CUBO *A Mathematical Journal*
Vol. 6, Nº 2, (1 - 14). August 2004.

On a Theorem of Arne Persson

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ABSTRACT

A geometrical description for the essential spectra of a large class of Schrödinger operators is presented. Persson's formula is obtained as a corollary.

RESUMEN

Se presenta una descripción geométrica del espectro esencial de una clase larga de operadores de Schrodinger. La fórmula de Persson es obtenida como un corolario.

Key words and phrases: *Persson theorem; essential spectrum; eigenvalues*
Math. Subj. Class.: *35P99;35J10;47A10*

[1] This paper was completed while the first author had been visiting the Department of Mathematical Sciences at Aalborg University. The financial support is gratefully acknowledged.

[2] H.C. is partially supported by MaPhySto – A Network in Mathematical Physics and Stochastics, funded by The Danish National Research Foundation

1 Introduction and the Result

It is by now a matter of common sense that the essential spectrum of a Schrödinger operator H is not influenced by localized perturbations, i.e. it is described by the potential's behavior at infinity. There are lots of geometrical descriptions of this phenomenon; probably the most popular one is the so called Persson formula (see [P] for the original article). As it is well known, this result expresses the bottom of the essential spectrum of H in terms of its mean energy on states which are farther and farther away from the origin. L. Gårding used Persson's result in [G] and gave a non-combinatorial proof for the "HVZ-theorem" (which describes the bottom of the essential spectrum for many body Schrödinger operators). More recently, G. Grillo (see [Gr]) generalized Persson's work to nonnegative, selfadjoint operators L defined on Hilbert spaces of type $L^2(X, m)$, where X is a locally compact, Hausdorff, separable space and m is positive Radon measure on it of full support. Finally, let us mention that nice textbook presentations of Persson's formula may be found in [C-F-K-S] and [H-S].

Let us stress from the beginning that for simplicity we only deal with operators of the form "$-\Delta + W$", where W is a multiplication operator. Similar results can be derived for more general second order elliptic differential operators (for example magnetic Schrödinger operators where W would be a first order differential operator), but we do not want to discuss this here.

To be more precise, let us fix some notation. First, we start with the potential, which is assumed to obey the following two conditions:

(A) $W : C_0^\infty(\mathbb{R}^n) \mapsto L^2(\mathbb{R}^n)$, $n \geq 1$, is a symmetric multiplication operator that admits a continuous extension from $H^1(\mathbb{R}^n)$ (the Sobolev space of square integrable functions whose distributional derivatives are also in L^2) to $H^{-1}(\mathbb{R}^n)$ (the dual of H^1);

(B) W is $-\Delta$ form bounded with bound less than 1.

The object of our study will be the Hamiltonian $H = -\Delta + W$ defined as a form sum on $H^1(\mathbb{R}^n)$, $n \geq 1$. Denote with $\mathcal{K}(\mathbb{R}^n)$ the set of all compact subsets of $\mathbb{R}^n$. If $K \in \mathcal{K}(\mathbb{R}^n)$, then H_{K^c} denotes the Friederichs extension of the symmetric operator $-\Delta + W$ defined on $C_0^\infty(K^c)$. In the particular case where K is a closed ball of radius $R > 0$ centered at the origin, the corresponding operator is denoted by H_R, and K^c with Ω_R. We shall naturally consider $L^2(K^c)$ embedded in $L^2(\mathbb{R}^n)$. Also, the $H_0^1(K^c)$ functions extended by 0 outside K^c, are $H^1(\mathbb{R}^n)$ functions. Thus, the resolvents $(H_{K^c} - z)^{-1}$ are bounded operators in $L^2(\mathbb{R}^n)$.

The last notation we need here is $\mathcal{F}_K := \{\varphi \in C_0^\infty(K^c),\ \|\varphi\|_{L^2(\mathbb{R}^n)} = 1\}$. Then the Persson formula reads as:

$$\inf \sigma_{ess}(H) = \sup_{K} \inf_{\varphi \in \mathcal{F}_K} \langle \varphi, (-\Delta + W)\varphi \rangle \ . \tag{1.1}$$

In order to formulate our result, we need two additional conditions on W:

(C) $W = V_0 + V$ and there exists $r_0 > 0$ such that $\text{supp}(V_0) \subseteq \{x \mid |x| < r_0\}$;

(D) The operator V is $-\Delta$ operator bounded with a bound less than one.

Remark

If the potential V_0 in (C) is zero, then one can replace the last three conditions with just one:

(B') W is $-\Delta$ operator bounded with a bound less than one.

We now can give the main result of our paper:

Theorem 1.1 *Assume that* W *satisfies the conditions* (A)-(D). *Then the essential spectrum of* H *admits the following representation:*

$$\sigma_{ess}(H) = \bigcap_{K \in \mathcal{K}(\mathbb{R}^n)} \sigma(H_{K^c}) \,. \tag{1.2}$$

Moreover,

$$\inf \sigma_{ess}(H) = \lim_{R \to \infty} \inf \sigma(H_R) \,. \tag{1.3}$$

Remark

We shall prove in Corollary 2.2 that the Persson formula follows easily from the above theorem. Of course, as we have already mentioned, Persson's formula is valid under much weaker assumptions on W (see a proof in [C-F-K-S] only requiring (A) and (B)), thus it would be interesting to obtain (1.2) without conditions (C) and (D).

2 The Proofs

Let us briefly describe the strategy of our proof. As H and H_{K^c} only differ on a finite region, one expects that their essential spectra to be equal, no matter of the choice we make for K. Even though have such results been known for a long time in the literature, we decided to give a proof in Proposition 2.1 for completeness. We mention here a result of M. Birman (see [B]) who proved that under certain conditions, not only are the essential spectra equal, but also H and H_{K^c} have unitarily equivalent absolutely continuous parts.

Now, as

$$\sigma(H_{K^c}) = \sigma_{\text{ess}}(H_{K^c}) \cup \sigma_{\text{disc}}(H_{K^c}) = \sigma_{\text{ess}}(H) \cup \sigma_{\text{disc}}(H_{K^c}) \,,$$

it follows that the r.h.s. of (1.2) can be written as

$$\sigma_{\text{ess}}(H) \bigcup \left(\bigcap_{K \in \mathcal{K}(\mathbb{R}^n)} \sigma_{\text{disc}}(H_{K^c}) \right) .$$

Clearly, if one shows that there are no common discrete eigenvalues for all H_{K^c}, then we are done. In fact, it is sufficient to prove that the smaller family $\{H_R\}$, $R > r_0$ cannot have any common discrete eigenvalues, and this is what we do in the second part of this section.

2.1 Invariance of the Essential Spectrum

In this subsection, only (A) and (B) are needed for W. With these assumptions, the main result here is contained in the following proposition:

Proposition 2.1 *For each compact set K in $\mathbb{R}^n$ we have:*
$$\sigma_{ess}(H) = \sigma_{ess}(H_{K^c}).$$

Proof. We shall need the following lemmas.

Lemma 2.2 (a) *If $u \in \mathrm{Dom}(H_{K^c})$ and $\mathrm{supp}(u)$ does not intersect the boundary of K^c, then u belongs to the domain of $H_{K_1^c}$ for each compact $K_1 \subseteq K$ (here K_1 may be the empty set, in which case $H_{K_1^c} = H$); moreover, $H_{K_1^c}u = H_{K^c}u$.*

(b) *Let $u \in \mathrm{Dom}(H)$ such that $\mathrm{supp}(u) \subset K^c$. Then $u \in \mathrm{Dom}(H_{K^c})$ and $Hu = H_{K^c}u$.*

Proof. We shall use the next characterization for the domains $\mathrm{Dom}(H)$ and $\mathrm{Dom}(H_{K^c})$:

(i) $u \in \mathrm{Dom}(H)$ if and only if $u \in H^1(\mathbb{R}^n)$ and $(-\Delta + W)u \in L^2(\mathbb{R}^n)$;

(ii) $v \in \mathrm{Dom}(H_{K^c})$ if and only if $v \in H_0^1(K^c)$ and $(-\Delta + W)v|_{K^c} \in L^2(K^c)$.

Now the statement (b) becomes obvious. For (a), let $\chi \in C^\infty(\mathbb{R}^n)$, $\chi(x) = 1$ on $\mathrm{supp}(u)$ and $\mathrm{supp}(\chi) \subset K^c$. Then $\chi u = u$ and, as distributions,

$$(-\Delta + W)(\chi u) = \chi(-\Delta u + Wu) \in L^2(\mathbb{R}^n).$$

Thus $u \in \mathrm{Dom}(H_{K_1^c}))$ ∎

Proposition 2.1 now follows if we prove the next lemma, which is a straightforward adaptation of Theorem 3.11 in [C-F-K-S].

Lemma 2.2 (a) *$\lambda \in \sigma_{ess}(H)$ if and only if there exists a sequence $\{\varphi_n\}$ in $\mathrm{Dom}(H)$ such that $\mathrm{supp}(\varphi_n) \subseteq \overline{B(0,n)}^c$, $\|\varphi_n\| = 1$, $\varphi_n \overset{w}{\to} 0$, $\|(H-\lambda)\varphi_n\| \to 0$;*

(b) *Let K be a compact in $\mathbb{R}^n$. Then, $\lambda \in \sigma_{ess}(H_{K^c})$ if and only if there exists a sequence $\{\varphi_n\}$ in $\mathrm{Dom}(H_{K^c})$ such that $\mathrm{supp}(\varphi_n) \subseteq \overline{B(0,nR_K)}^c$ (where $K \subseteq B(0,R_K)$), and $\|\varphi_n\| = 1$, $\varphi_n \overset{w}{\to} 0$, $\|(H_{K^c} - \lambda)\varphi_n\| \to 0$.*

Proof. We shall prove (b) (the proof of (a) is similar). The implication "$\Leftarrow$" is nothing but an improved Weyl criterion, therefore we are only left with the direct implication.

Now suppose that $\lambda \in \sigma_{\mathrm{ess}}(H_{K^c})$. The Weyl criterion assures the existence of a sequence $\{\psi_j\}$ in $\mathrm{Dom}(H_{K^c})$ such that $\|\psi_j\| = 1$, $\psi_j \overset{w}{\to} 0$ and $\|(H_{K^c} - \lambda)\psi_j\| \to 0$. Note at first that, for $\Phi \in C_0^\infty(\mathbb{R}^n)$, the operator $\Phi(H_{K^c}+i)^{-1}$ is compact in $L^2(\mathbb{R}^n)$. Let $\chi \in C^\infty(\mathbb{R}^n)$, $\chi(\mathrm{x}) = 0$ for $|\mathrm{x}| \le R_K$, $\chi(\mathrm{x}) = 1$ for $|\mathrm{x}| \ge R_K + 1$, $0 \le \chi \le 1$, and denote $\chi_n(\mathrm{x}) = \chi(\mathrm{x}/n)$. We have

$$\|\chi_n\psi_j\| \ge 1 - \|(1-\chi_n)\psi_j\|.$$

On the other hand,

$$\begin{aligned}\|(1-\chi_n)\psi_j\| &= \|(1-\chi_n)(H_{K^c}+i)^{-1}(H_{K^c}+i)\psi_j\| \\ &\leq \|(H_{K^c}-\lambda)\psi_j\| + |\lambda+i|\,\|(1-\chi_n)(H_{K^c}+i)^{-1}\psi_j\|.\end{aligned}$$

The first term in the right hand side converges to 0, by the hypotheses on ψ_j. The second term converges to 0 if n fixed, since $(1-\chi_n)(H_{K^c}+i)^{-1}$ is compact and the sequence ψ_j goes weakly to 0. Therefore, for each n there exists $j(n)$ (which can be chosen to be greater than n) such that

$$\|(1-\chi_n)\psi_{j(n)}\| \leq 1/n.$$

One now can take $\varphi_n = \chi_n\psi_{j(n)}/\|\chi_n\psi_{j(n)}\|$, and it is clear that $\|\varphi_n\| = 1$ and $\varphi_n \stackrel{w}{\to} 0$. On the other hand, computing in distributions on K^c,

$$\begin{aligned}(-\Delta+W+\lambda)(\chi_n\psi_{j(n)}) = {} & \chi_n(-\Delta+W-\lambda)\psi_{j(n)} - 2\textstyle\sum_k (1/n)\partial_k\chi(\cdot/n)\partial_k\psi_{j(n)} - \\ & -(1/n^2)(\Delta\chi)(\cdot/n)\psi_{j(n)}.\end{aligned} \tag{2.1}$$

Since $\psi_{j(n)} \in \mathrm{Dom}(H_{K^c})$, the right hand side belongs to $L^2(K^c)$. This fact, together with $\varphi_n \in H_0^1(K^c)$, ensure $\varphi_n \in \mathrm{Dom}(H_{K^c})$ (see (ii) in the proof of the Lemma 2.1). The first and the third term in the right hand side of (2.1) are norm convergent to 0. The norm of the second one can be bounded from above by

$$\frac{2}{n}\sum_k \|\partial_k\chi\|_\infty \|\partial_k\psi_{j(n)}\| \leq \frac{C_1}{n}\|(H_{K^c}-\lambda)\psi_{j(n)}\| + \frac{C_2}{n}\|\psi_{j(n)}\| \leq C/n,$$

where C_1, C_2 and C are constants and we used the continuous inclusion of the domain $\mathrm{Dom}(H_{K^c})$ into $H_0^1(K^c)$. Hence, this term also converges to 0 in L^2.

The equality (2.1) yields $\|(H_{K^c}-\lambda)\varphi_n\| \to 0$, since $\|\chi_n\psi_{j(n)}\| \to 1$. Thus $\{\varphi_n\}$ satisfies all the needed conditions. ■

2.2 Discrete Eigenvalues Cannot be Constant

We start proving that H_R cannot have constant discrete eigenvalues, using a proof by contradiction. Namely, assume that λ is a discrete eigenvalue of $H_{R-\delta}$, $-\infty < \delta \leq \delta_0$ (where $\delta_0 > 0$ is sufficiently small) and denote by P_R the finite dimensional projector ($\dim(P_R) = N$) of H_R corresponding to $[\lambda-\epsilon, \lambda+\epsilon]$, where $\epsilon > 0$ is small enough such that the interval contains only one eigenvalue, that is λ. Then we shall prove that λ either belongs to the resolvent set or to the essential spectrum of H_R, thus yielding a contradiction.

In order to motivate the reader for the technical parts which will follow, let us give a short and less rigorous overview of our strategy. First, we shall prove that $\lim_{\delta\searrow 0} H_{R-\delta} = H_R$ in the norm resolvent sense in $\mathbb{B}(L^2(\mathbb{R}^n))$. This implies that for $\delta > 0$ sufficiently small, the spectrum of $H_{R-\delta}$ in $[\lambda-\epsilon, \lambda+\epsilon]$ is purely discrete and $n-\lim_{\delta\searrow 0} P_{R-\delta} = P_R$. Moreover, there are exactly N discrete eigenvalues of $H_{R-\delta}$ in this interval and by assumption, λ is always one of them. Denote by φ_δ one of the normalized eigenvectors of $H_{R-\delta}$ for which $H_{R-\delta}\varphi_\delta = \lambda\varphi_\delta$. Then we shall prove that when $\delta \searrow 0$, the set $\{\varphi_\delta\}$ admits an adherent point $\varphi_0 \in \mathrm{Ran}(P_R)$, $\|\varphi_0\| = 1$, i.e. there exists a sequence $\{\varphi_j\}$ of such eigenvectors such that $\|\varphi_j - \varphi_0\| \to 0$.

Then for any j:

$$\lambda\langle\varphi_j,\varphi_0\rangle_{L^2(\mathbb{R}^n)} = \langle H_{R-\delta(j)}\varphi_j,\varphi_0\rangle_{L^2(\mathbb{R}^n)} = \langle\varphi_j, H_R\varphi_0\rangle_{L^2(\mathbb{R}^n)} \,. \tag{2.2}$$

Using the Green formula and the fact that W is a multiplication operator, one obtains

$$0 = \int_{|\mathbf{x}|=R} \varphi_j \overline{\frac{\partial\varphi_0}{\partial\nu}}\, ds_x \tag{2.3}$$

(here ν is the outer unit normal vector at $\partial\Omega_R$, $\nu(\mathbf{x}) = -\mathbf{x}/R$).

Since $\varphi_j(\mathbf{x}) = 0$ if $|\mathbf{x}| = R - \delta(j)$, one expects that for $|\mathbf{x}| = R$, $\frac{\varphi_j(\mathbf{x})}{\delta(j)}$ should be close to $-\frac{\partial\varphi_j}{\partial\nu}(\mathbf{x})$; in fact, we shall show that

$$\left\| \frac{\varphi_j}{\delta(j)} + \frac{\partial\varphi_j}{\partial\nu} \right\|_{L^2(\partial\Omega_R)} \leq C\,\delta(j)^{1/2}. \tag{2.4}$$

Clearly, (2.3) and (2.4) imply

$$\lim_{j\to\infty} \left| \int_{|\mathbf{x}|=R} \frac{\partial\varphi_j}{\partial\nu}\overline{\frac{\partial\varphi_0}{\partial\nu}} ds_x \right| = 0\,. \tag{2.5}$$

As we know that $\|\varphi_j - \varphi_0\|_{L^2(\mathbb{R}^n)} \to 0$, we shall use it in proving that

$$\lim_{j\to\infty} \left\| \frac{\partial\varphi_0}{\partial\nu} - \frac{\partial\varphi_j}{\partial\nu} \right\|_{L^2(\partial\Omega_R)} = 0\,. \tag{2.6}$$

We now conclude that $\frac{\partial\varphi_0}{\partial\nu}$ restricted to $\partial\Omega_R$ equals zero; this implies (via the Green formula) that $\varphi_0 \in \mathrm{Dom}(H)$, $H\varphi_0 = \lambda\varphi_0$ and of course, $\mathrm{supp}(\varphi_0) \subset \Omega_R$. If some unique continuation property holds, then $\varphi_0 \equiv 0$ (which means $P_R = 0$); otherwise, because $R > r_0$ can be chosen arbitrarily large, we can construct an infinite number of eigenvectors for H corresponding to λ with their supports going to infinity (which means $\lambda \in \sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ess}}(H_R)$).

Let us now give rigorous statements and proofs.

2.2.1 2.2.1. A Few Technical Estimates

Lemma 2.3 *Let $R_1 > R_2 > 0$ be fixed. Then there exists $C = C(R_1, R_2)$ such that*

$$\|u\|_{H^2(\Omega_R)} \leq C(\|-\Delta_R u\| + \|u\|), \tag{2.7}$$

for each $R \in [R_1, R_2]$, and for all $u \in \mathrm{Dom}(-\Delta_R) = H^2(\Omega_R) \cap H_0^1(\Omega_R)$.

Proof. Recall first that the result holds for $R = 1$ (see [H], Lemma 10.5.1). The lemma will be proved by the following simple observation. Consider the dilations $U_R : L^2(\Omega_R) \to L^2(\Omega_1)$, $U_R f(\mathbf{x}) = R^{n/2} f(R\mathbf{x})$. Then $U_R : H^k(\Omega_R) \to H^k(\Omega_1)$, $k = 0,1,2$ is bounded and $\nabla(U_R u) = R U_R \nabla u$, $\partial_j \partial_k (U_R u) = R^2 U_R(\partial_j \partial_k u)$ for all $u \in H^2(\Omega_R)$. Moreover, $U_R(H^2(\Omega_R) \cap H_0^1(\Omega_R)) = H^2(\Omega_1) \cap H_0^1(\Omega_1)$ and $-\Delta_1 U_R = R^2 U_R(-\Delta_R)$. The lemma now follows from the case $R = 1$. ∎

Lemma 2.4 *Let $R_2 > R_1 > r_0$ be fixed (see assumption (C)). Let θ be a $C_0^\infty(\mathbb{R}^n)$ function,* $\mathrm{supp}(\theta) \subset \Omega_{r_0}$. *Denote by χ_R the characteristic function of Ω_R. Then there exist $0 \le a < 1$ and $M > 0$ such that uniformly in $R \in [R_1, R_2]$:*

$$\|\chi_R (V\theta)(-\Delta_R + M)^{-1}\|_{BB(L^2(\Omega_R))} \le a\,. \tag{2.8}$$

Proof. Take $f \in C_0^\infty(\mathbb{R}^n)$; then we have (see e.g. [H-H]):

$$\left|(-\Delta_R + M)^{-1} f\right|(\mathbf{x}) \le \left((-\Delta + M)^{-1}|f|\right)(\mathbf{x}),$$

for any $\mathbf{x} \in \mathbb{R}^n$. If $\varphi \in C_0^\infty(\mathbb{R}^n)$, then

$$\begin{aligned} |\langle \chi_R (V\theta)\varphi, (-\Delta_R + M)^{-1} f\rangle_{L^2(\mathbb{R}^n)}| &\le \langle \chi_R(|V\theta|)|\varphi|, (-\Delta + M)^{-1}|f|\rangle_{L^2(\mathbb{R}^n)} \\ &\le \|\varphi\|\, \|V\theta(-\Delta + M)^{-1}\|\, \|f\|. \end{aligned} \tag{2.9}$$

Using that $\lim_{M\to\infty} \|V\theta(-\Delta + M)^{-1}\|_{BB(L^2(\mathbb{R}^n))} < 1$ (see assumption (D)), (2.8) is obtained after a density argument. ∎

Lemma 2.5 *With the same notation as in Lemma 2.4, take $\varphi \in \mathrm{Dom}(H_R)$. Then*

(a) $\theta\varphi \in H^2(\Omega_R) \cap H_0^1(\Omega_R)$ *for all* $R \in [R_1, R_2]$;

(b) $\|\theta\varphi\|_{H^2(\Omega_R)} \le C(\|H_R\varphi\| + \|\varphi\|), \quad \forall R \in [R_1, R_2]$.

Proof. If $\varphi \in \mathrm{Dom}(H_R)$, then $\varphi \in H_0^1(\Omega_R)$. Let θ_1 be a function θ like, but $\theta_1 = 1$ on $\mathrm{supp}(\theta)$. Computing in distributions on Ω_R, one has

$$(-\Delta + V\theta_1)(\theta\varphi) = \theta(-\Delta + W)\varphi - 2\nabla\theta \cdot \nabla\varphi - (\Delta\theta)\varphi, \tag{2.10}$$

hence the right hand side of the above equality belongs to $L^2(\Omega_R)$. Since $\varphi\theta \in H_0^1(\Omega_R)$, we get that $\varphi\theta$ belongs to the domain of the self-adjoint operator associated to the form $(-\Delta + V\theta_1)\big|_{H_0^1(\Omega_R)\times H_0^1(\Omega_R)}$. But by Lemma 2.4 (which says that $V\theta_1$ is $-\Delta_R$-bounded with bound less than 1), this domain is the same as the domain of $-\Delta_R$, namely $H_0^1(\Omega_R) \cap H^2(\Omega_R)$. The first part of the lemma thus follows.

Let us now prove the second part. We denote by g the right hand side of (2.10). Then, with M as in Lemma 2.4 (where θ must now be replaced with θ_1), we have $\theta\varphi = (-\Delta_R + M + V\theta_1)^{-1}(g + M\theta\varphi)$. By Lemma 2.3:

$$\begin{aligned} \|\varphi\theta\|_{H^2(\Omega_R)} &\le C\| -\Delta_R(-\Delta_R + M + V\theta_1)^{-1}(g + M\theta\varphi)\| + C\|\theta\varphi\| \\ &\le C(\|g\| + \|\varphi\|). \end{aligned} \tag{2.11}$$

On the other hand,

$$\|g\| \leq C\left(\|(-\Delta + W)\varphi\| + \|\varphi\| + \|\nabla\varphi\|\right). \tag{2.12}$$

But

$$\begin{aligned}\|\nabla\varphi\|^2 &= \langle -\Delta_R\varphi, \varphi\rangle \leq |\langle H_R\varphi, \varphi\rangle| + |\langle W\varphi, \varphi\rangle| \\ &\leq (\|H_R\varphi\| + \|\varphi\|)^2 + b\langle -\Delta_R\varphi, \varphi\rangle + C\|\varphi\|^2,\end{aligned}$$

where $b < 1$ and C do not depend on R. Hence

$$\|\nabla\varphi\| \leq C\left(\|H_R\varphi\| + \|\varphi\|\right). \tag{2.13}$$

Summing up (2.11), (2.12) and (21.3), one gets (b). ∎

2.2.2 Norm Convergence of the Resolvents

Lemma 2.6 *Let $r_0 < R_1 < R_2$ (see (C)) and define $R := (R_1 + R_2)/2$. Then, for $0 \leq \delta \leq \delta_0$, $\delta_0 < (R_2 - R_1)/2$, one has $R \pm \delta \in (R_1, R_2)$. Fix a constant $M > 0$ as that one obtained in Lemma 2.4, and such that $H_{R-\delta} \geq -M + 1$ for $0 \leq \delta \leq \delta_0$. Then as operators in $\mathbb{B}(L^2(\mathbb{R}^n))$,*

$$n - \lim_{\delta\to 0}(H_{R-\delta} + M)^{-1} = (H_R + M)^{-1}.$$

Proof. Before anything, let us mention that assumption (B) and Lemma 2.4 assure the existence of M with all the required properties.

The proof of this lemma will have two parts. In the first one we prove that $(\chi_R - 1)(H_{R-\delta} + M)^{-1} \to 0$, while in the second one we show that

$$\|\chi_R(H_{R-\delta} + M)^{-1} - (H_R + M)^{-1}\| \overset{\delta\to 0}{\longmapsto} 0. \tag{2.14}$$

The first step is easy, since

$$(\chi_R - 1)(H_{R-\delta} + M)^{-1} = (\chi_R - \chi_{R-\delta})(1 - \Delta_R)^{-1/2}(1 - \Delta_R)^{1/2}(H_{R-\delta} + M)^{-1},$$

where $\|(1 - \Delta_R)^{1/2}(H_{R-\delta} + M)^{-1}\| \leq C$ and $(\chi_R - \chi_{R-\delta})(1 - \Delta_R)^{-1/2}$ converges to 0, in norm.

Let us now prove the second step. Define $d(\mathbf{x}) = |\mathbf{x}| - 1$ on Ω_1. We shall recall a few facts concerning the continuity of the multiplication operator d^{-s}, $s \in (0, 1/2)$ in Sobolev spaces on Ω_1.

Let $\Omega_{1,2} = \{\mathbf{x} \in \Omega_1 \mid |\mathbf{x}| < 2\}$. Then

$$d^{-s} : H^s(\Omega_{1,2}) \mapsto L^2(\Omega_{1,2}) \tag{2.15}$$

is bounded, by Th. 11.7, Ch. I [L-M]. The Remark 11.7, Ch. I [L-M] says that by interpolating $H_0^2(\Omega_{1,2})$ and $L^2(\Omega_{1,2})$ we get the inclusion

$$[H_0^2(\Omega_{1,2}), L^2(\Omega_{1,2})]_{\frac{1-s}{2}} \subseteq \{u \in L^2(\Omega_{1,2}) \mid d^{-(s+1)}u \in L^2(\Omega_{1,2})\}. \tag{2.16}$$

On the other hand, (by Theorems 11.6 and 11.5, [L-M]), if $s < 1/2$

$$H_0^{s+1}(\Omega_{1,2}) = [H_0^2(\Omega_{1,2}), L^2(\Omega_{1,2})]_{\frac{1-s}{2}} \supseteq H^2(\Omega_{1,2}) \cap H_0^1(\Omega_{1,2}) . \tag{2.17}$$

If $u \in H^2(\Omega_{1,2}) \cap H_0^1(\Omega_{1,2})$, then

$$\begin{aligned} \|D_j(d^{-s}u)\|_{L^2(\Omega_{1,2})} &\leq \|d^{-s-1}u\|_{L^2(\Omega_{1,2})} + C\|d^{-s}D_j u\|_{L^2(\Omega_{1,2})} \\ &\leq C\|u\|_{H^2(\Omega_{1,2})}. \end{aligned} \tag{2.18}$$

The closed graph theorem, (2.15) and (2.18) assure that the operator

$$d^{-s} : H^2(\Omega_{1,2}) \cap H_0^1(\Omega_{1,2}) \mapsto H^1(\Omega_{1,2})$$

is bounded.

Consider $\eta \in C_0^\infty(\mathbb{R}^n)$, $\mathrm{supp}(\eta) \in B(0,2)$, and $\eta(\mathrm{x}) = 1$ on $B(0,3/2)$. If $u \in H^2(\Omega_1) \cap H_0^1(\Omega_1)$, then $\eta\, u \in H^2(\Omega_{1,2}) \cap H_0^1(\Omega_{1,2})$. Therefore

$$\|d^{-s}u\|_{H^1(\Omega_1)} \leq \|d^{-s}(\eta u)\|_{H^1(\Omega_{1,2})} + \|d^{-s}(1-\eta)u\|_{H^1(\Omega_1)} \leq C\|u\|_{H^2(\Omega_1)}. \tag{2.19}$$

Thus, the operator $d^{-s} : H^2(\Omega_1) \cap H_0^1(\Omega_1) \mapsto H^1(\Omega_1)$ is bounded.

Set $D := \mathrm{Dom}(-\Delta_R) \cap C^\infty(\overline{\Omega_R})$; it is known that D is an essential domain for $-\Delta_R$ and from assumption (B) and Lemma 2.5 one concludes that D is an essential domain for H_R, too. As a consequence, $I := (H_R + M)D$ is dense in $L^2(\Omega_R)$.

Let $g \in I$, $f \in L^2(\mathbb{R}^n)$ and denote $f_1 = (H_{R-\delta} + M)^{-1}f$, $g_1 = (H_R + M)^{-1}g$. Then $\theta f_1 \in H^2(\Omega_{R-\delta})$ for all $\theta \in C_0^\infty(\Omega_{r_0})$ (see Lemma 2.5). Take $\theta \in C_0^\infty(\Omega_{r_0})$, $\theta = 1$ if $|\mathrm{x}| \in [R_1, R_2]$. Then, using the Green formula, one has

$$|\langle[\chi_R(H_{R-\delta} + M)^{-1} - (H_R + M)^{-1}]f, g\rangle_{L^2(\mathbb{R}^n)}| = \Big| \int_{|\mathrm{x}|=R} (\theta f_1)\, \overline{\frac{\partial(\theta g_1)}{\partial \nu}}\, ds_x \Big|. \tag{2.20}$$

Set $g_2 = \theta g_1$, $f_2 = \theta f_1$. With $s < 1/2$, one has

$$\Big| \int_{|x|=R} f_2 \overline{\frac{\partial}{\partial \nu} g_2}\, ds_x \Big| \leq C\,\delta^s \|(|x| - (R-\delta))^{-s} f_2\|_{H^1(\Omega_{R-\delta})} \cdot \|g_2\|_{H^2(\Omega_R)}.$$

As in Lemma 2.3, we have

$$\|(|x| - (R-\delta))^{-s} f_2\|_{H^1(\Omega_{R-\delta})} \leq C\|(|y|-1)^{-s} f_2((R-\delta)\cdot)\|_{H^1(\Omega_1)}.$$

By (2.19, the right hand side can be bounded from above by $C\,\|f_2((R-\delta)\cdot\|_{H^2(\Omega_1)}$, and hence by $C\,\|f_2\|_{H^2(\Omega_{R-\delta})}$. Thus

$$\begin{aligned} \Big| \int_{|\mathrm{x}|=R} f_2 \overline{\tfrac{\partial g_2}{\partial \nu}}\, ds_x \Big| &\leq C\,\delta^s \|f_2\|_{H^2(\Omega_{R-\delta})} \cdot \|g_2\|_{H^2(\Omega_R)} \\ &\leq C\,\delta^s \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}, \end{aligned}$$

where for the second inequality one uses Lemma 2.5 Using the density of I and the fact that $\|\chi_R(H_{R-\delta}+M)^{-1}-(H_R+M)^{-1}\|\leq C$, it follows that if $0<s<1/2$, there exists $C(s)>0$ with

$$\|\chi_R(H_{R-\delta}+M)^{-1}-(H_R+M)^{-1}\|\leq C(s)\,\delta^s,$$

which concludes the lemma. ∎

Remark For other results of this type, see e.g. [S].

The next corollary is a direct consequence of the convergence in the norm resolvent sense (see e.g. [K]):

Corollary 2.7 *Let λ be a discrete eigenvalue of H_R and denote by P_R its corresponding finite dimensional projector. If $\delta>0$ is sufficiently small, then $H_{R-\delta}$ has purely discrete spectrum near λ, $n-\lim_{\delta\to 0}\|P_{R-\delta}-P_R\|=0$ and $\dim(P_R)=\dim(P_{R-\delta})$.*

2.2.3 Construction of φ_0

Proposition 2.8 *Let P_∞ be a finite dimensional orthogonal projector in a Hilbert space $\mathcal{H}$ and let $\{P_k\}$ be a sequence of orthogonal projectors such that*

$$n-\lim_{k\to\infty}P_k=P_\infty\,.$$

Choose any $\varphi_k\in\mathrm{Ran}(P_k)$ with $\|\varphi_k\|=1$; then the sequence $\{\varphi_k\}$ has an adherent point in $\mathrm{Ran}(P_\infty)$.

Proof. Starting from some k_0, the norm convergence condition implies that for $k\geq k_0$, $\|P_k-P_\infty\|<1$, hence $\dim P_k=\dim P_\infty$. Moreover, one can write down (see [K]) the Nagy unitary operator U_k which intertwines P_k and P_∞ (i.e. $U_kP_k=P_\infty U_k$):

$$U_k=\left[1-(P_k-P_\infty)^2\right]^{-1/2}\left[P_\infty P_k+(1-P_\infty)(1-P_k)\right]. \tag{2.21}$$

Define $\psi_k:=U_k\varphi_k$; then $\psi_k\in\mathrm{Ran}(P_\infty)$ and $\|\psi_k\|=1$.

Because $\mathrm{Ran}(P_\infty)$ is finite dimensional, the sequence $\{\psi_k\}$ admits an adherent point $\varphi_\infty\in\mathrm{Ran}(P_\infty)$, $\|\varphi_\infty\|=1$. In other words, there exists a subsequence $\{\psi_{k(n)}\}$ such that $\lim_{n\to\infty}\psi_{k(n)}=\varphi_\infty$.

From (2.21), one can see that $n-\lim_{k\to\infty}U_k=1$, hence $\lim_{k\to\infty}\|\psi_k-\varphi_k\|=0$. Therefore $\lim_{n\to\infty}\|\varphi_{k(n)}-\varphi_\infty\|=0$ and we are done. ∎

Remark In our case, we have to replace P_k with $P_{R-\delta(k)}$, P_∞ with P_R and φ_∞ with φ_0.

Proposition 2.9 *Let $\lambda\in\mathbb{R}$. Suppose that there exists a sequence $\{\varphi_\delta\}_{\delta\in I}$, $I\subseteq(0,\delta_0]$ such that $\|\varphi_\delta\|=1$ and φ_δ is a $H_{R-\delta}$ eigenfunction corresponding to λ. Moreover, suppose that $\varphi_\delta\to\varphi_0$ in $L^2(\mathbb{R}^n)$, $\|\varphi_0\|=1$ and φ_0 is a H_R eigenfunction corresponding to λ. Then*

$$\left.\frac{\partial\varphi_0}{\partial\nu}\right|_{|\mathbf{x}|=R}=0\,. \tag{2.22}$$

Proof. Note first that
$$\|\theta\varphi_\delta\|_{H^2(\Omega_{R-\delta})} \le C, \quad \|\varphi_\delta\|_{H^1(\Omega_{R-\delta})} \le C, \tag{2.23}$$
for each $\theta \in C_0^\infty(\Omega_{r_0})$, where C is a constant which does not depend on δ (see Lemma 2.5(b)). From now on we shall denote such constants by C. Since φ_δ and φ_0 are eigenfunctions corresponding to the same λ, one has
$$\langle(-\Delta + W)\varphi_\delta, \varphi_0\rangle_{L^2(\mathbb{R}^n)} - \langle\varphi_\delta, (-\Delta + W)\varphi_0\rangle_{L^2(\mathbb{R}^n)} = 0.$$

Choose $\{\psi_j\} \subset D$ a sequence such that $\|\psi_j - \varphi_0\|^2 + \|H_R(\psi_j - \varphi_0)\|^2 \to 0$. Let $\theta \in C_0^\infty(\Omega_{r_0})$, $\theta = 1$ if $|x| \in [R_1, R_2]$. Then accordingly to Lemma 2.5, $\theta\phi_0 \in H^2(\Omega_R)$ and $\|\theta\phi_0 - \theta\psi_j\|_{H^2(\Omega_R)} \to 0$. Using the Green formula, one has:
$$\langle(-\Delta + W)\varphi_\delta, \psi_j\rangle_{L^2(\mathbb{R}^n)} - \langle\varphi_\delta, (-\Delta + W)\psi_j\rangle_{L^2(\mathbb{R}^n)} = \int_{\partial\Omega_R} \varphi_\delta \frac{\overline{\partial\psi_j}}{\partial\nu}\, ds_x. \tag{2.24}$$

Replacing ψ_j with $\theta\psi_j$, tending j to the limit and dividing by δ, we get:
$$0 = \left|\int_{\partial\Omega_R} \frac{1}{\delta}\varphi_\delta \,\overline{\frac{\partial(\theta\varphi_0)}{\partial\nu}}\, ds_x\right|. \tag{2.25}$$

We shall show that $\frac{1}{\delta}\varphi_\delta\big|_{|\mathbf{x}|=R}$ converges to $-\frac{\partial(\theta\varphi_0)}{\partial\nu}\big|_{|\mathbf{x}|=R}$ in $L^2(\partial\Omega_R)$. We begin this by proving that
$$\left\|\frac{1}{\delta}\varphi_\delta + \frac{\partial(\theta\varphi_\delta)}{\partial\nu}\right\|_{L^2(\partial\Omega_R)} \le C\delta^{1/2}.$$

For $\varphi \in C^\infty(\overline{\Omega}_{R-\delta}) \cap \mathrm{Dom}(-\Delta_{R-\delta})$, set $\tilde\varphi(r,\omega) = \varphi(r\omega)$, if $r > 0$ and $\omega \in \mathbb{B}S^{n-1}$ (the function $\tilde\varphi$ is just φ in polar coordinates). We have
$$\begin{aligned}\tfrac{1}{\delta}\varphi(R\omega) + \left(\tfrac{\partial\varphi}{\partial\nu}\right)(R\omega) &= \tfrac{1}{\delta}\tilde\varphi(R,\omega) - \partial_1\tilde\varphi(R,\omega)\\ &= \tfrac{1}{\delta}\int_{R-\delta}^{R} [\partial_1\tilde\varphi(t,\omega) - \partial_1\tilde\varphi(R,\omega)]\, dt.\end{aligned}$$

Then,
$$\begin{aligned}\int_{\partial\Omega_R}\left|\tfrac{1}{\delta}\varphi(x) + \left(\tfrac{\partial\varphi}{\partial\nu}\right)(x)\right|^2 ds &= R^{n-1}\int_{|\omega|=1}\left|\tfrac{1}{\delta}\int_{R-\delta}^{R}[\partial_1\tilde\varphi(t,\omega) - \partial_1\tilde\varphi(R,\omega)]\,dt\right|^2 ds_\omega\\ &\le \tfrac{1}{\delta}\int_{R-\delta}^{R}\int_{|\omega|=1} R^{n-1}\,|[\partial_1\tilde\varphi(t,\omega) - \partial_1\tilde\varphi(R,\omega)]|^2\, ds_\omega\, dt\\ &\le \tfrac{1}{\delta}\int_{R-\delta}^{R}\int_{|\omega|=1} R^{n-1}\left|\int_t^R \partial_1^2\tilde\varphi(\tau,\omega)\,d\tau\right|^2 ds_\omega\, dt\\ &\le \tfrac{1}{\delta}\int_{R-\delta}^{R}(R-t)\int_{|\omega|=1} R^{n-1}\int_t^R |\partial_1^2\tilde\varphi(\tau,\omega)|^2\, d\tau\, ds_\omega\, dt\\ &\le C\delta\|\varphi\|^2_{H^2(R-\delta<|x|<R)} \le C\delta\|\varphi\|^2_{H^2(\Omega_{R-\delta})},\end{aligned}$$

where we have repeatedly employed the Cauchy-Schwarz inequality. We now approximate $\theta\varphi_\delta$ in $H^2(\Omega_{R-\delta}) \cap H_0^1(\Omega_{R-\delta})$ with a sequence $\{\varphi_j\}$ of $C^\infty(\overline{\Omega}_{R-\delta}) \cap$ Dom$(-\Delta_{R-\delta})$ functions. Then, we use the above estimate for $\theta\varphi_j$, and let $j \to \infty$. We get

$$\left\| \frac{1}{\delta}\varphi_\delta + \frac{\partial \varphi_\delta}{\partial \nu} \right\|_{L^2(\partial\Omega_R)} \leq C\delta^{1/2} \|\theta\varphi_\delta\|_{H^2(\Omega_{R-\delta})} \leq C\delta^{1/2}$$

(by (2.23)). Thus,

$$\left| \int_{\partial\Omega_R} \frac{\partial \varphi_\delta}{\partial \nu} \overline{\frac{\partial \varphi_0}{\partial \nu}} ds \right| \leq C\delta^{1/2}.$$

On the other hand, for any $3/2 < s < 2$:

$$\left\| \frac{\partial \varphi_0}{\partial \nu} - \frac{\partial \varphi_\delta}{\partial \nu} \right\|_{L^2(\partial\Omega_R)} \leq C(s)\ \|\theta(\varphi_0 - \varphi_\delta)\|_{H^s(\Omega_R)}. \tag{2.26}$$

To handle this, one uses the interpolation inequality (Prop. 2.3. [L-M]), that is

$$\|\theta(\varphi_0 - \varphi_\delta)\|_{H^s(\Omega_R)} \leq C(s) \|\theta(\varphi_0 - \varphi_\delta)\|_{L^2(\Omega_R)}^{1-s/2} \|\theta(\varphi_0 - \varphi_\delta)\|_{H^2(\Omega_R)}^{s/2}, \tag{2.27}$$

since $H^s(\Omega_R) = [H^2(\Omega_R), L^2(\Omega_R)]_{1-s/2}$. Here $\|\theta(\varphi_0 - \varphi_\delta)\|_{L^2(\Omega_R)}^{1-s/2} \to 0$ when $\delta \to 0$, and $\|\theta(\varphi_0 - \varphi_\delta)\|_{H^2(\Omega_R)}^{s/2} \leq C$. Thus

$$\int_{\partial\Omega_R} \left| \frac{\partial \varphi_0}{\partial \nu} \right|^2 ds \to 0,$$

which concludes the proposition. ∎

2.3 Proof of (1.3)

As we have already said, (1.3) is not a new result, but a reformulation of a part of the "classical" proof of the Persson formula (see e.g. [C-F-K-S]). Nevertheless, without including it in Theorem 1.1, Persson's formula would not be a consequence of the theorem.

In order to simplify notation, set $\lambda := \inf \sigma_{\text{ess}}(H)$. Denote by E_H the spectral measure associated to H. From now on, ϵ and δ denote infinitesimally small positive numbers.

Fix $\delta > 0$. We know that for every $\epsilon > 0$ the projector $P := E_H((-\infty, \lambda - \epsilon])$ is compact, therefore there exists $R(\epsilon)$ sufficiently large such that for any $R \geq R(\epsilon)$, one has (χ_R denotes the characteristic function of Ω_R)

$$\|\chi_R P\| \leq \epsilon.$$

Then for any normalized $\varphi \in \mathcal{F}_{\Omega_R}$,

$$\begin{aligned} \langle \varphi, H_R\varphi \rangle_{L^2(\Omega_R)} &= \langle \varphi, H\varphi \rangle_{L^2(\mathbb{R}^n)} \geq (\lambda - \epsilon)\|(1-P)\chi_R\varphi\|^2 - |\lambda - \epsilon| \|P\chi_R\varphi\|^2 \\ &\geq (\lambda - \epsilon)(1-\epsilon)^2 - |\lambda - \epsilon|\epsilon^2 . \end{aligned}$$

Now choose $\epsilon = \epsilon_\delta$ small enough so that the r.h.s. of the above inequality is larger than $\lambda - \delta$. Using the min-max principle, we have $\inf \sigma(H_R) \geq \lambda - \delta$; because $\{\inf \sigma(H_R)\}$ is an increasing sequence with R, then

$$\lim_{R\to\infty} \inf \sigma(H_R) \geq \lambda .$$

The reversed inequality comes from the inclusion $\sigma_{ess}(H) \subset \sigma(H_R)$. ■

Corollary 2.10 *The Persson formula is a direct consequence of Theorem* 1.1.

Proof. Firstly, as $\sigma_{ess}(H) \subset \sigma(H_{K^c})$ for any K, one has

$$\inf \sigma_{ess}(H) \geq \sup_K \inf \sigma(H_{K^c}) .$$

Secondly, from the obvious inequality $\inf \sigma(H_R) \leq \sup_K \inf \sigma(H_{K^c})$ and (1.3):

$$\inf \sigma_{ess}(H) \leq \sup_K \inf \sigma(H_{K^c}) ,$$

and we are done. ■

Received June 2003. Revised Oct 2003

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