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Metonic Cycles, Classical and Non-Classical, and Chinese Calendrical Calculations (104 BC - 618 AD)

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ABSTRACT

First, we survey Metonic cycles used in all sorts of calendrical calculations; second we give a general description of fourteen Chinese systems of calendrical calculations based on such cycles (104 BC - 618 AD). Lastly, we apply this description to the example of the determination of the fundamental calendrical elements of the Chinese year 450, a year having a special interest because one of its calendar manuscript is still extant.

RESUMEN

Primero inspeccionamos ciclos Metonic utilizados en todos los tipos de cálculos de calendarios; en segundo lugar damos una descripción general de catorce sistemas de cálculos de calendarios chinos basados en tales ciclos (104 BC - 618 AD). Finalmente, aplicamos esta descripción a un ejemplo de determinación de los elementos fundamentales del calendario chino del año 450, un año que tiene especial interés debido a que uno de los manuscritos del calendario está aún en existencia.

Key words and phrases: *China, mathematical astronomy, calendar*

1 Introduction

The so-called Metonic cycle, or nineteen-year cycle, is classically defined by the theoretical equivalence between 19 solar years and 235 lunar months[1] so that over this common lapse of time, new moons, and other phases of the moon, occur on approximately the same calendar dates.

This famous cycle borrows its name to the Athenian Meton (around 430 BC)[2], a Greek astronomer believed to have discovered it although the same cycle is also attested in Babylonian documents around the middle of the fifth century BC or later[3]. Its practical value lies in its simplicity as well as the degree of its approximation: from an astronomical point of view, the mean lengths of the solar year and lunar months being respectively equal to 365.24220 days and 29.530588 days, the difference between 235 lunar months and 19 solar year is approximately equal to one hour an a half, a quantity which amounts to something like a day in a little more than three centuries, as a trivial calculation shows.

Such a deviation between the solar and lunar cycles is not necessarily too large for astronomical calculations, at least from the point of view of ancient astronomy. The same is even more true in the case of calendrical calculations, at least over a period of time limited to a few centuries, where differences of one, two or perhaps three days between the phases of the moon, observed and calculated, are not necessarily deemed that much important. The Metonic cycle has thus been often considered more useful by computists - calendar makers - than by astronomers, save those of the most ancient periods[4]. It belongs to numerous calendar systems in use all over the world such that the Julian and Gregorian calendar systems, the modern Jewish calendar[5] and ancient Chinese calendar systems from the Han (206 BC - 220 AD), the Three Kingdoms (220-280), Liu Song (420-479) and Toba Wei (386-564) periods.

Other relations between the solar and lunar cycles quite similar to the classical Metonic cycle, but based on constants other than 19 years and 235 months are also attested. Among these, the most often quoted one is the Callipic cycle composed of 76 years and 940 months, from the Greek Callipus of Cyzicus [6] (around 350 BC). Others, from the Mediterranean basin and Western Europe, are based on a number of years equal to 8, 11, 16, 28, 84, 95, 100, 112 or 532 years[7].

Numerous Chinese examples of such non-classical Metonic cycles also exist. They rely on a number of solar year equal to 391, 448, 505, 562, 600, 619, 657 and 686 years

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[1] The terms 'solar year' and 'lunar months' are taken here as approximate computistic equivalents of the more scientific astronomical terms "tropical year" and "synodical month". They are less precise because they both mean either "the integer number of days contained in the solar year or lunar month" or "the decimal (or fractional) values of the solar year and lunar months on which calendrical calculations are based". In the sequel, when using these, the context will show the intended meaning.

[2] Cf. D.R. Dicks, 1970, p. 88; G.J. Toomer, 1974, p. 337-340.

[3] O. Neugebauer, 1975, I, p. 365 and 541; J.P. Britton, 1999, pp. 239-242.

[4] J. P. Britton, 1999, p. 239-242, for example, gives an analysis of the nineteen-year cycle with respect to the Babylonian lunar theory.

[5] E.G. Richards, 1998, p. 223.

[6] Callipus was contemporary with Aristotle. Cf. D.R. Dicks, 1970, p. 190.

[7] W. M. Stevens, 1995, p. 50-51; G. Rocca-Serra, 1980, p. 28.

(Table 5). But they are almost entirely unknown, save to a handful of specialists of the history of the Chinese calendar and astronomy - more generally, everything presently available in most Western sources concerning the historical Chinese calendar and its techniques of calculation is extremely wanting - .

Incidentally, the Chinese 600 years cycle from the Northern Liang dynasty (5-th century AD, cf. Table 4 below, entry no. 5), which occurs in this list, has 600 years and 7421 lunar months, exactly as the much more ancient, and numerically identical, 600 years cycle attributed to Pliny and Hipparchus[8]. Could this coincidence be the consequence of a transmission of knowledge between Greece, the Roman Empire and China? The unknown answer might be positive or negative but it surely cannot be given on the mere basis of anteriority, even when measured by an impressive number of centuries, as it is the case here. The difficulty comes from the fact that the Chinese source[9] from which the constants 600 and 7421 are extracted do not give any further precision while no other known documents refer to the same question.

Most fundamentally, it is important to realise that there is more than meet the eye in Metonic cycles, classical and non-classical: in themselves, the equality between a number of years and a number of months do not determine the usage such an equality is made of, no more than the structure of the underlying calendrical computations.

In the Western world, the classical Metonic cycle is, *inter alia*, but fist of all, the basis of the lunar cycle on which complex Easter calculations were erected[10]. With its underlying 'ecclesiastical moon' and the dependence of numerous Christian religious festivals on the fixation of the date of Easter, it almost becomes a kind of Christian concept. Seen under such an angle, the Metonic cycle is not applied to the determination of calendrical lunar months because the calendar contains no such months.

By contrast, in certain luni-solar calendar systems, Western of not, Metonic cycles depend on the cycle of the moon for the determination of lunar months, ordinary and intercalary. Considered in this way, Metonic cycles are intercalary-cycles and their purpose is the synchronisation of the lunar and solar cycles of the calendar. For example, systems based on the 19/235 equivalence admit 7 intercalary months because 235 months are composed of $12 \times 19 + 7$ months. Their lunar year is thus composed of 12 months, by analogy with the solar year, so that 7 lunar year out of 19 will have 13 and not 12 lunar months. But, more precisely, what is the rank of the intercalary years? And within a given intercalary year what is the rank of the intercalary month? It is impossible to tell without a knowledge of the underlying calendrical calculations.

When the available historical documentation is limited to the knowledge of the existence of a Metonic relation, reconstructions of underlying calendrical systems are open to doubt or even theoretically impossible. Precisely for this reason, Classical scholars have produced mutually incompatible reconstructions of the antique Athenian calendar, where, in particular, the rank of intercalary months within the nineteen year cycle changes from one author to the other. Some authors write that the intercalary

[8] B. Goldstein, 1995, p. 155-158.

[9] *Kaihuang zhanjing*, ch. 105, p. 754.

[10] Coyne G. V. S.J., Hoskin, M. A. and Pederson, O. (ed.). 1983, p. 29 sq.

years have the following ranks: 3, 5, 8, 11, 13, 16, 19, others 3, 6, 9, 11, 14, 17, 19 or 3, 6, 8, 11, 14, 17, 19 among other possibilities [11].

Like their Athenian counterparts, Chinese antique calendar systems are poorly understood even if published chronologies and concordance tables of the antique Chinese calendar give an opposite impression: some of these take us back to the foundation of the Zhou dynasty, in 841 BC, others still earlier, but in all cases the available tables looks impressive.

Even without going as far back in time, the less ancient Qin period (221-207 BC) and the beginning of the Han dynasty (206 BC - 220 AD) are not well understood too. The analysis of recently found manuscript calendars from these periods do in no way confirm the traditional idea according to which antique Chinese calendar systems follow the classical Metonic cycle within six different systems of calendrical calculation (*gu liu li*, literally, 'the six ancient calendar systems'), differing the one from the other by the choice of the origin of time (or epoch), but all admitting the same values for the solar and lunar year: $365\frac{1}{4}$ days and $29\frac{499}{940}$ days, respectively [12].

Recently, in a revolutionary study, the Swiss sinologist R. H. Gassmann explains that, contrary to all admitted beliefs, the Chinese calendar from the Spring and Autumn period (722- 481) do not follows a Metonic pattern but rather some kind of empirical pattern admitting both the insertion of intercalary months and intercalary days [13] in order to keep the solar and lunar components of the calendar roughly in agreement the one with the other [14]. If the result of this revolutionary study is accepted, the history of the antique Chinese calendar will need a drastic revision.

For much less ancient periods, the situation is quite different however: with the Han calendar reform of 104 BC, Chinese calendrical calculations are documented for the first time in Chinese history. Special treatises on calendar systems from this period are extant, whereas for more ancient historical periods nothing of the sort exists.

These special treatises which have been handed down to us are mainly the monographs on the calendar preserved in the successive Chinese official annals [15] (See below their partial list, limited to the needs of the present article, in the bibliography of primary sources). Through their medium, an impressive set of mathematical

[11] Cf. the well-documented and very interesting footnote 6, p. 99, in Pritchett, 2001.

[12] Cf. Chen Zungui, 1984, p. 1419-1422; Martzloff, 2002; Arrault, 2002 (overall description of extant antique Chinese calendar manuscripts).

[13] R. H. Gassmann, 2002. The outcome of this recent and outstanding research has not yet been discussed by sinologists.

[14] The notion of intercalary days has never heard of before in the context of Chinese studies and these new ideas have not yet been critically evaluated by specialists.

[15] The fact that these technical treatises are not independent treatises (like, say, Ptolemy's *Almagest*) but only later monographs inserted in historical annals, depends on the fact that, in China, calendrical science was intimately connected with chronology and history as well as astronomy and astrology. China's greatest historian, *Sima Qian* (around 145-86 BC), and author of the *Shiji* (Chinese Annals), for example, was at the same time and astrologer, astronomer and specialist of calendrical calculations. Transposed in the Western context, it is as though we had indirectly access to ancient science through the medium of the writings of later historians of science : for the study of Kepler, for example, we would be in a position to conduct researches on the sole basis of A. Koyré's historical studies, not on Kepler's original works. Hence the baffling difficulty of such studies.

techniques specialised in calendrical calculations is theoretically accessible.

Of course, this does not amount to saying that owing to these texts everything becomes suddenly clear and simple, quite the contrary: the usual problems inherent in the transmission of ancient texts are intensely complex. Extant editions are mostly late, or very late, and often corrupted; the context of their production is not well understood; original editions and manuscripts are unavailable and still worse, the sources on which they are based are difficult to track back.

In addition, there is not doubt too that Chinese technical texts tends to be more difficult to apprehend than their literary counterparts: everything is stated in the form of abstract and general algorithms, using numerous technical terms almost never defined and listed in no dictionary. Mathematical proofs are absent and specific numerical examples of calculations are never given. Overall, understanding these texts is an activity closer to the decoding of an unknown language that one akin to the reading of ancient Chinese, even if the kind of Chinese classical language involved in these technical texts is fairly stereotyped and limited from the point of view of its lexicon and syntax. Everything seems written in plain language but it is not in fact: technical terms lurks everywhere but appear without warning and look like ordinary words; mnemotechnical formulations abound and consequently, words do not really mean what they seem to mean at first. In a word, underlying linguistic conventions are in no way those of a literary text and their nature is *a priori* unknown.

After years of painstaking studies published in confidential publications difficult to access, a general picture of the subject now begins to gradually emerge [16]. The following synthetic presentation is based on the author's work in progress and aims at giving an general description of the most fundamental aspects of the subject for the period 104 BC - 618 AD.

2 Chinese calendrical calculations (104BC-618AD).

2.1 The Chinese calendar and its reforms

Like all calendars, the Chinese calendar is an extremely conservative system: it does not easily integrate new elements and change is its enemy. For example, it is striking to note that popular Chinese calendar manuscripts found in the Dunhuang caves (780-1035 AD) at the turn of the twentieth century [17] have a factual content practically identical with that of Chinese calendars presently in use in Eastern Asia[18].

But this does not amount to saying that in all its aspects the Chinese calendar is static. Like all other human accomplishments, it has its own history which allows us to state that, for example, it began to integrate weekdays (Sunday, Monday, etc.) from the seventh century AD, approximately, probably as a result of Christian and manichean religious influences[19]. In the same order of ideas, it is also obvious that

[16] Cf. the extensive bibliography in J.C. Martzloff 2000, p. 403-407.

[17] On this fascinating discovery cf. P. Hopkirk 1980.

[18] Deng Wenkuan, 1998

[19] Cf. A. Arrault, 2002, p. 25.

the festivals observed in the present Chinese calendar are not the same as those in force two thousand years earlier. But, as a rule, the calendar changes very slowly. Its history makes sense only over extremely long periods of time and its fundamental structure is so to speak almost 'unbreakable'. The numerous superstitious elements it contains are left unaffected by all sort of conflicting political events taking place over long strings of centuries.

Paradoxically, however, Chinese calendrical calculations have been submitted to innumerable reforms [20] over short - or even very short - periods of time. In two thousand years, approximately, not less than fifty such reforms have been recorded. Even within the limited period considered in the present article we are obliged to take fourteen such reforms into consideration, some being applied during more than two centuries while others do not exceed 10 or 20 years, with various other intermediary length of duration between these extremes (Table 4). For this reason, the Chinese case is highly unusual, it constitute a kind of record, nothing of the sort being attested elsewhere.

So, on the one hand, the calendar itself remains quasi-static and changes only over 'geological' periods of time, on the other hand underlying calendrical calculations constantly change over short periods of time. How is this possible?

In fact, changes brought into the calendar by calendrical calculations are quite real but they concern elements so inherently irregular (or seemingly irregular) that they are extremely difficult to detect for calendars users: even after many reforms, everything looks as if nothing had changed.

Let us take, for example, he distribution of long and short months and of the position of intercalary months. None of these follow an easily predictable pattern, they are inherently irregular, or rather seemingly irregular, even within a given calendar system. Should it change, nobody would notice it save calendar makers.

In other words, what is valid for a given year does not give in itself an indication about the structure of any another future year. This situation is very different, for example, of that prevailing in the Gregorian calendar where the beginning of the year always happen on January 1^{st} and where the number of days of each months always follows an easily recognisable pattern from year to year: January 31 days, February 28 (or 29) days, March 31 days, etc. In the Chinese calendar the date of the lunar new year is already variable from year to year (see below) and if it is true than a reform of the calendar will modify the pattern of succession of dates in different ways according to the underlying system of calculation, the resulting change is absolutely not perceptible on the mere basis of a general knowledge of the visible structure of the calendar.

But, if changes are that much imperceptible why were they needed after all? The answer to this question could lead us very far; it has to do with the tremendous political importance of the symbolic value of a correct calendar for the Imperial power,

[20] These reforms are generally called 'reform of the calendar' by historians of China but, in fact, it is extremely important to realise that these so-called reforms of the calendar have never affected the calendar itself in its most fundamental structure but only its underlying calculations. The so-called reforms of the calendar are in fact reforms of calendrical calculations or, better, reforms of the computus.

that is, a calendar astronomically consistent: calculations had to be consistent with astronomical phenomena, especially the phases of the moon and luni-solar eclipses conjunctions. Since it appeared that the construction of a 'correct calendar' was an extremely difficult task, Chinese official calenderists were asked to create new systems of calculation in order to get a better agreement between the calendar and the heavens. In that way, the changes brought into the structure of the calendar became ascertainable by comparison with what could be observed in the heavens while the visible structure of the calendar remained apparently immobile.

To sum up, the Chinese calendar has both an invariant visible structure left untouched by the successive reforms of the calendar (at least within the limits of the period studied in the present article) and a highly variable invisible mathematical structure whose effects are imperceptible save by comparison of the calendar with the lunar phases and other phenomenon less easy to locate such as solstices and equinoxes.

2.2 The invariant structure of the Chinese calendar

Considered in its entirety, the Chinese calendar is a complex structure relating to highly particular socio-cultural aspects of Chinese civilisation, including religion, popular beliefs, divination, medicine, social-life, festivals, agriculture, astrology astronomy and mathematical astronomy, among others. In the sequel, we only considers its most essential astronomical aspect, its backbone, so to speak, and only for the period 104 BC - 618 AD: what is stated here is not necessarily wholly valid for subsequent periods.

Fundamentally, the Chinese calendar is a luni-solar calendar where the lunar and solar year are mutually synchronised through the regular insertion of an intercalary lunar month, by means of a special rule explained hereafter, in the mathematical part of the article. Its fundamental time-unit is the day. It begins with the instant of midnight and lasts until the next midnight. Days are cyclically and continuously enumerated by groups of sixty, by means of sixty special elements, known as the 'sexagesimal cycle', a very ancient cycle (already present in the oracle-bone from the Shang dynasty, around 1400-1500 BC) determined by the simultaneous enumeration of two sets of symbols respectively composed of ten and twelve elements, called the ten celestial trunks *tiangan* and the twelve earthly branches *dizhi* - Further details on this cycle are omitted here, because they are not needed in the special set of calculations explained here, but the reader will easily find everything he would like in practically any source mentioning the Chinese calendar -. For the needs of the present article, it will be quite sufficient to denote these 60 elements in a very simple way by integers from 0 to 59, sometimes also noted #0, #1, ..., #59 in order to distinguish them from ordinary integers - integers from 1 to 60 would also be suitable but the first solution is more convenient because, from the point of view of calendrical calculations, these number are in fact the result of reductions of ordinary integers modulo 60 - .

The solar year begins with the theoretical Winter solstice and covers the span of time included between two successive theoretical Winter solstices. It is composed of 366 days and divided into 24 equal intervals of time [21] whose real day-length is nevertheless equal to 15 or 16 days because the division of the theoretical value of the solar year into 24 equal intervals leads to decimals, or rather fractions, while the final calendar only retain the day as its fundamental time-unit.

These 24 intervals determine the 24 solar 'breaths' *qi*[22]. Although the term 'breath' evokes a process taking place over a certain period of time, the 24 *qi* are also associated with the idea of nodes[23], or special instant of time, by analogy with nodes on a bamboo trunk *jie*. Both meanings are attested, but from the point of view of calendrical calculations, only nodes should be considered, the 'solar breaths' being defined by the interval whose extremities are two consecutive nodes.

The 24 solar nodes are divided into even *jie* and odd *zhong* nodes according to their parity when enumerating them from their first element, i.e., the Winter solstice. We define the interval of time between two consecutive odd nodes as a solar month although, strictly speaking, this notion does not occur in the Chinese original sources.

We skip the enumeration of the Chinese names of the 24 nodes because it is unnecessary for the needs of the present article. We will nevertheless note that the nodes no. 1 and 13 are respectively the theoretical Winter and Autumn solstice while the no. 7 and 19 are the Spring and Autumn equinox. Here, it is important to note that these calendrical solstices and equinoxes are not exactly astronomical (save the Winter solstice, which is chosen in such a way as to coincide with the true astronomical Winter solstice) because they are theoretically based on an arithmetical division of the year into 24 equal intervals, not variable intervals, as should be the case if these divisions were fully astronomical. Hence, within the variety of Chinese calendar considered here, day and night are never equal at the time of the calendrical equinoxes; there generally exists a difference of two or three days between astronomical and calendrical solstice and equinoxes.

We also note that the Chinese seasons do not coincide with the astronomical seasons because the four beginnings of the seasons of the Chinese calendar (nodes no. 4, 10, 16 and 22) occur one month and a half earlier than these. Consequently, from the point of view of the Chinese calendar, springs begins in mid-winter and autumn in mid-summer. The Chinese calendar shares this astonishing characteristic with the Celtic calendar[24].

The Chinese lunar year is composed of 12 or 13 lunar months *yue* containing either 29 or 30 days and respectively called 'short months' *xiao yue* and 'long months' *da*

[21] Generally called the '24 solar periods'.

[22] The term *qi* refers to the antique Chinese pneumatic theory. According to the context it can mean 'air', 'atmosphere', 'gaz' and even 'weather'. In the context of the calendar, the Chinese year is so to speak analogous with a kind of respiration with periods of increasing or decreasing activity.

[23] The term 'node' used here is an exact equivalent of the Chinese term *jie* but not of the term *qi* which globally designates the 24 solar nodes, also referred to by the compound *jieqi* in the context of the modern Chinese calendar. In this respect, many authors rather use the neutral expression 'solar terms', but it seems better to insist on the idea that, from the point of view of Chinese calendrical calculations, the *qi* are in fact instants of time, points, extremities of intervals. The precision is all the more important that usually (for example in Chinese dictionaries) the distinction between the 24 'solar terms' considered as periods of time and as time-instants is practically never made.

[24] D. Laurent, 2002, p. 113-127 (Celtic calendar).

yue. Years composed of 12 months are 'ordinary', others 'intercalary'. The months begins with the calculated new moon *shuo* and extends until the next new moon.

Within the span of time considered here, the distribution of short and long months over a lunar year follows two distinct patterns showing (a) an uninterrupted regular alternation of long and short months (b) an alternation of long and short months interrupted by the apparition of two successive long months. Thus, the lunar year may be composed either of 6 short and 6 long months, 5 short and 7 long months or 6 short and 7 long months. The lunar year has thus either 354, 355 or 384 days.

Despite the simplicity of these patterns, the nature of a given year cannot be predicted without calculations because a year can begin either with a short or a long month and the position of couples of successive long months do not follow an immediately obvious pattern.

In addition, it is certain, than the calendar cannot only follow the most simple alternating pattern between short and long months, because, if it were the case, the lunar year would contain 6 long months and 6 short months regularly alternated; this would imply that the underlying duration of the lunar month would be equal to 29.5 days, whereas its true mean length is a little longer (around 29.530589 days in fact), a fact known by the Chinese of antiquity, although not with the modern degree of precision of course. A simple calculation, omitted here, shows that, over sufficiently long periods of time, and given that the calendar respects as much as possible the actual astronomical mean length of the lunation, the actual proportion of long and short months in the Chinese calendar is respectively equal to 53% and 47%, approximately.

Whatever the distribution of long and short months, the 12 lunar months $N_1, N_2, \cdots, N_{12}$, are determined by the set of new moons $n_1, n_2, \cdots, n_{12}$ and are by construction bijectively coupled with the 12 odd solar nodes, $s_1, s_3, \cdots, s_{23}$ in an invariable way so that a given odd node can only belong to the same lunar month, never to another one, and conversely.

By convention, the 11-th lunar month always contain the calendrical winter solstice, s_1, and this initial solstice cannot belong to any another month. In the same way, the 12-th month contain s_3, the 1-st month contains s_5, and so on. But the position of these odd solar nodes within their corresponding month is variable: they can occur on any day, either the first, the last or any other day.

More mathematically, the 11-th month, N_{11}, is equal to the interval $[n_{11}, n_{12}[$, the 12- th,N_{12}, to $[n_{12}, n_1[$, the first, N_1, to $[n_1, n_2[$, etc. and

$$\begin{array}{lll} n_{11} \leq s_1 < n_{12}, & s_1 \text{ is coupled with } & N_{11} = [n_{11}, n_{12}[\\ n_{12} \leq s_3 < n_1, & s_3 \text{ is coupled with } & N_{12} = [n_{12}, n_1[\\ n_1 \leq s_5 < n_2 & s_5 \text{ is coupled with } & N_1 = [n_1, n_2[\\ \cdots & & \end{array}$$

By contrast, no such a coupling between nodes and lunar months exists in the case of even nodes: a given even solar node can sometimes belong to two different lunar months. The last node s_{24}, the one preceding s_1, for example, can either belong to the 11-th or to the 10- month.

It sometimes happens that a calculated lunar month cannot be coupled with any odd node. The corresponding month is then said to be intercalary and the lunar year is also said to be intercalary. In that case, the lunar year is composed of 13 months but the new set of months is not enumerated from 1 to 13 but always from 1 to 12. To achieve this, the intercalary month bears conventionally the same number as the month which immediately comes before it and is declared 'intercalary'.

The rank of the intercalary month within the sequence of ordinary months is not fixed once for all: it can occur anywhere. For examples, the following years BC: 88, 50, 80, 61, 72, 45, 83, 56, 94, 86, 78, 59 have an intercalary month respectively located after any of the twelve ordinary months of the lunar year[25].

Calendrical calculations are conceived in such a way that the occurrence of an intercalary month can only occur when the shift between the lunar and solar year (precisely defined hereafter) has attained one lunar month.

The coupling between lunar months and solar nodes also determine the beginning of the lunar year. By definition, the first lunar month is the month containing the odd solar node s_5 (i.e. the fifth solar node which is also the third odd solar node) - other possible definitions have been also adopted over various historical periods but the present one is by far the most common and we will not consider other possibilities here -.

Lastly, given the limitation of the luni-solar shift to one lunar month, the definition of the first lunar month also imply that the new year, i.e. the first day of the first lunar month can vary within an interval of one month, between January 21-th and February 20-th, when using dates of the Gregorian calendar projected into the past (proleptic Gregorian calendar), before the Gregorian calendar reform in 1582, as terms of comparison.

3 Calendrical calculations

3.1 Calendrical constants

From the first documented official reform of Chinese calendrical calculations, in 104 BC to the reform promulgated in 618 AD (Table 4), all known Chinese calendrical systems belong to the Metonic type, classical or non-classical. In the sequel, These types will be denoted by the abbreviation α/β, meaning that a solar years are equal to $12\alpha+\beta$ lunar months, where β denotes the number of intercalary months included in α solar years. For example, the system no. 12 in the Table 5 below belongs to the 448/165 type, that is 448 years contain $448\times 12+165$ lunar months, or 5376 ordinary months and 165 intercalary months. Again, always in the same table the systems no. 1, 2, 3, 4 and 6 all belong to the 19/7 type and, consequently they obey to the nineteen-year classical cycle, composed of 235 months, 7 of which are intercalary.

Obviously, the system 448/165 just mentioned is different from the others defined by the constants 19/7, but the reverse is not true: systems with the same Metonic constants are not necessarily identical, and in actual fact, they are not. For example,

[25]This can be checked from any chronological table of the Chineses calendar, for example Zhang Peiyu, 1997.

systems no. 1, 2, 3, 4 and 6 all belong to the same 19/7 type but they are nonetheless all different: the Metonic 19 and 7 constants do not determine the underlying calendrical calculations. Others parameters must be taken into account.

The fundamental parameters determining Chinese calendrical calculations include, among other things, the fractional lengths of the solar year and of the lunar months, mostly referred to in the original sources in a special way, by fractions a/b and c/d (denoting a number of days), where a, b, c and d are integers chosen in such a way that they respect a Metonic relation, classical or non-classical (Table 5) and where the denominators b and d are not necessarily equal the one with the other (in fact, they are never equal).

The form of the constants a, b, c, d listed in Table 5 below looks surprising, because they contradict the admitted and oversimplified notion of the decimal and positional simplicity of the Chinese antique numeration system[26]; decimals such as 365.25 days or 29. 53 days or other similar, more direct notations, never occur. In actual fact, the numerators and denominators of the fractions a/b and c/d involve numbers ending with unexpected digits, such as 81, 1581749, 53563, 116321 or 3939.

One wonders why the Chinese have deliberately chosen denominators such as 116321 or 3939 rather than the more simple 116320 or 3940: the last units '1' and the '9' are certainly non-significant for the precision of calculations. And also why different systems of calendrical calculation are not based on fractions with the same denominators, chosen in a regular way? Why some systems (no. 8, 9, 10, 11) have values of the solar year and lunar month reduced to the same numerator - yes, 'numerator', not 'denominator' -, to the detriment of the simplicity of calculations? Such characteristics are very unusual but they are never mentioned by historians of Chinese mathematics. They can be explained, however, by the close interlocking between rational and irrational mathematics in Chinese calenderics.

As a rule, in the antique Chinese context, numbers are both arithmetical quantities and mystical entities. For example, 81 is not only plainly equal to 9×9 but is also a symbolic number, related to the volume of the fundamental musical tube, created and standardised by the legendary Yellow Emperor. It is a tube mystically related to official measures of length and capacity by piling up inside medium sized black grains of millet. Chinese antique 'physics' also defines its fundamental quantities as the result of as the result of numerological considerations[27]. A full analysis of numbers used in Chinese calendar systems would thus lead us far from purely rational mathematics. But it is also a fact that this aspect of the question has no influence on the results of the calculations: once defined, the fundamental constants of calendrical calculations wholly follow the universal laws of arithmetic. Here, we shall limit ourselves to the mathematical side of the question.

3.2 Calendrical calculations

The set of elements liable to be calculated in Chinese calendrical systems is extremely wide but what follows is restricted to the mathematical kernel common to all systems,

[26] J. Needham, 1959, p. 5-17.
[27] H. A. Vogel, 1994, p. 137.

i.e. the 24 solar nodes and the 12 or 13 lunar months.

Like all other such systems, Chinese calendrical systems depend on the determination of the origin of time (technically called the 'epoch'). The Chinese epoch of systems conceived between 104 BC and 618 AD is called the 'Great Origin' *shangyuan*. It represents a purely fictitious origin, fixed, by definition, at the instant of midnight of the first day of the calendar taken into account into calendrical calculations, in such as way that it simultaneously coincide with :

- the initial Winter solstice;
- the initial new moon of the 11-th month, n_{11}, i.e. that located at the beginning of the lunar month containing the Winter solstice, according to the conventional numbering of lunar month and the way they are coupled with the solar nodes, as explained above;
- the first element of the sexagesimal cycle,#0.

These initial conditions being fulfilled, the year to be calculated is located in time on the basis of a time-parameter t equal to the integer number of solar year elapsed from the Great Origin so that $t = os_1$, where o is, in particular, the Winter solstice of the Great Origin, and s_1 the Winter solstice immediately preceding the lunar year to be calculated (Fig. 1). In actual calculations, the value of t is trivially derived from a constant giving the number of year contained in the interval of time beginning with the Great Origin and ending with a conventional year x_0, fixed once and for all (see below IV-3):

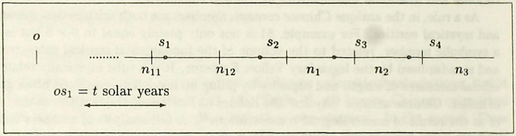

Fig. 1 The fundamental elements of Chinese calendrical calculation: o = the Great origin, s_1 = the Winter solstice preceding the lunar year to be calculated, and beginning with the new moon n_1. The calculations concern the determination of the set of new moons $n_{11}, n_{12}, n_1, \ldots$ and solar nodes $s_1, s_2, \ldots$, likewise equally distant the ones from the others, given the value of $t = os_1$.

Given the above initial conditions and value of t for the year to be calculated, the calculations consists in the determination of :

(i) the 24 solar nodes, $s_1, s_2, \cdots, s_{24}$

(ii) the 12 (or 13) corresponding calendrical new moons - noted $n_{11}, n_{12}, n_1, n_2, \cdots$ as has been already said above.

(iii) the nature of the lunar year (ordinary or intercalary) and the rank of the intercalary month, when one exists.

Then, all the calculations are done in such a way that two successive solar nodes (respectively new moons) are separated the one from the other by a constant, but non-integer, number of days, respectively equal to $a/24b$ days and $c/12d$ days. For this reason, the calculations are said to rely on mean elements *ping* (= mean) by contrast with those which would use true elements *ding* (literally 'determined elements' but meaning in fact 'true elements', in an astronomical sense).

Now, let a calendar system characterised by the Metonic relation α/β whose non-independent parameters, α, β, γ are such that

$$\alpha \text{ solar years } = \gamma \text{ lunar months} = 12\alpha + \beta \text{ lunar months} \tag{1}$$

so that α solar years are composed of 12α ordinary lunar months and β intercalary months.

Then, with the same notations as above, the calculations are determined by the following formula[28]:

$$\begin{aligned} m(t) &= \lfloor \gamma t/\alpha \rfloor & (2)\\ e(t) &= \gamma t \bmod \alpha & (3)\\ j(t) &= \lfloor cm/d \rfloor & (4)\\ s_1(t) &= < \rfloor at/b \rfloor \bmod 60; at \bmod b > & (6)\\ n_{11}(t) &= < j \bmod 60; cm \bmod d > & (5) \end{aligned}$$

where, by definition :

- $\lfloor a/b \rfloor \stackrel{def}{=}$ the integer part of the fraction a/b ;
- $a \bmod b \stackrel{def}{=} a - b \times \lfloor a/b \rfloor$, where a and b are integers;

  (in other words, when dividing a by b, the resulting integer quotient is equal to $\lfloor a/b \rfloor$ and the integer remainder to $a \bmod b$).

and where

- $< a; b >$ represents the result of a calculation under the form <integer part modulo 60; fractional part>, where a is an integer modulo 60 and b a fraction whose function is the precise fixation of the n_i and s_i inside the day whose integer part is equal to a.

Such a decomposition of the results of calculations into two elements is in no way arbitrary and is justified by the fact that the Chinese original sources do draw such a distinction by calling a the 'great remainder' *da yu* and b the 'small remainder' *xiao yu*, i. e. the quotient and the remainder of some division.

If need be, $< a; b >$ can also be transformed into a single composite number $a + b/x$ where x is the quotient of the division leading to a and b. (cf. the example of calculations developed in the next paragraph).

[28] In the sequel we use the abbreviations m, e, j, ... rather than $m(t), e(t), j(t)$ ...

The meaning of the other undefined symbols is as follows :

- $m=$ the integer number of months contained in t years ;
- $j=$ the integer number of days contained in m months ;
- $e=$ the age of the moon with respect to the instant of the Winter solstice, i.e. the time elapsed between the last new moon and the instant of the winter solstice or the length of the interval $[n_{11}s_1[$, the unit of time being such that, converted into a number of lunar months, it becomes equal to e/α lunar months. In the more general theory of the calendar, Chinese or not, e corresponds to what is called technically the epact. Although variously defined as the case may be, it always represents the fundamental parameter measuring the luni-solar shift. In the Chinese case, at the instant of the origin, for example, $t = 0$ years and, since, in virtue of the above initial conditions at the instant of the Great Origin, the new moon coincide with the winter solstice, $e = 0$. One year later $t = 1$, and when the underlying calendrical system belongs to the classical Metonic cycle 19/7 ($\alpha= 19$; $\beta= 7$ and $\gamma= 235$, $e= 235 \bmod 19 = 7$; so that the epact corresponding to $t = 1$ is equal to 7/19 months. A month containing approximately 29.53 days, $e \cong 10.88$ days, as could be expected, since, as can be easily checked, the difference between 12 solar months and 12 lunar months is precisely equal to such a number of days.

Once n_{11} and s_1 determined, the following new moons and solar nodes are successively determined by repeated additions of the constant value of the lunar month and of the solar period (solar year/24).

Then, the possible existence of an intercalary month is determined by the following criterion:

$$\text{If } e \geq \alpha - \beta \text{ then an intercalary month necessarily occurs} \qquad (7)$$

When (7) is verified, the rank of the intercalary month with respect to other lunar months is obtained by checking the coupling between lunar months and odd solar nodes so that the intercalary month is precisely the one which cannot be coupled with any odd node, as already stated above. It should be noted, however, that (7) does not guarantee that an intercalary month will necessarily occur during the next lunar year but only that it could be any of those determined by the following new moons: n_{11}, n_{12}, n_1, and so on.

Lastly, the final calendar is obtained by restricting the results of the calculations to their integer part so as to get a day number without decimals (or rather, fractional parts). Usually, only what concerns the lunar year beginning with n_1 is retained because the lunar component of the calendar is considered by the Chinese as the most fundamental one. Thus, the results of calculations concerning the lunar and solar elements preceding n_1 (such as n_{11}, n_{12}, s_1, s_2, etc.) are discarded.

3.3 Justifications

(2) to (6) readily follow from the definition of the algorithm of the division between two integers:

(2) and (3): With the above notations (Fig. 1), the interval os_1 from the Great Origin o to the Winter solstice s_1 on which the calculations depends, contains an integer number of solar years equal to t. From the Metonic relation (1) α solar years $= \gamma$ lunar months and hence each solar year contains γ/α lunar months. Thus, t solar years contain a non-integer number of lunar months equal to $(\gamma t)/\alpha$. When dividing γt by α, the quotient of the division gives the corresponding integer number of months, m, and its remainder the epact e.

(4): a month containing c/d days, m months contain (cm/d) days. When dividing cm by d, the quotient gives the integer number of days j contained in the interval on_{11} and the remainder, cm mod d, the remaining fractional part of a month.

(5): the reduction of j modulo 60 serves to locate the day of the winter solstice within the range 0 ... 59, while the remaining fractional part, cm mod d, locates n_{11} within the day in question.

(6): Same reasoning, but with n_{11} replaced by s_1 and c/d by a/b.

(7): Given that α solar years $= \beta$ intercalary months, the luni-solar shift increases by β/α lunar months per solar year. Converted into lunar months, the epact is equal to e/α months, as explained above. In order to get an intercalary month, the total shift must not exceed one lunar month, that is $e/\alpha + \beta/\alpha \geq 1$. Hence the sought result.

4 The calculation of the calendar of the year 450:

4.1 The designation of the lunar year

The Chinese lunar year 450 calculated in the present paragraph refers to that which is listed in chronological tables of the Chinese calendar under the number '450' which also designates a year of the Christian era. Such a designation is convenient because most days of the corresponding Chinese lunar year overlap with those of the Christian year 450. In the present case, for example, the Chinese corresponding lunar year begins on January 29/1/450 and its last days occurs on 16/2/451. Hence, apart from the greatest part of January 450 together with the January and the first half of February of the next year, 451, all the days of the Chinese lunar year '450' belong to the Christian year 450[29]. The same remark also apply in the case of any other Chinese lunar year.

In the Chinese historical and chronological context, the lunar year 450 is labelled *Taiping zhenjun shiyi nian* or " True Lord of Enduring Peace Eleventh Year reign-period ", where the era so called belongs to the Toba Wei dynasty (386-534) and where the Taoist appellation "True Lord" designates the Emperor Tai Wudi (reign: 423-452).

[29]Cf. Zhang Peiyu's chronological table, 1997, p. 170.

4.2 The content of the calendar of the year 450

Unlike any other year within the range 104 BC - 618 AD, the year 450 has a special interest because one of its manuscript calendar has reached us, together with that of the following year, 451[30]. For the period studied in the present article, not a single other such manuscript is extant and, more generally, extant antique and even medieval Chinese calendar manuscripts are extremely rare.

Restricted to the solar nodes, the new moons of the year 450, and the length of lunar months (29 or 30 days) the content of the manuscript is as indicated in the following table, where successive days are numbered in two ways: first, according to the sexagesimal cycle (column 2), second, separately for each month, by their day-number, from 1 to 29 or 30 (column 3) :

new moon no.	new moon (sexagesimal)	day-number	new moon no.	new moon (sexagesimal)	day-number
n_1	#59	1	s_{16}		15
s_4		9	s_{17}		30
s_5		25	n_{7*}	#26	1
n_2	#28	1	s_{18}		15
s_6		10	n_8	#55	1
s_7		25	s_{19}		2
n_3	#58	1	s_{20}		17
s_8		11	n_9	2#5	1
s_9		26	s_{21}		2
n_4	#28	1	s_{22}		17
s_{10}		12	n_{10}	#54	1
s_{11}		27	s_{23}		4
n_5	#57	1	s_{24}		19
s_{12}		13	n_{11}	#23	1
s_{13}		28	s_1		4
n_6	#27	1	s_2		19
s_{14}		14	n_{12}	#53	1
s_{15}		29	s_3		5
n_7	#56	1	s_4		21

Table 1. The fundamental elements (new moons and solar nodes) contained in the calendar manuscript of the year 450. Here we note n_{7*} the initial new moon of the intercalary month N_{7*}, the star meaning 'intercalary'. As explained above, the intercalary month bears the same number as the month immediately preceding it. In the Chinese original, the seventh month is called *qiyue* (*qi* = 7 and *yue* = lunar month) and the seventh intercalary month *run qiyue* (*run* = intercalary) or literally, 'intercalary seventh month'.

[30] A photocopy of the manuscript of these two calendars have been published in Ren Jiyu, vol. 1, 1997, p. 275-276 ; its quality is so bad that it is impossible to give a new reproduction here ; it remains nonetheless still legible. An analysis of the content of the manuscript, but without the calendrical calculations, also occurs in Deng Wenkuan, 1996, p. 360-372.

From the above table, the sexagesimal numbers of the successive solar nodes can be readily obtained. For example since s_4 has 9 for day-number, its sexagesimal number is 59 + 9 - 1 or #7.

4.3 Calculation of the year 450

It is particularly interesting to check whether or not the calendar is conform with the result of the theoretical calendrical calculations supposed to be valid for the year 450. As shown here, the answer is positive[31]. The detail of such calculations have never been published anywhere.

As the dates mentioned in Table 4 and the bibliography of primary sources show, the calendrical system in use in 450 was the *Jingchu li*, a system expounded in the calendrical treatise of the *Jinshu*[32] (The *Jin* dynasty history), written 644, by the historiographers of the Tang dynasty, long after the fall of the *Jin*.

In the Chinese context, systems of calendrical calculation receive a special name, here *Jingchu li*, an expression approximately meaning 'Luminous inception computus' (computus = system of calendrical calculations). Usually, *li* means 'calendar' but, here, it should be taken as 'system of calendrical calculations', or computus, and nothing else. In fact, in the context of Chinese calendrical treatises, *li* is an abbreviation of the term *lifa* (literally, [mathematical] methods *fa* for the calendar *li*]. In its turn, the term 'calendar' corresponds to Chinese terms like *liri* or *lipu*, among others (*ri* = days and *li* = calendar, *pu* = tables).

As already noted above, the parameter t to be used for the calculation of the year 450 depends on the number t_0 of years elapsed between the Great Origin and a fixed reference year x_0, generally corresponding to the year of adoption of a reform of calendrical calculations.

In the present case, $t_0 = 4045; x_0 = 237$ and

$$t = 4045 + (x - x_0) \tag{8}$$

Hence, for $x = 450, t = 4258$ solar years. There is a 'subtelty' however: t concerns the number of solar years elapsed between the Great Origin and the theoretical Winter solstice of the Christian year 449, not 450, because the Winter solstice determining the value of t occurs in the month of December of the year immediately preceding the Chinese lunar year to be calculated.

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4.3.1 The calculation of s_1, n_{11} and e

Given that the calendrical parameters of the *Jingchu li* are

α	β	γ	a	b	c	d
19	7	235	562120	1843	134630	4559

it follows that:

$s_1 =< 21; 397 >$

$n_{11} =< 59; 2079 >$

$e = 14$

The first result show that the first solar node, or theoretical Winter solstice, occurs on a day whose sexagesimal number is equal to #21, 397/1843 days after midnight, or at 5h 10mn 11 sec A.M., approximately, the day being divided into an unit of time u such that 1 day = $1843u$. Considered under the angle of the representation of the time elapsed since the beginning of the last sexagesimal cycle, the couple <21; 397> also gives rise to the quantity 21 + 397/1843 whose unit of measure is the day.

The second result has a similar meaning but the unit of time into which the day is divided is now equal to 4559 (= d) and not 1843 (= b). Thus, the day is divided differently according to what is computed, a solar node or a new moon. Here, the new moon n_{11} occurs 2079/4559 days after the beginning of the sexagesimal day no. 59, the last of the cycle. Hence its similar representation: 59 + 2079/4559.

The third result, $e = 14$, is such that $e \geq 19 - 7 = 12$. Consequently, from the criterion (7) above, the year 450 possesses an intercalary month and has 13 lunar months, not just 12.

4.3.2 Calculation of the 24 solar nodes:

The 24 solar nodes are computed by repeatedly adding $a/(24b)$ to s_1 and reducing the integer part of the result modulo 60, the calculation starts from s_1, taken in the form $21 + \frac{397}{1843}$.

The value of $a/(24b)$ is however expressed in a very special way, by using a different set of fractions as those in use in the preceding calculation: $\frac{a}{24b} = 15 + \frac{402}{1843} + \frac{11}{12 \times 1843}$ days. as though the day were divided into 1843 'minutes' and each 'minute' into 12 'seconds'.

These fractions being quite cumbersome, we shall use instead the simplified notation: $a; b, c$ where a is the first integer while b and c represent the respective numerator of the subsequent fractions. For example, with this notation, the above decomposition of $a/24b$ becomes equal to 15; 402, 11. Insofar as the context makes clear which denominators are intended, not confusion will arise.

With this simplified notation, the whole set of solar nodes needed for the calculation of the year 450 is:

solar nodes	values			solar nodes	values		
s_1	21;	397		s_{13}	23;	1546	
s_2	36;	799,	11	s_{14}	39;	105,	11
s_3	51;	1202,	10	s_{15}	54;	508,	10
s_4	6;	1605,	9	s_{16}	9;	911,	9
s_5	22;	165,	8	s_{17}	24;	1314,	8
s_6	37;	568,	7	s_{18}	39;	1717,	7
s_7	52;	971,	6	s_{19}	55;	277,	6
s_8	7;	1374,	5	s_{20}	10;	680,	5
s_9	22;	1777,	4	s_{21}	25;	1083,	4
s_{10}	38;	337,	3	s_{22}	40;	1486,	3
s_{11}	53;	740,	2	s_{23}	56;	46,	2
s_{12}	8;	1143,	1	s_{24}	11;	449,	1

Table 2. The 24 solar nodes needed for the determination of the lunar year 450.

4.3.3 Calculation of the 13 lunar months:

The set of new moons necessary for the calculation of the 13 lunar months $N_1, N_2, \cdots, N_{13}$, of the year 450 beginning with the two last ones of the preceding year, 449, the following 15 new moons should be taken into account: $n_{11}, n_{12}, n_1, n_2, \cdots, n_{11}, n_{12}, n_{13}$. They are determined by the repeated addition of the constant length of the lunar month, c/d, to n_{11}, written in the form $59 + 2079/4559$, the result being expressed in the form of a binomial $< a; b >$ where a is an integer reduced modulo 60 and b the numerator of a fraction whose denominator is equal to 4559:

new moons temporary no.		calculated value		new moons temporary no.		calculated value	
1	n_{11}	59;	2079	9	n_7	55;	3915
2	n_{12}	28;	4498	10	n_8	25;	1055
3	n_1	58;	2358	11	n_9	54;	3474
4	n_2	28;	218	12	n_{10}	23;	1334
5	n_3	57;	2637	13	n_{11}	53;	3753
6	n_4	27;	497	14	n_{12}	23;	1613
7	n_5	56;	2916	15	n_{13}	52;	4032
8	n_6	26;	776				

Table 3. The new moons needed for the determination of the lunar year 450. Here, two different numberings are used: the first corresponds to the natural order of enumeration, the second to the numbering of new moons beginning with that of the eleventh month belonging to the lunar year preceding the one to be calculated. Both numberings are temporary. The first because it is not used in the final calendar, the second because it will be modified by the possible insertion of an intercalary month.

Beyond the determination of the new moons determining the sought lunar months, this table also indicates the nature of each month, short or long, by merely checking the value of the fractional part f of each new moon by applying the following simple criterion[33]:

When the new moon $< a, f >$ is such that $f \geq 2140$ then the corresponding month is long, short otherwise. (9)

Justification: $c/d = \frac{134630}{4559} = 29 + \frac{2419}{4559}$ days and when $f \geq 2140$, i.e. $\geq \frac{2140}{4559}$ days, then the addition of the two fractional parts 2419 and f becomes greater than 2419 + 2140 = 4559. Converted in days, the result is thus greater than one day. Hence, the month in question must have 29 + 1 = 30 days and is necessarily long.

4.3.4 Determination of the intercalary month

According to the table giving the elements of the calendar manuscript of the year 450 (cf. IV-2 above), the year 450 the month N_{7*} is intercalary.

Indeed, thisN_{7*} is really intercalary, because it respects the definition of intercalary months since it contains no one solar node, being preceded as it is by the odd node s_{17}, the following odd node, s_{19} occurring after its end, as Table 1 above clearly shows.

With our above temporary second numbering, this N_{7*} corresponds to $N_8 = [n_8 n_9[$.

Now, to check that our calculations indicate that the new moon n_8, is indeed the initial one of the intercalary month N_8 , all we have to do is to verify that the month in question do not contain any odd node.

This is indeed the case because[34]:

$s_{17} < n_8 < n_9 < s_{19}$ since $25 + \frac{1314}{1843} + \frac{8}{12\times1843} < 26 + \frac{1055}{4559} < 55 + \frac{3474}{4559} < 56 + \frac{277}{1843} + \frac{6}{12\times1843}$

We can also check than the luni-solar shift taken at the instant of the solar node s_{19} with respect to the immediately preceding new moon, n_8 - that is the length of the interval $n_8 s_{19}$ - is equal to $56 + \frac{277}{1843} + \frac{6}{12\times1843} - (26 + \frac{1055}{4559}) = 29.9$ days, approximately, or more than one lunar month.

Discarding the temporary notations N_8 and n_8, the intercalary month becomes now N_{7*}. and the following months $N_8, N_9, \cdots$ instead of N_9,N_{10} and so on.

The final labelling of the sequence of the 13 lunar months of the year 450 is thus $N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_{7*}, N_8, N_9, N_{10}, N_{11}, N_{12}$.

With the preceding results, the full calendar of the year 450 - limited to the solar nodes and new moons - can be readily obtained and the same calculations can also be applied in the case of any other year within the range of validity of the *Jingchu li* system of calendrical calculations.

When only the result of the calculations is needed, independently of Chinese number representations based on very unusual systems of fractions, the intermediary steps

[33]This criterion is explicitly stated in the Chinese original sources.

[34]The present inequality makes sense because all integers (25, 26, 55, 56) fall into the range 0..59. If it were not the case, 60 should be added to some of these.

can be sensibly simplified for it suffices to perform the various arithmetical operations with plain decimals and integers reduced modulo 60. In that case, programming the calculations of the Chinese calendar becomes easier and the characteristics of other years can be obtained at will.

Tables

no.	name of the system (*li*)	Approximate meaning	Periods	Dates
1	Taichu or Santong	Grand inception or Three periods[35]	Han, Former and Later	104 BC-84 AD
2	Sifen	Quarter day[36]	Later Han Shu Wei	85-220 221-263 220-236
3	Qianxiang	Supernatural manifestation	Wu	223-280
4	Jingchu	Luminous inception	Wei Jin, Former and Later Liu Sung Toba Wei	237-265 265-420 420-444 398-451
5	Yuanshi	Profound beginning	Northern Liang	412-439 452-522
6	Yuanjia	Epochal prosperity	Liu Sung Qi Liang	445-479 479-503 502-509
7	Daming	Great enlightenment	Liang Chen	510-557 557-589
8	Zhengguang	Orthodox Brilliance	Toba Wei Western Wei Northern Zhou	523-534 535-556 557-565
9	Xinghe	Thriving harmony	Eastern Wei Northern Qi	540-550 550
10	Tianbao	Celestial preservation	Northern Qi	551-577
11	Tianhe	Celestial harmony	Northern Zhou	566-578
12	Daxiang	Great manifestation	Northern Zhou Sui	579-581 581-583
13	Kaihuang	Opening magnificence	Sui	584-596
14	Daye	Great enterprise	Sui	597-618

Table 4: Calendrical systems in usage during the period 104 BC-618 (partial list).

[35]The 'three periods' are supra-annual cycles respectively composed of 27, 81 and 1539 solar years.
[36]So called because it admits a value of the solar year equal to 365.25 days (a quarter day).

no.	name	Metonic type α/β	solar year (a/b days)	lunar month (c/d days)
1	Santong li	19/7	$\frac{5621201}{539}$	$\frac{2392}{81}$
2	Sifen li	19/7	$\frac{1461}{4}$	$\frac{27759}{940}$
3	Qianxiang li	19/7	$\frac{215130}{589}$	$\frac{43026}{1457}$
4	Jingchu li	19/7	$\frac{673150}{1843}$	$\frac{134630}{4559}$
5	Yuanshi li	600/221	$\frac{2629759}{7200}$	$\frac{2629759}{89052}$
6	Yuanjia li	19/7	$\frac{111035}{304}$	$\frac{22207}{752}$
7	Daming li	391/144	$\frac{14423804}{39491}$	$\frac{116321}{3939}$
8	Zhengguang li	505/186	$\frac{2213377}{6060}$	$\frac{2213377}{74952}$
9	Xinghe li	562/207	$\frac{6158017}{16860}$	$\frac{6158017}{208530}$
10	Tianbao li	676/249	$\frac{8641687}{23660}$	$\frac{8641687}{292635}$
11	Tianhe li	391/144	$\frac{8568631}{23460}$	$\frac{8568631}{290160}$
12	Daxiang li	448/165	$\frac{4745247}{12992}$	$\frac{1581749}{53563}$
13	Kaihuang li	429/158	$\frac{37605463}{102960}$	$\frac{5372209}{181920}$
14	Daye li	410/151	$\frac{15573963}{42640}$	$\frac{33783}{1144}$

Table 5. Main fundamental constants (Metonic constants, solar year, lunar month) of the 14 calendrical systems in use during the period 104 BC-618. The consistency of the data can be checked by verifying that, in each case, α solar years $= (19\alpha + \beta)$ lunar months.

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