CUBO *A Mathematical Journal*
Vol. 7, N^o 1, (101 - 115). April 2005.

The complex Plateau Problem

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1 Introduction

Let γ be a closed curve in $\mathbb{R}^3$.

The classical Plateau problem is the problem of constructing a minimal surface M in $\mathbb{R}^3$ which has γ as its boundary.

We treat the analogue of this problem where we replace $\mathbb{R}^3$ by the space $\mathbb{C}^n$ of n complex variables and fix a smooth closed oriented curve γ in $\mathbb{C}^n$.

We seek a Riemann surface Σ in $\mathbb{C}^n$ having γ as its boundary. You recall that a Riemann surface in $\mathbb{C}^n$ is a two-manifold which is locally parametrized by complex-analytic functions. We shall allow a discrete set of singular points on Σ, so that the strictly correct term for Σ is "one-dimensional complex analytic variety".

Two approaches have been used for a solution to this problem:

Road (I) uses the ideas of analytic continuation in the complex plane and was taken by R. Harvey and B. Lawson in 1975 in their paper [3]. There they deal with a much more general situation. In Section 3 we shall use their method for the special case of a smooth curve in $\mathbb{C}^2$,where the argument is especially elegant and transparent.

Road (II) uses the theory of commutative Banach algebras and was developed in the late 1950'2 and 1960's by a number of authors.(See [10],[2],[9],[8],[1]).We shall describe this road, more briefly, in Section 4.

We shall begin, in Section 2, with the classical background from the study of analytic continuation in the complex plane.

[1] The author thanks Thomas Banchoff and Larry Larrivee for help with the diagrams. He is also grateful to John Anderson for helpful suggestions.

2 Analytic Continuation

Let α be a smooth closed arc in the complex plane and let z_0 be a point in the interior of α. We fix an orientation on α and we denote by Ω^+ a neighborhood of α to the left of α and by Ω^- a neighborhood of α to the right of α.

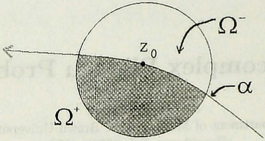

Figure 1.

We consider a smooth function ϕ defined on α and we define a function Φ on the complement of α by

$$\Phi(z) = \frac{1}{2\pi i}\int_\alpha \frac{\phi(\zeta)d\zeta}{\zeta - z}$$

for z in $\mathbb{C}\backslash\alpha$.

We denote by Φ^+ the restriction of Φ to Ω^+ and by Φ^- the corresponding function on Ω^-. Since the integrand is an analytic function of the parameter z,the function Φ is analytic on $\mathbb{C}\backslash\alpha$ and Φ^+ is analytic on Ω^+ and similarly for Φ^-. What happens as the variable z approaches z_0 ?

Theorem 2.1 *The limit of $\Phi^+(z)$ as z approaches z_0 within Ω^+ exists, and we denote it $\Phi^+(z_0)$. Similarly, we define $\Phi^-(z_0)$. Then*

$$\Phi^+(z_0) - \Phi^-(z_0) = \phi(z_0) \tag{1}$$

For this theorem, see Plemelj (1908),[7].

We now consider a simple closed curve β in the complex plane and let ϕ be a smooth function defined on β.Under what condition on ϕ does there exist an analytic continuation Φ of ϕ from β to the domain Ω bounded by β ?

If such an extension Φ exists, then we have

$$\int_\beta \phi(\zeta)d\zeta = \int_\beta \Phi(\zeta)d\zeta = 0$$

by Cauchy's theorem. Similarly we have

$$\int_\beta \zeta^n\phi(\zeta)d\zeta = 0 \tag{2}$$

for n = 0,1,2,...

So (2) is a necessary condition on ϕ for the existence of an analytic extension. Is (2) sufficient ? We define $\Phi(z)$,for z in Ω, by

$$\Phi(z) = \frac{1}{2\pi i}\int_\beta \frac{\phi(\zeta)d\zeta}{\zeta - z}$$

Fix a point ζ on β. For $| z |$ sufficiently large, the series

$$-\sum_{n=0}^{\infty} \frac{\zeta^n}{z^{n+1}}$$

converges and equals $\frac{1}{\zeta - z}$. It follows that

$$\Phi(z) = \int_\beta \frac{\phi(\zeta)d(\zeta)}{\zeta - z} = -\sum_{n=0}^{\infty}\int_\beta \phi(\zeta)\zeta^n d\zeta \frac{1}{z^{n+1}}$$

and the last term vanishes by (2). So $\Phi(z) = 0$ for large $| z |$ and in consequence $\Phi = 0$ outside of β. We now fix a point z_o on β and chose a subarc α of β which contains z_0.

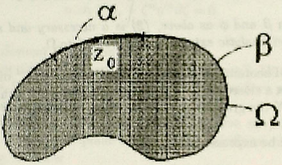

Figure 2.

We write, for each z in $\mathbb{C}\backslash\beta$,

$$\Phi(z) = \Phi_1(z) + \Phi_2(z),$$

where

$$\Phi_1(z) = \int_\alpha \frac{\phi(\zeta)d\zeta}{\zeta - z}$$

$$\Phi_2(z) = \int_{\beta\backslash\alpha} \frac{\phi(\zeta)d\zeta}{\zeta - z}.$$

It is clear that Φ_2 has a continuous extension from Ω to the outside of β across the arc α. For Φ_1 we have by the Jump Theorem, that

$\Phi_1^+(z_0) - \Phi_1^-(z_0) = \phi(z_0)$, where Φ_1^+ and Φ_1^- are defined as earlier. Also

$$\Phi_2^+(z_0) - \Phi_2^-(z_0) = 0.$$

Adding equations, we get at z_0,

$$\Phi_1^+ + \Phi_2^+ - (\Phi_1^- + \Phi_2^-) = \phi$$

Since

$$\Phi_1^- + \Phi_2^- = \Phi^-$$

and since Φ vanishes outside of β, $\Phi^-(z_0) = 0$. So

$$\Phi_1^+ + \Phi_2^+ = \phi(z_0)$$

and so

$$\Phi^+(z_0) = \phi(z_0)$$

So $\phi(z_0) = lim \Phi(z)$ as z approaches z_0 from within Ω.
So Φ, restricted to Ω, is the required analytic extension of ϕ. We have proved

Theorem 2.2 *Given β and ϕ as above, (2) is a necessary and sufficient condition for the existence of an analytic extension of ϕ from β to Ω.*

We can interpret Theorem 2.2 geometrically in the space $\mathbb{C}^2$ of two complex variables, z and w. We fix a closed curve β in $\mathbb{C}$ and consider a smooth function ϕ defined on β. We denote by X the graph of ϕ in $\mathbb{C}^2$, so that X is the set of all points $(z, \phi(z))$ with z in β.

Condition (2) can be expressed as

$$\int_X wz^n dz = 0, n = 0, 1, 2 \tag{3}$$

since $\int_X wz^n dz = \int_\beta \phi(z) z^n dz$.

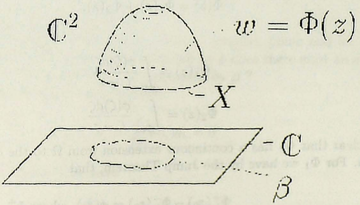

Figure 3.

3 The Moment Condition

Formula (3) gave us a necessary condition on a graph X in $\mathbb{C}^2$ to be the boundary of a Riemann surface. We now wish to do the corresponding thing for a given smooth closed curve β in $\mathbb{C}^2$. Suppose that γ bounds a Riemann surface Σ.

Fix non-negative integers n and m and let α denote the differential form $\zeta^n\eta^m d\zeta$ on $\mathbb{C}^2$. We claim that the restriction of α to Σ is a closed form on Σ. We denote by ζ and η the complex coordinates on $\mathbb{C}^2$. Let p be a point on Σ and let t be a local coordinate on Σ at p. Then near p we have $\zeta = g(t)$ and $\eta = h(t)$, where g and h are analytic functions of t. So $\alpha = (g(t))^n((h(t))^m g'(t)dt$ and so $d\alpha = (k(t)dt) \wedge dt = 0$, proving our Claim. Applying Stokes' theorem to Σ, with boundary γ, we get

$$\int_\gamma \alpha = \int_\Sigma d\alpha = 0$$

So we have the following necessary condition on γ :
For each pair of integers

$$n, m \geq 0, \int_\gamma \zeta^n \eta^m d\zeta = 0. \tag{4}$$

We call (4) the Moment Condition on γ.

Is the moment condition sufficient as well as necessary? To arrive at an answer, we assume there is a Riemann surface Σ bounded by γ and we project Σ and γ into the complex plane by the map $\pi : (z, w) \to z$.

The image of γ under π in $\mathbb{C}$ is a smooth closed curve $\pi(\gamma)$ in general with self-intersections, which divides the plane into a finite or infinite number of connected components. See Fig. 5 below

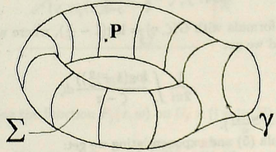

Figure 4.

We fix one of the components U of $\mathbb{C}\backslash\pi(\gamma)$. The inverse image $\pi^{-1}(U)$ of U in Σ lies over U as a finite-sheeted cover, possibly branched. We denote by n the number of sheets of this cover. For each point z in U there are n points $(z, w_j(z)), j = 1, 2, ..., n$, lying over z. Locally, except at branch points, each w_j is an analytic function of z. In general, w_j is multiple-valued on U.

We define a function F of z and w in $U \times \mathbb{C}$ by setting

$$F(z,w) = \prod_{j=1}^{n}(w - w_j(z))$$

for z in U and w in $\mathbb{C}$.

If we expand this product, we get an n'th degree polynomial in w whose coefficients are elementary symmetric functions of the w_j, and hence single-valued analytic on U. So F is an analytic function on $U \times \mathbb{C}$. Furthermore, the zeros of F are the points $(z, w_j(z))$ with z in U, and so exactly the points in $\pi^{-1}(U)$. Our next goal is to express $F(z,w)$ in terms of data on γ. We put $R = max \mid \eta \mid$ taken over all points (ζ, η) on $\Sigma \cup \gamma$.

We have

$$F(z,w) = \prod_{j=i}^{n}(w - w_j(z)) = w^n \prod_{j=1}^{n}(1 - \frac{w_j(z)}{w})$$

for all (z,w) in $U \times \mathbb{C}$. If $\mid w \mid \geq R$, $\log(1 - \frac{w_j(z)}{w})$ is locally well-defined and for z in U $\sum_{j=1}^{n} \log(1 - \frac{w_j(z)}{w})$ is single-valued analytic on U. It follows that

$$\log F(z,w) = n \log w + \sum_{j=1}^{n} \log(1 - \frac{w_j(z)}{w}) + 2\pi i N \tag{5}$$

for some integer N.Fix a point z in U and consider the meromorphic differential form $\frac{G(\zeta,\eta)d\zeta}{\zeta - z}$ on Σ, where G is an analytic function on Σ. The poles of this form occur at the points $((z, w_j(z))$, j = 1,2,...n, and they are simple poles with residue $G(z, w_j(z))$ at the jth point. The residue theorem, applied to Σ then gives

$$\frac{1}{2\pi i}\int_\gamma \frac{G d\zeta}{\zeta - z} = \sum_{j=1}^{n} G(z, w_j(z))$$

We apply this formula with $G(\zeta, \eta) = \log(1 - \frac{\eta}{w})$, where w is a complex number with $\mid w \mid > R$, and we get

$$\frac{1}{2\pi i}\int_\gamma \frac{\log(1 - \frac{\eta}{w}))}{\zeta - z} d\zeta$$

equals $\sum_{j=1}^{n} \log(1 - \frac{w_j(z)}{w})$.

Recalling formula (5) and exponentiating we get:

$$F(z,w) = w^n exp[\frac{1}{2\pi i}\int_\gamma \frac{log(1 - \frac{\eta}{w})}{\zeta - z} d\zeta], \quad z \in U, \quad \mid w \mid > R. \tag{6}$$

We note that, by the residue theorem again, the number of sheets n of $\pi^{-1}(U)$ is also expressed in terms of γ, by the relation:

$$n = \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - z}, \quad z \in U. \tag{7}$$

We have reached our goal of expressing $F(z,w)$ in terms of data on γ. However, this expression holds only for $|w|>R$. The zeros of $F(z,w)$, on the other hand, which give us the Riemann surface Σ, are contained in $U\times(|w|<R)$ as is shown in Fig. 7, below:

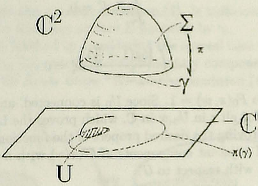

Figure 5.

Can we reverse the procedure, i.e. given a curve γ is $\mathbb{C}^2$, can we use the right-hand side of (6), which is defined in terms of γ, to construct a Riemann surface Σ ?

We choose a simple closed curve γ in $\mathbb{C}^2$ and assume that γ satisfies the moment condition (4) We list the complementary components of $\pi(\gamma)$ in the complex plane as $U_0, U_1, U_2, \ldots$, with U_0 denoting the unbounded component.

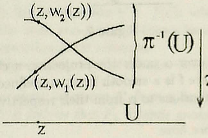

Figure 6.

For given j, we define the function $F_j(z,w)$ on $U_j\times(|w|>R)$ by the formula

$$F_j(z,w)=w^{n_j}exp[\int_\gamma \frac{\log(1-\frac{\eta}{w})}{\zeta-z}d\zeta]$$

Here n_j is defined by the formula (7).

We have the task now to show that, for each j, $F_j(z,w)$ admits analytic continuation from $U_j\times(|w|>R)$ to $U_j\times\mathbb{C}$ and that the zero sets $F_j(z,w)=0$ in $U_j\times\mathbb{C}$ fit together to an analytic variety Σ in $\mathbb{C}^2\backslash\gamma$ having γ as its boundary.

Lemma 3.1 *$F_0(z,w)$ is identically 1 for z in U_0 and w in $\mathbb{C}$.*

Proof Fix z, w with $|z| > R$ and $|w| > R$, and put $G(\zeta,\eta) = \frac{\log(1-\frac{\eta}{w})}{\zeta - z}$. Then G is an analytic function of ζ and η in some bidisk containing γ, and hence we can find a sequence of polynomials $(P_\nu(\zeta,\eta))$ which tends uniformly to G on γ as ν approaches infinity. Since each P_ν is a linear combination of monomials $\zeta^n\eta^m$, the moment condition gives that

$$\int_\gamma P_\nu d\zeta = 0,$$

and so

$$\int_\gamma \frac{\log(1-\frac{\eta}{w})}{\zeta - z} d\zeta = 0.$$

Also $n_0 = 0$ and so $F_0(z,w) = 1$. Since U_0 is connected, analytic continuation yields that $F_0(z,w) = 1$ for all z in U_0,w in $\mathbb{C}$, which proves the Lemma. ∎

A key tool in proving the desired propert of the functions F_j is the following: Let U_i, U_j be a pair of adjacent components of $\mathbb{C}\backslash\pi(\gamma)$ with a common boundary arc α oriented positively with respect to U_i

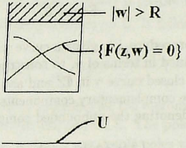

Figure 7.

The portion of γ lying over α under the projection π can be represented in the form: $\eta = f(\zeta)$, ζ in α, where f is a smooth function defined on α. The functions F_i and F_j have continuous extensions to α from their respective regions.

Lemma 3.2 *For z_0 in α, we have*

$$F_j(z_0,w) = (w - f(z_0))F_i(z_0,w), \; |w| > R. \tag{8}$$

Proof We put

$$\Phi_i(z,w) = \frac{1}{2\pi i}\int_{\pi(\gamma)} \frac{log(1-\frac{f(\zeta)}{w})}{\zeta - z} d\zeta,$$

z in U_i, and we define Φ_j similarly for z in U_j.Fix w ,$|w| \geq R$.
The limit of $\Phi_i(z,w)$, as z approaches z_o within U_i, exists and we denote it $\Phi_i(z_0,w)$.
Similarly we define $\Phi_j(z_0,w)$.
By the Jump Theorem, we have

$$\Phi_j(z_0,w) - \Phi_i(z_0,w) = log(1 - \frac{f(z_0)}{w}).$$

Also $F_i(z_0, w) = w^{n_i} exp\Phi_i(z_0, w)$, and similarly for F_j. Also, $n_j = n_i + 1$. So

$$\frac{F_j(z_0, w)}{F_i(z_0, w)} = w^{n_j - n_i} exp[\Phi_j(z_0, w) - \Phi_i(z_0, w)] = w[1 - \frac{f(z_0)}{w}].$$

Thus (8) holds. ∎

We wish to prove that the moment condition,(4) ,is sufficient (as well as necessary)for the existence of a Riemann surface Σ bounded by γ.

We shall illustrate the method by considering the special case when γ is a smooth curve in $\mathbb{C}^2$ such that $\pi(\gamma)$ has exactly 3 complementary components U_0, U_1, and U_2, and $\pi(\gamma)$ is as shown in the figure below. We assume condition (4) is satisfied by γ.

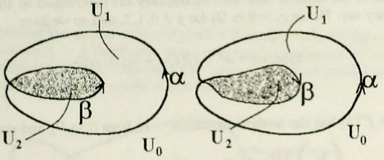

Figure 8.

γ is given parametrically by an equation $\eta = f(\zeta)$, where f is a smooth function defined on the union of the open arcs α and β shown in the figure. Thus f is defined on $\pi(\gamma)$ with the self-intersection point removed.

Warning: what now follows omits a lot of details. See the cited literature for more.

Claim 3.1 *f admits an analytic extension* f^* *from* α *to* U_1, *and* $F_1(z, w) = w - f^*(z)$ *for z in* U_1 *and* $| w |> R$. *It follows that* F_1 *extends analytically to all of* $U_1 \times \mathbb{C}$.

By Lemma 3.2, $F_1(z_0, w) = (w - f(z_0))F_0(z_0, w)$, for z_0 in α and $| w |> R$. By Lemma 3.1, $F_0 = 1$ on U_0, and so $F_0(z_0, w) = 1$. It follows that

$$F_1(z_0, w) = w - f(z_0).$$

Fix w in $| w |> R$. By its definition, $F_1(z, w)$ is analytic in z for z in U_1 and extends continuously to α. It follows that f admits an analytic extension from α to U_1. We call this extension f^*. Then $F_1(z, w)$ and $w - f^*$ agree, as functions of z, on α and hence agree on U_1.

Thus $F_1(z, w) = w - f^*(z)$ for z in U_1, proving the Claim.

We denote by Σ_1 the Riemann surface: $F_1(z, w) = 0$ in $U_1 \times \mathbb{C}$. Then Σ_1 has equation: $w = f^*(z)$ and so fits over α onto the arc of γ which lies over α. This provides the piece of our desired Riemann surface Σ which lies over U_1.

We next consider Case 1 in Fig. 8 above. Here the arc β is oriented positively with respect to U_2. Then $n_2 = n_1 + 1 = 2$, and by Lemma 3.2 we have for each point

λ in β and $|\ w\ |> R$, the relation $F_2(\lambda, w) = (w - f(\lambda))F_1(\lambda, w)$ and so

$$F_2(\lambda, w) = (w - f(\lambda))(w - f^*(\lambda)). \tag{9}$$

Thus $F_2(\lambda, w) = w^2 - (f(\lambda) + f^*(\lambda))w + f(\lambda)f^*(\lambda)$. On the other hand we have the representation $F_2(z, w) = \sum_{-\infty}^{\infty} c_j(z)w^j$ for (z,w) in $U_2\times\ |\ w\ |> R$ where each c_j is analytic on U_2 and continuous on $U_2 \cup \beta$. In particular, this equation holds at a point λ on β. So we have $F_2(\lambda, w) = \sum_{-\infty}^{\infty} c_j(\lambda)w^j$ equals $w^2-(f(\lambda)+f^*(\lambda))w+f(\lambda)f^*(\lambda)$, $|\ w\ |> R$.

It follows that $c_2(\lambda) = 1, c_1(\lambda) = -(f(\lambda) + f^*(\lambda))$, and $c_0(\lambda) = f(\lambda)f^*(\lambda)$, and $c_j(\lambda) = 0$ for $j \neq 0, 1, 2$.

Since the c_j are analytic functions on U_2, they are determined by their values on any boundary arc. Hence $c_j = 0$ in U_2 for $j \neq 0, 1, 2$, and so we have

$$F_2(z, w) = w^2 + c_1(z)w + c_0(z), z\epsilon U_2, \tag{10}$$

and

$f(\lambda) + f^*(\lambda)$ has the analytic extension $-c_1$ from β to U_2 and similarly,

$$f(\lambda)f^*(\lambda) \text{ extends as } c_0 \text{ to } U_2. \tag{11}$$

At this point , we know from (10) that F_2 has an analytic extension from $U_2 \times (|\ w\ |> R)$ to $U_2 \times \mathbb{C}$, and is a second degree polynomial in w. What can be said about the zero-set of F_2 in $U_2 \times \mathbb{C}$?

We factor F_2 into factors linear in w and obtain

$$F_2(z, w) = (w - W_1(z))(w - W_2(z)), \text{ for } z\epsilon U_2 \cup \beta, \tag{12}$$

where W_1, W_2 are branches of a two-valued analytic function on U_2. The zero-set of F_2 is the graph in $\mathbb{C}^2$ of this two-valued function.

Further, by (9), for $\lambda\epsilon\beta$, we have

$$F_2(\lambda, w) = (w - f(\lambda))(w - f^*(\lambda)). \tag{13}$$

If W_1 and W_2 are not coincident, we conclude that, outside a small singular set on β, W_1 coincides with f on β and W_2 coincides with f^* on β. We now define the Riemann surface Σ_2 over U_2 to be the two-sheeted graph of the two-valued analytic function (W_1, W_2) on U_2. Then Σ_2 continues Σ_1 analytically from U_1 across β to U_2, since $W_2 = f^*$ on β, and also Σ_2 fits onto the arc:$\eta = f(\zeta)$ of γ over β.

Finally, we take Σ to be the Riemann surface obtained in $\mathbb{C}^2$ by joining together Σ_1 and Σ_2, and the arc over β along which Σ_1 connects with Σ_2. Then Σ is the desired Riemann surface having γ as its boundary. A schematic sketch of Σ is given in the following picture.

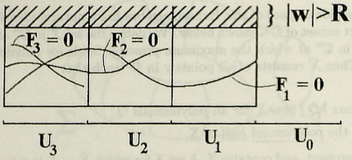

Figure 9.

Case 2 is handled in a similar way, and yields a Riemann surface lying one-sheeted over U_1 with boundary γ. A third case, where $\pi(\gamma)$ has 3 complementary components is shown in the following figure. One can show, using the moment condition, that this case cannot occur, since γ is a simple closed curve.

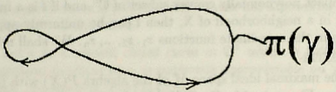

Figure 10.

The method we have sketched here can be carried out in general, and yields the following result:

Theorem 3.3 *Let γ be a smooth simple closed curve in $\mathbb{C}^2$ which satisfies the Moment Condition. If γ is suitably oriented, then there exists a 1-complex dimensional complex-analytic subvariety Σ of $\mathbb{C}^2\setminus\gamma$ which has γ as its boundary in the sense of Stokes' Theorem, i.e.*

$$\int_\gamma \sigma = \int_\Sigma d\sigma \text{ for every smooth closed } 1-\text{ form } \sigma \text{ on } \mathbb{C}^2.$$

This is a special case of Theorem I in [3]. That theorem also gives more information on the sense in which Σ is attached to γ.

4 Banach algebras

Let X be a compact subset of $\mathbb{C}^n$. The polynomials in the coordinate functions $z_1, z_2, \ldots, z_n$, restricted to X, form an algebra of continuous functions on X. We denote the uniform closure of this algebra on X by $P(X)$. With norm: $\| f \| = max \mid f \mid$, $P(X)$ is a Banach algebra.

The maximal ideal space of this Banach algebra has a natural identification with a certain compact subset of $\mathbb{C}^n$, shown below. We define the set $\hat{X}$ in $\mathbb{C}^n$ as the collection of all points y in $\mathbb{C}^n$ at which the maximum principle holds for polynomials, relative to the set X. Thus $\hat{X}$ consists of all points y in $\mathbb{C}^n$ such that

$\mid Q(y) \mid \leq max \mid Q \mid overX$ for all polynomials Q.

$\hat{X}$ is called the *polynomial hull* of X.

$\hat{X}$ is itself compact, and contains X. A set X for which $\hat{X} = X$ is called *polynomially convex*.

Let K be a compact subset of $\mathbb{C}$. Then K is polynomially convex if and only if the complement of K in $\mathbb{C}$ is connected. Runge's theorem states that if K is polynomially convex, and if f is a function defined and holomorphic in some neighborhood of K, then f can be uniformly approximated on K by polynomials in z. A generalization of Runge's theorem for $\mathbb{C}^n, n > 1$ was given by Andre Weil and Kiyoshi Oka. It states that if X is a compact polynomially convex subset of $\mathbb{C}^n$ and if f is a function defined and holomorphic in a neighborhood of X, then f can be uniformly approximated on X by polynomials in the coordinate functions $z_1, z_2, ..., z_n$. We shall refer to it as the *Oka – Weil Theorem*.

We identify the maximal ideal space M of the algebra $P(X)$ with the polynomial hull $\hat{X}$, as follows. Fix m in M. By Gelfand's theory, there exists a non-zero homomorphism τ sending $P(X) \to \mathbb{C}$ whose kernel is m.Also the norm of τ as a linear functional on $P(X)$ is 1.Denote by ζ the point $(\tau(z_1), \tau(z_2), ..., \tau(z_n))$ in $\mathbb{C}^n$. If P is a polynomial on $\mathbb{C}^n$, then $P(\zeta) = \tau(P(z_1, ..., z_n)$. So $\mid P(\zeta) \mid \leq \| P \|$. So ζ belongs to $\hat{X}$. Conversely, every point ζ in $\hat{X}$ arises from some m in this way. So we have the identification of M and $\hat{X}$.

We give some examples of polynomial hulls.

Ex.1: β is a simple closed curve in the complex plane. Then $\hat{\beta}$ is the union of β and the region bounded by β

Ex.2: S is the 3-sphere $\mid z \mid^2 + \mid w \mid^2 = 1$ in $\mathbb{C}^2$. Then $\hat{S}$ is the closed ball bounded by S.

Ex.3: Y is the circle on the complex line $w = 0$ in $\mathbb{C}^2$ given by $\mid z \mid = 1$.Then $\hat{Y}$ is the closed disk on $w = 0$ bounded by Y.

Ex.4: R^n denotes the subspace of $\mathbb{C}^n$ consisting of those points all of whose coordinates are real. Let K be a compact subset of R^n. Then $\hat{K} = K$.

Let now γ be a simple closed smooth curve in $\mathbb{C}^n$.Suppose that there exists a Riemann surface Σ lying in the domain $\mathbb{C}^n \backslash \gamma$ whose boundary is γ, such that $\Sigma \cup \gamma$ is compact. Choose a point a in Σ . If P is a polynomial, then the restriction of P to Σ is analytic on Σ By the maximum principle on Σ, then, $\mid P(a) \mid \leq max \mid P \mid$ over γ. Since this holds for every polynomial P, the point a lies in $\hat{\gamma}$.

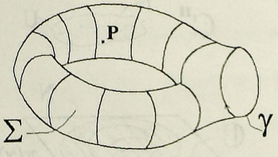

Figure 11.

The question now arises: what other point of $\mathbb{C}^n$ belong to $\hat{\gamma}$? We shall answer this question in the Curve Theorem below.

We note that it may happen that $\hat{\gamma} = \gamma$, i.e. that γ is polynomially convex. This is the case when γ is the curve: $z = e^{i\theta}, w = e^{-i\theta}, 0 \leq \theta \leq 2\pi$. It is a nice exercise to prove that, indeed, this curve is polynomially convex.

The fact that a given curve is polynomially convex has a powerful consequence for uniform approximation on the curve.

Theorem 4.1 *Let γ be a smooth closed curve in $\mathbb{C}^n$ which is polynomially convex. Then every continuous function on γ is uniformly approximable on γ by polynomials in the coordinates. (In symbols, $P(\gamma) = C(\gamma)$).*

Helson and Quigley in [5] gave a proof of this theorem.

Proof We make use of the following result of approximation on plane sets, due to Hartogs and Rosenthal, [4], (1931): Let K be a compact plane set of two-dimensional measure 0. Then every continuous function on K can be uniformly approximated on K by rational functions which are analytic on a neighborhood of K. ∎

Consider the coordinate function z_1. We claim that the complex conjugate $\overline{z_1}$ of z_1, restricted to γ, lies in $P(\gamma)$.Let π_1 denote the map which projects $\mathbb{C}^n$ to $\mathbb{C}$ with $\pi_1(z_1, z_2, ..., z_n) = z_1$. Since γ is smooth ,by assumption, and π_1 is a smooth map, the image $\pi_1(\gamma)$ in $\mathbb{C}$ has 2-dimensional measure 0. Given $\epsilon > 0$, the Hartogs-Rosenthal theorem allows us to find a rational function r , analytic in a neighborhood N of $\pi_1(\gamma)$,such that $| r(\zeta) - \overline{\zeta} | < \epsilon$ for each ζ in $\pi_1(\gamma)$. We now choose a neighborhood U of γ in $\mathbb{C}^n$ with $\pi_1(U) \subset N$. $r \circ \pi_1$ is then analytic in U. Also, we have for $z = (z_1, z_2, ..., z_n)$ in γ

$$| r(\pi_1(z)) - \overline{\pi_1(z)} | < \epsilon,$$

since $\pi_1(z)$ is in $\pi_1(\gamma)$. Because $r \circ \pi_1$ is analytic on U, the Oka-Weil theorem gives that the restriction of $r \circ \pi_1$ to γ lies in $P(\gamma)$. Since ϵ is arbitrary, we have that the restriction of $\overline{z_1}$ lies in $P(\gamma)$.The similar statement holds for z_j for all j.Thus $P(\gamma)$ contains all the coordinate functions and their conjugates, restricted to γ, and so the Stone-Weierstrass theorem gives us that $P(\gamma) = C(\gamma)$, and we are done.

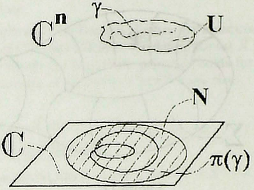

Figure 12.

The converse of this theorem is true, because if $P(\gamma) = C(\gamma)$, then the maximal ideal space of $P(\gamma)$ coincides with the maximal ideal space of $C(\gamma)$, and the maximal ideal space of $C(X)$, for every compact Hausdorff space X, equals X. Since the maximal ideal space of $P(\gamma)$ is $\hat{\gamma}$, we conclude that $\hat{\gamma} = \gamma$.

Let γ be a simple closed smooth curve in $\mathbb{C}^n$ with $\hat{\gamma} \neq \gamma$. We have the following result, which we call the <u>Curve Theorem</u>:

Theorem 4.2 *If $\hat{\gamma} \neq \gamma$, then the set $\Sigma = \hat{\gamma}\backslash\gamma$ is a one-dimensional complex analytic subvariety of $\mathbb{C}^n\backslash\gamma$.*

See Stolzenberg, [9], for a proof of the Curve Theorem.

It still remains to show that the variety Σ has γ as its boundary. This is true in the sense of Stokes' theorem, i.e. if ω is a smooth one-form of $\mathbb{C}^n$, then

$$\int_\gamma \omega = \int_\Sigma d\omega$$

For a proof of this, see Mark Lawrence,[6] and Harvey-Lawson,[3].

Material related to this article can be found, in particular, in the book The Theory of Uniform Algebras, by E.L Stout, Bogden and Quegley, (1971), Chapter 6,and in the book, Several Complex Variables and Banach Algebras, 3rd edition, by H. Alexander and J. Wermer, Springer-Verlag (1998), Chapters 12 and 19.

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