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Tartaglia's Bet

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ABSTRACT

We discuss an episode in the early history of ballistics and employ elementary calculus to settle an old bet (at least in a simple case). Specifically, for a model of projectile motion in which the resistive force is proportional to the velocity an explicit formula is derived for the optimal angle of projection. This formula involves a log-like function and gives the optimal angle in terms of a single dimensionless parameter that incorporates all of the physical constants in the model.

1 Introduction

Stillman Drake ([2], p.26) relates a charming tale of a barroom wager that Niccolo Tartaglia entered into with some military men of Verona in 1531. Tartaglia had previously invented a simple device, which he called the gunner's square, for gauging the angle of inclination of a cannon. In a dispute with the Verona gunners he claimed that the optimal firing angle for a cannon – that angle causing the ball to be cast the farthest – was at the "sixth point" of his gunner's square, that is, at 45° to the horizontal. On the other hand, the Verona gunners bet that the optimal angle was somewhat below Tartaglia's sixth mark. According to Drake, Tartaglia was vindicated by an actual field test and won the bet. About a century later Galilei ([3], p.275) published a proof that in a resistanceless medium the optimal firing angle is indeed 45°. A discussion of this result is now standard fare in a first course in calculus. But the Verona gunners were dealing with real media that offer resistance to the ball's flight, not the resistanceless Galilean model, leaving the Verona contest unsettled, at least on a theoretical, if not on a practical, level.

In [4] an elementary proof is given that in a medium that resists motion in proportion to the velocity (an assumption that Newton ([7], p.244) called "more a mathematical hypothesis than a physical one") the optimal angle is, in agreement with the Verona gunners, less than 45°. We provide another proof of this result for the simplest model of a resisting medium. This proof, unlike that given in [4], is a consequence of an *explicit* representation of the sine of the optimal angle in terms of a single physical parameter, a logarithm-like function introduced recently by D. Kalman [5], and some elementary calculus. Instructors in honors calculus or basic analysis classes may find the discussion we provide useful as an illustration of the ability of elementary analytical arguments to shed light on a basic physical problem with interesting historical roots. We set the stage for our analysis by reviewing the elementary Galilean model for a point projectile in a resistanceless medium.

2 The textbook case

Suppose a point projectile of unit mass is launched from the origin at an angle θ to the horizontal and with initial speed v. Assuming a lack of resistance and a vertically acting gravitational force engendered by a uniform gravitational acceleration g, the equations of motion of the projectile are:

$$\ddot{x}(t) = 0, \qquad \dot{x}(0) = v\cos\theta, \quad x(0) = 0$$

$$\ddot{y}(t) = -g, \quad \dot{y}(0) = v\sin\theta, \quad y(0) = 0.$$

These differential equations may be routinely integrated to yield the parametric equations

$$y(t) = -\frac{g}{2}t^2 + (v\sin\theta)t, \qquad x(t) = (v\cos\theta)t.$$

Eliminating the time parameter reveals the parabolic nature of the trajectory:

$$y(x) = \frac{-g}{2v^2\cos^2\theta}x^2 + (\tan\theta)x = (\tan\theta)x\left(1 - \frac{g}{2v^2\sin\theta\cos\theta}x\right)$$

The range, $R(\theta)$, of the projectile is then the positive x-intercept of the trajectory:

$$R(\theta) = \frac{2v^2\sin\theta\cos\theta}{g} = \frac{v^2}{g}\sin 2\theta.$$

From this one sees immediately that the optimal angle of projection, that is, the angle producing a maximum range, is given explicitly by $\theta = 45°$.

In Section 4 below we show that in the case of the simplest model for a resisting medium, while the range has no simple explicit representation, the sine of the optimal angle of projection has an explicit representation in terms of an elementary, but non-standard, log-like function. Furthermore, some elementary analysis involving this log-like function, that is well within the grasp of a good calculus student, shows that the sine of the optimal angle of projection is strictly less than $1/\sqrt{2}$ and hence the optimal angle is definitely below 45°. Before treating the model for projectile flight with resistance proportional to the velocity we first introduce the log-like function and establish a basic fact about it.

3 A Log-Like function

A particular branch of the log-like function L, defined below (the one that Kalman calls "*glog*$_-$", see [5]) suffices for our purposes. The ordinary (natural) logarithm solves the equation $y = e^x$ for x. The log-like function we employ is a tool for solving $xy = e^x$, and other "exponential-linear" equations.

For $x \in (-\infty, 0) \cup (0, 1]$, let

$$E(x) = \frac{e^x}{x}.$$

Then E is a one-to-one continuous function which is strictly decreasing on each of the intervals $(-\infty, 0)$ and $(0, 1]$ and

$$\mathrm{Range}(E) = E((-\infty, 0)) \cup E((0, 1]) = (-\infty, 0) \cup [e, \infty).$$

An explicit representation of the optimal firing angle in a linearly resisting medium will be obtained in terms of the inverse function $L = E^{-1}$. The pertinent property of L is summarized in the statement

For each $p \in (-\infty, 0) \cup [e, \infty)$, the equation $x = e^{x/p}$ has a unique solution $x = x(p) \in (0, e]$. This solution is given by $x(p) = e^{L(p)}$.

To see why this is so, note that if $p \in (-\infty, 0) \cup [e, \infty)$ and $x \in (0, e]$, then $x/p \in (-\infty, 0) \cup (0, 1]$ and the equation

$$x = e^{x/p}$$

is equivalent to

$$E\left(\frac{x}{p}\right) = p.$$

Applying the inverse function L we obtain

$$x = pL(p).$$

But then

$$x(p) = e^{x(p)/p} = e^{L(p)}.$$

We also note that $p \longrightarrow x = e^{L(p)}$ is a one-to-one mapping of the set

$$\{p : -\infty < p < 0 \quad \text{or} \quad e \le p < \infty\}$$

onto the set

$$\{x : 0 < x \le e\}.$$

Additional properties of L and a comprehensive survey of exponential-linear equations may be found in [5].

4 Linearly resisted projectiles

In [4] a simple model for the motion of a point projectile of unit mass fired with a muzzle velocity v and subject to a vertically acting constant gravitational force g and a tangentially acting resistive force that is proportional to the velocity, with constant of proportionality k, is investigated. The equations of motion modeling the projectile,

$$\ddot{x}(t) = -k\dot{x}(t), \qquad \dot{x}(0) = v\cos\theta, \quad x(0) = 0$$

$$\ddot{y}(t) = -g - k\dot{y}(t), \quad \dot{y}(0) = v\sin\theta, \quad y(0) = 0$$

are linear and may be solved explicitly. Indeed, a single integration of each equation yields

$$\dot{x}(t) = (v\cos\theta)e^{-kt}, \qquad x(0) = 0$$

$$\dot{y}(t) = -\tfrac{g}{k} + (\tfrac{g}{k} + v\sin\theta)e^{-kt}, \quad y(0) = 0$$

and an additional integration produces the parametric equations

$$\begin{aligned} x(t) &= (v\cos\theta)(1 - e^{-kt})/k \\ y(t) &= (\tfrac{g}{k} + v\sin\theta)(1 - e^{-kt})/k - \tfrac{g}{k}t \end{aligned}$$

for the trajectory of the resisted projectile.

We therefore make the substitutions

$$(1 - e^{-kt})/k = \frac{x}{v\cos\theta} \quad \text{and} \quad t = -\frac{1}{k}\ln\left(1 - \frac{kx}{v\cos\theta}\right)$$

in the second equation to obtain the following form of the trajectory:

$$y = \left(\frac{g}{kv}\sec\theta + \tan\theta\right)x + \frac{g}{k^2}\ln\left(1 - \frac{k}{v\cos\theta}x\right).$$

The range, $R(\theta)$, is the positive x-intercept of this trajectory and hence

$$\ln\left(1 - \frac{k}{v\cos\theta}R(\theta)\right) = -\left(\frac{k}{v}\sec\theta + \frac{k^2}{g}\tan\theta\right)R(\theta)$$

or equivalently

$$R(\theta) = \frac{v\cos\theta}{k}(1 - e^{-A(\theta)R(\theta)}) \tag{1}$$

where $A(\theta) = a\sec\theta + b\tan\theta$, $a = k/v$ and $b = k^2/g$. So unlike in the "textbook" case, the range of the projectile in the linearly resisting medium has no simple explicit representation but is instead characterized by the implicit relationship (1). Nevertheless, this implicit expression for the range allows the sine, s, of the optimal firing angle to be characterized. Indeed, using the fact that the optimal angle θ^* satisfies

$R'(\theta^*) = 0$ (the fact that R is differentiable on $(0, \pi/2)$ is an easy consequence of the implicit function theorem [1]), we obtain upon differentiation of (1)

$$\frac{\sin\theta^*}{a}(1 - e^{-A(\theta^*)R(\theta^*)}) = \frac{\cos\theta^*}{a}(a\sec\theta^*\tan\theta^* + b\sec^2\theta^*)R(\theta^*)e^{-A(\theta^*)R(\theta^*)}. \quad (2)$$

However, from (1)

$$1 - e^{-A(\theta^*)R(\theta^*)} = a\sec\theta^* R(\theta^*) \quad (3)$$

Substituting this into (2) and rearranging gives us

$$\frac{s}{s+c} = e^{-A(\theta^*)R(\theta^*)} \quad (4)$$

where $c = b/a = vk/g$ and $s = \sin\theta^*$. Using this result in (3) gives

$$R(\theta^*) = \frac{(c/a)\cos\theta^*}{s+c},$$

and hence

$$A(\theta^*)R(\theta^*) = (a\sec\theta^* + b\tan\theta^*)R(\theta^*) = \frac{c + c^2 s}{s+c}$$

and therefore, by (4), the sine s, of the optimal firing angle satisfies

$$\frac{es}{s+c} = \exp\left(\frac{(1-c^2)s}{s+c}\right) \quad (5)$$

where the constant $c = kv/g > 0$ encapsulates *all* of the physical parameters of the model.

We now use (5) to give an explicit representation of the sine of the optimal angle in terms of the function L introduced in Section 3. Before proceeding with the analysis we dispatch the special case $c = 1$ by noticing that in this instance

$$s = \frac{1}{e-1} < \frac{1}{\sqrt{2}}.$$

and hence $\theta^* < 45°$. For $0 < c \neq 1$, we set

$$p = \frac{e}{1-c^2} \in (-\infty, 0) \cup (e, \infty) \quad \text{and} \quad x = \frac{es}{s+c} \in (0, e)$$

and we observe that (5) is equivalent to

$$x = e^{x/p}.$$

Therefore, by the result of the previous section

$$\frac{es}{s+c} = x = e^{L(p)}. \quad (6)$$

The definition of p gives

$$c = \sqrt{1 - e/p}$$

and hence, by solving (6) for s, we get an *explicit* representation

$$s(p) = \frac{e^{L(p)}}{e - e^{L(p)}} \sqrt{1 - e/p} \tag{7}$$

for the sine of the optimal firing angle in terms of the log-like function L and the single dimensionless physical parameter $p = eg^2/(g^2 - k^2v^2)$.

Finally, we set $x = e^{L(p)}$, noting that $\ln x = L(p)$, and hence

$$p = E(\ln x) = \frac{x}{\ln x}.$$

Making these substitutions in (7), we obtain

$$s = s(x) = \frac{\sqrt{x^2 - ex \ln x}}{e - x}$$

where $0 < x < e$. The rest is elementary calculus.

First we note that $s(x)$ is always positive. A routine application of l'Hospital's rule (it's a bit easier to deal with $s(x)^2$ for this) gives

$$\lim_{x \to e^-} s(x) = \frac{1}{\sqrt{2}}.$$

(Note that $x \to e^-$ is equivalent to $k \to 0^+$, representing the limiting case of motion without resistance.) So our job is finished if we can show that $s(x)$ is a strictly increasing function. Readily available technology, for example a graphing calculator, shows that this is the case. But while a picture is worth a thousand words, it is a poor substitute for a proof. Fortunately, a simple "bootstrapping" proof shows that $s(x)$ is increasing. Since $s(x)$ is positive, it is sufficient to show that $\frac{d}{dx}(s(x)^2)$ is positive. A routine calculation shows that $\frac{d}{dx}(s(x)^2)$ has the same sign as the function

$$f(x) = 3x - e - (x + e)\ln x \qquad 0 < x < e.$$

Note that $f(0) = \infty$ and $f(e) = 0$, so if $f(\eta) = 0$ for some $\eta \in (0, e)$, then

$$0 = f'(\xi) = 2 - \ln \xi - e/\xi$$

for some $\xi \in (0, e)$. But $f'(e) = 0$ and hence, for some $\zeta \in (0, e)$,

$$0 = f''(\zeta) = -\frac{1}{\zeta} + \frac{e}{\zeta^2}$$

i.e., $\zeta = e$ for some $\zeta < e$, which is absurd. So $s(x)$ is increasing for $0 < x < e$ and $s(x) \to 1/\sqrt{2}$ as $x \to e^-$ and hence $s(x) < 1/\sqrt{2}$ for all $x \in (0, e)$.

Therefore, the sine of the optimal angle of launch satisfies $\sin \theta^* = s < 1/\sqrt{2}$ for all positive values of $c = kv/g$, confirming that the optimal firing angle is always less than 45°. Arguing within the confines of the simple model we have investigated, the Verona gunners may therefore be forgiven if they make a posthumous plea to Seniore Tartaglia to pay them their due!

5 Concluding remarks

The result derived above illustrates the use of classroom calculus to analyze an interesting historical problem in projectile motion under the assumption of a particularly simple model of resistance. The difficulties inherent in modeling resisted motion were recognized from the very beginning of the science of dynamics. Indeed, Galileo himself remarked that air resistance "...does not, on account of its manifold forms, submit to fixed laws and exact descriptions", ([3], p.252). Accurate modeling of aerodynamic drag is still a challenge to this day. A primer on resisted motion containing much useful information for mathematicians can be found in [6]. The model treated in this note is an instance of a power law (the first power) of resistance. For this law we have proved that the optimal angle of projection is below 45°, however for certain higher order power laws there is numerical evidence that optimal angle may be greater than 45° [8].

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