

Since, by the Stokes theorem, the integral over M of the right hand sideis equal to $\int_M \omega^n$, this normalization is necessary for the existence of a solution. Observe that in any open set where $\omega = dd^c v$ the equation is not really different from the one studied so far:

$$(dd^c v + \varphi)^n = f(dd^c v)^n,$$

with $v + \varphi$ psh. The point is that $v + \varphi$ is not psh on the whole manifold since the only globally defined psh functions on M are constants.

The volume form associated to the Hermitian metric (5.1) is given by n-th wedge product $\frac{1}{n!}\omega^n$. Note that the solution of (5.2) provides us with a Kähler metric whose volume form has been prescribed. The equation has been studied since fifties when E. Calabi observed that its solution gives a new Kähler metric with presigned Ricci curvature. Recall that for a Kähler manifold (M, ω) the Ricci curvature form is given by

$$Ric(\omega) = -\frac{1}{2}dd^c[\log\det(g_{j\bar{k}})].$$

The Calabi conjecture says that given a $(1,1)$ closed form R' on (M,ω) representing the first Chern class one can find a Kähler metric ω' (in given Kähler class) with $Ric(\omega') = R'$. A short calculation and the fact that any pluriharmonic function on M must be constant lead to the conclusion that to prove the Calabi conjecture one needs to solve (5.2) for smooth (strictly) positive f. Calabi [CA] proved the uniqueness of the solution up to an additive constant and suggested that the continuity method, which works in case of the real Monge-Ampère equation, should be employed for the proof of the existence of a solution. He proved some a priori estimates for the derivatives of the solution. His work was completed only twenty years later by S.-T. Yau who derived the missing L^∞ estimates.

Theorem 5.1 [YA] *Let $f > 0, f \in C^k(M), k \geq 3$. Then there exists a solution to (5.2) belonging to Hölder class $C^{k+1,\alpha}(M)$ for any $0 \leq \alpha < 1$.*

In contrast to the case of the Monge-Ampère equation in a strictly pseudoconvex domain we have no "pluripotential" proof of the existence part of this theorem. An interesting task is to find one. However, we can use the methods of pluripotential theory to generalize this result and obtain weak solutions of the equation under similar assumption to those imposed in case of the Dirichlet problem in a pseudoconvex domain.

Let us call an upper semicontinuous function φ on M *ω-plurisubharmonic* if $dd^c\varphi + \omega \geq 0$ on M. For such φ we write $\varphi \in PSH(\omega)$. Given a Borel set $E \subset M$ one can define its capacity by

$$cap_\omega(E) = \sup\{\int_E (dd^c\varphi + \omega)^n : \varphi \in PSH(\omega), 0 \leq \varphi \leq 1\}.$$

In terms of this capacity we define some families of functions on M.

$$\mathcal{F}(A,h) = \{f \in L^1(M) : f \geq 0, \int_M f\omega^n = \int_M \omega^n,$$
$$\int_E f\omega^n \leq F(cap_\omega(E)) \text{ for any Borel set } E \subset M\},$$

where F is given in (3.2). One can show that the solutions of the Monge-Ampère equation (5.2) when f varies over $\mathcal{F}(A,h)$ are uniformly bounded. This L^∞ a priori estimate is a crucial step in the proof of the following existence result.

Theorem 5.2 [KO3][KO7] *If F is given in (3.2) and $1 \in \mathcal{F}(A,h)$, then for any $f \in \mathcal{F}(A,h)$ there exists a continuous solution of (5.2). Moreover there exists a constant $a(A,h) > 0$ such that any solution of*

$$(dd^c\varphi + \omega)^n = f\omega^n, \quad \max_M \varphi = 0,$$

with $f \in \mathcal{F}(A,h)$ satisfies $\varphi \geq -a(A,h)$.

Again as in Section 3 we have the inclusion

$$L^{\psi_h}(c_0) \subset \mathcal{F}(A,h),$$

where

$$L^{\psi_h}(c_0) = \{f \in L^1(M) : f \geq 0, \int_M f\omega^n = \int_M \omega^n, \int_\Omega \psi_h(f)\omega^n \leq c_0\},$$

with

$$\psi_h(t) = |t|(log(1+|t|))^n h(log(1+|t|)),$$

and h satisfying the conditions in (3.2). In particular this gives us continuous solutions to (5.2) for suitably normalized $f \in L^p(M)$ for any $p > 1$. Furthermore the solutions corresponding to the data from fixed $L^{\psi_h}(c_0)$ are equicontinuous. They are also stable with respect to small variation of the data f in L^1 norm. Here is an example of a stability theorem.

Theorem 5.3 [KO7] *Suitably normalized ω-psh solutions of the equations*

$$(dd^c\varphi + \omega)^n = f\omega^n, \quad (dd^c\psi + \omega)^n = g\omega^n$$

for $f, g \in L^{\psi_h}(c_0)$ and $h(x) = x^n$ satisfy

$$||\varphi - \psi||_\infty \leq c||f - g||_1^{1/(2n+3)},$$

with c depending only on c_0.

6. The complex equation of Monge-Ampère type on a compact Kähler manifold

We have seen that the study of the equation (5.2) originated in an effort to prove the Calabi conjecture. Another vital geometrical problem is to find a Kähler-Einstein metric on a given Kähler manifold. By definition such a metric obeys the equation

$$Ric(\omega) = const.\omega.$$

The solution boils down to solving the equation of Monge-Ampère type

$$(\omega + dd^c\varphi)^n = \exp(c\varphi + f)\omega^n, \tag{6.1}$$

where the constant c depends on the first Chern class $c_1(M)$. If $c_1(M)$ is negative then $c = 1$, if $c_1(M)$ is positive then $c = -1$. In the latter case the equation is not solvable in general (see [AU2][SIU][TI]). Thus there may be no Kähler-Einstein metrics on M. For $c = 1$ a solution always exists which follows from Theorem 6.1

(below) proved by T. Aubin and S.-T. Yau independently. We shall consider the equation of Monge-Ampère type in its general form

$$(\omega + dd^{\varphi})^n = f(\varphi, \cdot)\omega^n, \tag{6.2}$$

with a normalizing condition

$$\int_M f(t_0, \cdot)\omega^n = \int_M \omega^n$$

for some real t_0.

Theorem 6.1 **[AU1][YA]** *If f in* (6.2) *is positive, smooth and its partial derivative with respect to the first variable is strictly positive then there exists a smooth solution of* (6.2).

This result can be generalized along the same lines as Theorem 5.1. Suppose that f from the equation is non negative, increasing and continuous in the first variable, and

$$f(t, z) \leq const.g(z),$$

with $g \in \mathcal{F}(A, h)$, where $\mathcal{F}(A, h)$ is a family from the previous section.

Theorem 6.2 **[KO5]** *For f satisfying the above assumptions there exists a continuous solution solution of* (6.2).

Under some natural assumptions the solutions are stable when f varies slightly (see [KO7]).

Bibliography

[AU1] T. Aubin, *Equations du type Monge-Ampère sur les varietes kählériennes compactes,* Bull. Sci. Math. **102** (1978), 63-95.

[AU2] T. Aubin, *Nonlinear Analysis on Manifolds. Monge-Ampère Equations,* Grundl. d. math. Wiss. 244, Springer-Verlag, Berlin-Heidelberg-New-York, 1982.

[BE] E. Bedford, *Survey of pluripotential theory.* Several complex variables: Proceedings of the Mittag-Leffler Inst. 1987-1988. Edited by John Eric Fornaess. Mathematical Notes 38. Princeton University Press, 1993.

[BT1] E. Bedford and B.A. Taylor, *The Dirichlet problem for the complex Monge-Ampère operator,* Invent. Math. **37** (1976), 1-44.

[BT2] E. Bedford and B.A. Taylor, *A new capacity for plurisubharmonic functions,* Acta Math. **149** (1982), 1-40.

[CNS] L. Caffarelli, L. Nirenberg, J. Spruck, *The Dirichlet problem for nonlinear second order elliptic equations. I. Monge-Ampère equation.* Comm. of Pure and Appl. Math. **37** (1984), 369-402.

[CKNS] L. Caffarelli, J.J. Kohn, L. Nirenberg and J. Spruck , *The Dirichlet problem for nonlinear second order Elliptic Equations. II. Complex Monge-Ampère, and Uniformly Elliptic, Equations.* Comm. of Pure and Appl. Math. **38** (1985), 209-252.

[CA] E. Calabi, *On Kähler manifolds with vanishing canonical class,* Algebraic Geometry and Topology, A Symposium in Honor of S. Lefschetz, Princeton Univ. Press, (1955) 78-89.

[CE1] U. Cegrell, *Discontinuité de l'opérateur de Monge-Ampère complexe.* C.R. Acad. Sci. Paris, Ser. I Math., 296 (1983), 869-871.

[CE2] U. Cegrell, *Pluricomplex Energy,* Acta Math. **180** (1998), 187-217.

[CE3] U. Cegrell, *The general definition of the complex Monge-Ampère operator,* preprint.

[CP] U. Cegrell and L. Persson, *An energy estimate for the complex Monge-Ampère operator.* Ann. Polon. Math. **67** (1997), 95-102.

[CLN] S.S. Chern, H.I. Levine and L. Nirenberg, *Intrinsic norms on a complex manifold.* Global Analysis, University of Tokyo Press, 1969, 119-139.

[DE] J.-P. Demailly, *Measures de Monge-Ampère et caractérisation géométrique des variétés algébraiques affines,* Mem. Soc. Math. France (N.S.), **19** (1985) 1-124.

[GT] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order.* Grundl. d. math. Wiss. 244, Springer-Verlag, Berlin-Heidelberg-New-York, 1983.

[GS] C. Goffman, J. Serrin, *Sublinear functions of measures and variational integrals,* Duke Math. J. **31** (1964), 159-178.

[HO1] L. Hörmander, *The analysis of linear partial differential operators I,* Springer-Verlag, Berlin-New York, 1983.

[HO1] L. Hörmander, *Notions of convexity,* Birkhäuser, 1996.

[KI1] C. O. Kiselman, *Sur la définition de l'opérateur de Monge-Ampère complexe,* Proc. Analyse Complexe, Toulouse 1983, Lecture Notes in Math. **1094**, 139-150.

[KI2] C. O. Kiselman, *Plurisubharmonic functions and potential theory in several complex variables,* in *Development of mathematics 1950-2000,* J.-P. Pier editor, Birkhäuser, 2000.

[KL] M. Klimek, *Pluripotential Theory,* Oxford University Press, New York, 1991.

[KO1] S. Kolodziej, *Some sufficient conditions for solvability of the Dirichlet problem for the complex Monge-Ampère operator,* Ann. Polon. Math. **65** (1996), 11-21.

[KO2] S. Kolodziej, *The rangè of the complex Monge-Ampère operator II,* Indiana U. Math. J. **44,** 3 (1995), 765-782.

[KO3] S. Kolodziej, *The complex Monge-Ampère equation,* Acta Math. **180** (1998), 69-117.

[KO4] S. Kolodziej, *A sufficient condition for solvability of the Dirichlet problem for the complex Monge-Ampère operator,* Contemporary Mathematics **222** (1998) 241-243.

[KO5] S. Kolodziej, *Weak solutions of equations of complex Monge-Ampère type,* Ann. Polon. Math. **73** (2000), 59-67.

[KO6] S. Kolodziej, *Equicontinuity of families of plurisubharmonic functions with bounds on their Monge-Ampère masses,* to appear in Math. Z.

[KO7] S. Kolodziej, *Stability of solutions to the complex Monge-Ampère equation on compact Kähler manifolds,* to appear in Indiana Univ. Math. J.

[KO8] S. Kolodziej, *Lecture notes in the complex Monge-Ampère equation,* www.im.uj.edu.pl/gamma/kolodziej.

[LE1] P. Lelong, *Plurisubharmonic functions and positive differential forms,* Gordon and Breach, 1969.

[LE2] P. Lelong, *Discontinuité et annulation de l'opérateur de Monge-Ampère complexe.*
Sèminaire P. Lelong, P.Dolbeault, H.Skoda (Analyse) 1981/1983, Lecture Notes in Math. **1028** 219-224, Springer-Verlag, Berlin-Heidelberg-New York 1983.

[PE] L. Persson, *On the Dirichlet Problem for the Complex Monge-Ampère operator,* Doctoral Thesis No 1, 1992, University of Umeå.

[SIC] J. Siciak, *Extremal plurisubharmonic functions and capacities in* $\mathbb{C}^n$, Sophia University, Tokyo 1982.

[SIU] Y.-T. Siu, *Lectures on Hermitian-Einstein Metrics for stable bundles and Kähler-Einstein metrics,* Birkhäuser, 1987.

[TI] G. Tian, *Canonical metrics in Kähler geometry,* Lecture notes in mathematics, ETH Zürich, Birkhäuser, 2000.

[WA] J. B. Walsh, *Continuity of envelopes of plurisubharmonic functions,* J. Math. Mech. **18** , (1968), 143-148.

[XI] Y. Xing, *Continuity of the complex Monge-Ampère operator*, Proceed. of AMS **124** (1996), 457-467.

[YA] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation,* Comm. Pure and Appl. Math. **31** (1978), 339-411.