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Semi-classical measures and the Helmholtz Equation

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ABSTRACT

In this expository article, we review and improve uniform resolvent estimates for a family of operators and show how such results can be applied to the study of the high frequency Helmholtz equation by means of semi-classical measures. For the source term concentrated near a point, we provide a complete solution to determine the semi-classical measure.

RESUMEN

En este artículo revisamos y mejoramos estimaciones resolventes uniformes para una familia de operadores y mostramos como tales resultados pueden ser aplicados al estudio de la ecuación de Helmholtz de alta frecuencia por medio de medidas semi clásicas. Para el término fuente concentrado cerca de un punto, proveemos una solución completa para determinar la medida semi clásica.

Key words and phrases: *Mourre's method depending on a parameter, semi-classical resolvent estimates in Besov spaces, the Helmholtz equation, semi-classical measure.*

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1 Introduction

The Helmholtz equation describes the propagation of light wave in material medium. Recently, there is a renewed interest in the study of the Helmholtz equation related to the design of very high power laser devices such as Laser Méga-Joule in France or the National Ignition Facility in the USA. The laser field, $A(x)$, can be very accurately modelled and computed by the solution of the Helmholtz equation

$$\Delta A(x) + k_0^2(1 - N(x))A(x) + ik_0\nu(x)A(x) = 0 \tag{1.1}$$

where k_0 is the wave number of laser in vacuum, $N(x)$ is a smooth positive function representing the adimensional electronic density of material medium and $\nu(x)$ is positive smooth function representing the absorption coefficient of the laser energy by material medium. Since laser can not propagate in the medium with the electronic density bigger than 1, it is assumed that $0 \leq N(x) < 1$. The equation (1.1) may be posed in an unbounded domain with known incident excitation A_0. The equation is then complemented by a so-called radiation condition on the boundary. The high oscillatory behavior of the solution to the Helmholtz equation makes the numerical resolution of (1.1) rather expensive. Fortunately, the wave length of laser in vacuum, $\frac{2\pi}{k_0}$, is much smaller than the scale of N. It is therefore naturel and important to study the Helmholtz equation in the high frequency limit $k_0 \to \infty$. In this expository article, instead of studying boundary value problem related to a non-self-adjoint operator, we study the high frequency Helmholtz equation with a source term

$$(\Delta + \epsilon^{-2}n(x)^2 + i\epsilon^{-1}\alpha_\epsilon)u_\epsilon(x) = -S_\epsilon(x) \tag{1.2}$$

in $\mathbb{R}^d$, $d \geq 1$. Here $\epsilon \sim \frac{1}{k_0} > 0$ is regarded as a small parameter, $\alpha_\epsilon > 0$ is a regularizing constant verifying

$$\lim_{\epsilon \to 0} \alpha_\epsilon = \alpha \geq 0 \quad \text{and if } \alpha = 0, \exists \gamma \in]0,1[\text{ such that } \alpha_\epsilon \geq \epsilon^\gamma. \tag{1.3}$$

$n(x)$ is the refraction index. Here we only discuss the case the source term is concentrated near a point $x = 0$

$$S_\epsilon(x) = \epsilon^{-\frac{3+d}{2}} S(\frac{x}{\epsilon}) \tag{1.4}$$

See [3, 6]. If the source term presents a concentration-oscillation phenomenon near a submanifold Γ of $\mathbb{R}^d$, the problem is much more difficult due to the lack of decay in tangent direction of Γ. See [7] for constant refraction index and [35] for the variable refraction index under some technical conditions. Equation (1.2) can be put into the form of semi-classical Schrödinger equation

$$(-h^2\Delta + V(x) - E - i\kappa)u_h(x) = h^{\frac{1-d}{2}} S(\frac{x}{h}) \tag{1.5}$$

in $\mathbb{R}^d$, $d \geq 1$, where

$$h = \epsilon, \quad V(x) = E - n(x)^2, \quad \kappa = \kappa(h) = h\alpha_h.$$

Remark that if $S(0) = 1$, the right hand side can be regarded as a weak approximation of $h^{\frac{1}{2}}\delta_0(x)$. We shall use tools from semi-classical scattering theory to study it by the approach of E. Wigner, *i.e.*, the approach of semi-classical measure or Wigner measure.

The purpose of the work is to present the resolvent estimates depending on a parameter in a unified way and to apply them in the study of the Helmholtz equation by means of semi-classical measures.

In Section 2, we recall basic properties of Wigner transform and semi-classical measures. The results of this part are due to [11, 12, 21]. We present in Section 3 a parameter-dependent version of the well-known Mourre's method in quantum scattering theory and give uniform microlocal and Besov-space resolvent estimates. Some results in this Section are contained in [35]. These estimates are applied in Section 4 to the high frequency Helmholtz equation. The main result Theorem 4.1 is new and was conjectured in [3] and proved in [6] under some assumption on the self-intersection manifold of the Hamiltonian flow near zero.

2 Wigner transform and semi-classical measures

Semi-classical measures or Wigner measures were introduced by Wigner in 1932 in the study of semi-classical limit of quantum mechanics from the point of views of thermodynamic equilibrium. See [36]. For $\psi \in L^2(\mathbb{R}^d)$, the Wigner transform of ψ is defined by

$$W(\psi)(x,\xi) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-iy\xi}\psi(x+\frac{y}{2})\overline{\psi(x-\frac{y}{2})}\,dy, \tag{2.1}$$

for $(x,\xi) \in \mathbb{R}^{2d}$. $W(\psi)$ is quadratic in ψ, but is linear with respect to the density function $\rho(x,y) = \psi(x)\overline{\psi(y)}$, *a.e.* in x, y. A remarkable property of Wigner transform is that if $\psi = \psi_h(t)$ is solution to the Schrödinger equation

$$\begin{cases} ih\frac{\partial\psi}{\partial t} &= -\frac{h^2}{2}\Delta\psi \\ \psi_{|t=0} &= \psi_0, \end{cases} \tag{2.2}$$

then the scaled Wigner transform, $W_h(x,\xi;t)$ of ψ:

$$W_h(x,\xi;t) = \frac{1}{h^d}W(\psi)(x,\frac{\xi}{h})$$

is solution to the Liouville equation

$$\begin{cases} \frac{\partial W_h}{\partial t} + \xi\cdot\nabla_x W_h &= 0 \\ W_{h|t=0} &= \frac{1}{h^d}W(\psi_0)(x,\frac{\xi}{h}) \end{cases} \tag{2.3}$$

More generally, if there is an appropriate potential $V(x)$, it was expected that the Wigner transform, $W_h(t)$, of the solution $\psi_h(t)$ to the Scrödinger equation

$$ih\frac{\partial\psi_h(t)}{\partial t} = (-\frac{h^2}{2}\Delta + V(x))\psi_h(t) \tag{2.4}$$

converges to some limit f as $h \to 0$, which satisfies the associated Liouville equation

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla_x f - \nabla V(x) \cdot \nabla_\xi f = 0 \quad \text{in} \quad \mathbb{R}^d_x \times \mathbb{R}^d_\xi \times \mathbb{R}_t. \tag{2.5}$$

It is worth to notice that the solution of (2.5) can be written down explicitly in terms of solution of the Hamiltonian system of $p(x,\xi) = \frac{\xi^2}{2} + V(x)$. The approach of E. Wigner allows to relate formally the quantum mechanics to classical mechanics. However, the limit f is, in general, not a function, but only a measure. Rigorous justification of Wigner's approach requires the study of measures obtained as week limit of the Wigner transform of a family of wave functions. This approach was justified for many linear and nonlinear evolution equations. See [21, 11, 12, 38, 39].

Another interest for studying semi-classical measure originates from the compensated compactness principle in methods of variations ([20]). Let $\{u_k\}$ be a bounded sequence in $L^2(\mathbb{R}^d)$. Extracting a subsequence if necessary, we can assume without loss that this sequence converges weekly to some limit $u \in L^2(\mathbb{R}^d)$. The defect measure ν, associated with the sequence $\{u_k\}$ is a Radon measure defined as the vague limit in sense of measure of the sequence $|u_k(x) - u(x)|^2$ as $k \to \infty$. The defect measure is used to describe the loss of the compactness in elliptic variational problems, and can not be used to distinguish different oscillation directions. A simple example is that for $v \in L^2(\mathbb{R}^d)$ and $\xi \in \mathbb{R}^d \setminus \{0\}$, the defect measure of the sequence $u_k(x) = v(x)e^{ikx\cdot\xi}$ is $\nu = |v(x)|^2 dx$, which is independent of ξ. Semi-classical measure, which may be regarded as refined version of defect measure, appears to be very useful in many problems. See [4].

2.1 Basic properties of Wigner transform

Let $\psi \in L^2(\mathbb{R}^d)$. Denote

$$\rho(x,y) = \psi(x)\overline{\psi(y)}, \quad \tilde{\rho}(x,y) = \rho(x + \frac{y}{2}, x - \frac{y}{2}), \quad a.\ e.\ \text{in}\ (x,y) \in \mathbb{R}^d \times \mathbb{R}^d.$$

It is clear that

$$\tilde{\rho} \in L^2(\mathbb{R}^{2d}) \cap C_\infty(\mathbb{R}^d_y; L^1(\mathbb{R}^d_x)) \cap C_\infty(\mathbb{R}^d_x; L^1(\mathbb{R}^d_y))$$

where $C_\infty(\mathbb{R}^d_y; L^1(\mathbb{R}^d_x))$ denotes the space of L^1_x-valued functions on $\mathbb{R}^d_y$ which tend to 0 as $y \to \infty$. $C_\infty(\mathbb{R}^d_y; L^1(\mathbb{R}^d_x))$ is equipped with the natural norm. The Wigner transform, $W_\epsilon(\psi)$, of ψ depending on a small parameter $\epsilon > 0$, is defined by

$$\begin{aligned} W_\epsilon(\psi)(x,\xi) &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-iy\cdot\xi} \psi(x + \frac{\epsilon y}{2}) \overline{\psi(x - \frac{\epsilon y}{2})}\, dy \\ &= (2\pi\epsilon)^{-d} \int_{\mathbb{R}^d} e^{-iy\cdot\xi/\epsilon} \tilde{\rho}(x,y)\, dy \end{aligned} \tag{2.6}$$

Proposition 2.1 ([12]) *One has*

$$\|W_\epsilon(\psi)\|^2_{L^2} = (2\pi\epsilon)^{-d}\|\tilde\rho\|^2_{L^2} = (4\pi\epsilon)^{-d}\|\rho\|^2_{L^2} = (4\pi\epsilon)^{-d}\|\psi\|^4_{L^2}, \tag{2.7}$$

$$\int_{\mathbb{R}^d_\xi} W_\epsilon(\psi)(x,\xi)\,d\xi = \rho(x,x), \quad a.e.\ in\ x, \tag{2.8}$$

$$\int_{\mathbb{R}^d_\xi} W_\epsilon(\psi)(x,\xi)e^{-\epsilon\xi^2/2}\,d\xi = (2\pi\epsilon)^{-d/2}\int_{\mathbb{R}^d_y}\tilde\rho(x,\epsilon y)e^{-y^2/(2\epsilon)}\,dy. \tag{2.9}$$

Proof. Remark that

$$W_\epsilon(\psi)(x,\xi) = (2\pi\epsilon)^{-d}\mathcal{F}_{y\to\xi}\tilde\rho(x,\xi/\epsilon)$$

where $\mathcal{F}_{y\to\xi}$ is Fourier transform

$$\mathcal{F}_{y\to\xi}u(\xi) = \int_{\mathbb{R}^d} e^{-iy\cdot\xi}u(y)\,dy.$$

(2.7) follows from Plancherel formula for Fourier transform. (2.8) is trivial. (2.9) follows from the same calculation and the inverse Fourier transform of $\xi\to e^{-\epsilon\xi^2/2}$. ■

With a direct calculation (see [21]), one can check that

$$\begin{aligned} &W_\epsilon(\psi) * ((\pi\epsilon)^{-d}e^{-(x^2+\xi^2)/\epsilon}) \\ &= (\sqrt{2}\pi\epsilon)^{-d}\left|\int_{\mathbb{R}^d_z} e^{-(i\xi\cdot z+|x-z|^2/2)/(2\epsilon)}\psi(z)(2\pi\epsilon)^{-d/4}dz\right|^2 \geq 0 \end{aligned} \tag{2.10}$$

and

$$\int\int_{\mathbb{R}^{2d}} W_\epsilon(\psi) * ((\pi\epsilon)^{-d}e^{-(x^2+\xi^2)/\epsilon})dxd\xi = \|\psi\|^2. \tag{2.11}$$

It is useful to introduce the bilinear mapping associated with Wigner transform which is quadratic in ψ. Define

$$w_\epsilon(f,g)(x,\xi) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-iy\cdot\xi}f(x+\epsilon\frac{y}{2})\overline{g(x-\epsilon\frac{y}{2})}\,dy.$$

Clearly, $w_\epsilon(f,f) = W_\epsilon(f)$. By the properties of Fourier transform on temperate distributions, w_ϵ extends to a continuous bilinear mapping from $\mathcal{S}'(\mathbb{R}^d)\times\mathcal{S}'(\mathbb{R}^d)$ to $\mathrm{S}'(\mathbb{R}^{2d})$. One has, for f and g in L^2,

$$\int_{\mathbb{R}^d_\xi} w_\epsilon(f,g)(x,\xi)\,d\xi = f(x)\overline{g(x)} \tag{2.12}$$

$$\int_{\mathbb{R}^d_x} w_\epsilon(f,g)(x,\xi)\,dx = \frac{1}{(2\pi\epsilon)^d}\hat f(\xi/\epsilon)\bar{\hat g}(\xi/\epsilon) \tag{2.13}$$

$$\mathcal{F}_{\xi\to v}(w_\epsilon(f,g))(x,v) = f(x-\epsilon v/2)\overline{g(x+\epsilon v/2)} \tag{2.14}$$

a.e. in x, ξ and v. For $f, g \in \mathcal{S}'$, one has

$$\begin{aligned} < w_\epsilon(f,g), a > &= < a^w(x,\epsilon D)f, \overline{g} >, \quad \forall a \in \mathcal{S}(\mathbb{R}^{2d}), & (2.15)\\ w_\epsilon(f,g) &= \overline{w_\epsilon(g,f)}, \text{ in } \mathcal{S}'(\mathbb{R}^{2d}) & (2.16)\end{aligned}$$

where $< \cdot, \cdot >$ denotes the $\mathcal{S}' - \mathcal{S}$ dual product. In this work, we denote $b^w(x,\epsilon D)$ the pseudo-differential operator depending on the parameter ϵ, called a semi-classical pseudo-differential operator, with Weyl symbol b:

$$(b^w(x,\epsilon D)u)(x) = \frac{1}{(2\pi)^d}\int\int e^{i(x-y)\cdot\xi} b(\frac{x+y}{2}, \epsilon\xi)u(y)\,dyd\xi.$$

For the theory of semi-classical pseudo-differential operators, see [26].

Proposition 2.2 ([12]) *(a). For $f, g \in L^2(\mathbb{R}^d_x)$, one has*

$$\begin{aligned} \mathcal{F}_{\xi\to v} w_\epsilon(f,g)(x,v) &\in C_0(\mathbb{R}^d_v; L^1(\mathbb{R}^d_x)) & (2.17)\\ \mathcal{F}_{x\to\eta} w_\epsilon(f,g)(\eta,\xi) &\in C_0(\mathbb{R}^d_\eta; L^1(\mathbb{R}^d_\xi)) & (2.18)\end{aligned}$$

and their respective norms are uniformly bounded by $\|f\|\|g\|$.
(b). Let $a, b \in \mathcal{S}(\mathbb{R}^{2d})$. Then,

$$< w_\epsilon(f,g), ab >_{\mathcal{S}',\mathcal{S}} = < a^w(x,\epsilon D)f, b^w(x,\epsilon D)\overline{g} >_{\mathcal{S}',\mathcal{S}} + r_\epsilon \quad (2.19)$$

where $|r_\epsilon| \le \epsilon C(a,b)\|f\|\|g\|$ for some $C(a,b)$ independent of f, g and ϵ.

Proof. (a). (2.17) follows from (2.14) and

$$\sup_{v\in\mathbb{R}^d} \|f(\cdot - \epsilon v/2)\overline{g(\cdot + \epsilon v/2)}\|_{L^1(\mathbb{R}^d_x)} \le \|f\|_{L^2}\|g\|_{L^2}.$$

(2.18) can be deduced from the following relation

$$\mathcal{F}_{x\to\eta} w_\epsilon(f,g)(\eta,\xi) = \frac{1}{(2\pi\epsilon)^d}\hat{f}(\frac{\xi}{\epsilon}+\frac{\eta}{2})\overline{\hat{g}(\frac{\xi}{\epsilon}-\frac{\eta}{2})}$$

and the Parseval formula.

(b). By (2.15), $< w_\epsilon(f,g), ab > = < \overline{g}, (ab)^w(x,\epsilon D)f >$. By the calculus of semi-classical pseudo-differential operators, $(ab)^w(x,\epsilon D) = b^w(x,\epsilon D)a^w(x,\epsilon D) + \epsilon R^w(x,\epsilon D;\epsilon)$, where $R(\epsilon)$ is a bounded family in $\mathcal{S}(\mathbb{R}^{2d})$. Since $b^w(x,\epsilon D)$ is invariant by transposition, we obtain

$$< w_\epsilon(f,g), ab > = < a^w(x,\epsilon D)f, b^w(x,\epsilon D)\overline{g} > + r_\epsilon$$

where $r_\epsilon = \epsilon < \overline{g}, R^w(x,\epsilon D;\epsilon)f >$ satisfies the desired estimate, due to the uniform L^2-boundedness for semi-classical pseudo-differential operators with bounded symbol. ∎

2.2 Semi-classical measures

Let **X** denote the space

$$\mathbf{X} = \{\varphi \in C_\infty(\mathbb{R}^{2d}_{x,\xi}; \mathcal{F}_{\xi\to z}\varphi(x,z) \in L^1(\mathbb{R}^d_z; C_\infty(\mathbb{R}^d_x))\}$$

equipped with the norm

$$\|\varphi\|_{\mathbf{X}} = \int_{\mathbb{R}^d_z} \sup_x |\mathcal{F}_{\xi\to z}\varphi(x,z)|dz.$$

X is a Banach algebra and $\mathcal{S}(\mathbb{R}^{2d})$, $C_0^\infty(\mathbb{R}^{2d})$ are dense in **X**.

Let $\{u_n\}$ be a bounded sequence in $L^2(\mathbb{R}^d)$. Denote

$$\begin{aligned} U_{\epsilon,n}(x,\xi) &= W_\epsilon(u_n)(x,\xi) \\ \tilde{U}_{\epsilon,n}(x,\xi) &= U_{\epsilon,n} * (\frac{1}{(\pi\epsilon)^d} e^{-(|x|^2+|\xi|^2)/(4\epsilon)}). \end{aligned}$$

Theorem 2.3 ([11]) *There exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$, $\epsilon_k \to 0$ and a positive bounded Radon measure μ on $\mathbb{R}^d$ such that for any $a \in C_0^\infty(\mathbb{R}^{2d})$*

$$\lim_{k\to\infty} < a^w(x,\epsilon_k D)u_{n_k}, u_{n_k} >= \int\int a(x,\xi)\mu(dx,d\xi). \tag{2.20}$$

μ is called the semi-classical measure (or Wigner measure) associated with $\{u_{n_k}\}$.

Proof. Let $U_{\epsilon,n}$ be defined as above. For any $f \in \mathbf{X}$, one has

$$\int_{\mathbb{R}^{2d}} U_{\epsilon,n}(x,\xi)f(x,\xi)dxd\xi = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d}} \mathcal{F}_{\xi\to z}f(x,z)u_n(x+\frac{\epsilon z}{2})\overline{u_n(x-\frac{\epsilon z}{2})}\, dxdz.$$

It follows that

$$|\int_{\mathbb{R}^{2d}} U_{\epsilon,n}(x,\xi)f(x,\xi)dxd\xi| \le \frac{1}{(2\pi)^d}\|f\|_{\mathbf{X}}\|u_n\|^2 \le C\|f\|_{\mathbf{X}}.$$

This proves that $\{U_{\epsilon,n}\}$ is bounded in $\mathbf{X}^*$. Since **X** is separable, there exists a subsequence $\{U_{\epsilon_k,n_k}\}$ of $\{U_{\epsilon,n}\}$ and $\mu \in \mathbf{X}^*$ such that $\epsilon_k \to 0$ and $\{U_{\epsilon_k,n_k}\}$ converges $*$-weakly to μ:

$$\lim_{k\to\infty}\int U_{\epsilon_k,n_k} f dxd\xi = \int_{\mathbb{R}^{2d}} f(x,\xi)\mu(dx,d\xi), \quad \forall f \in \mathbf{X}.$$

By (2.15), for $a \in C_0^\infty(\mathbb{R}^{2d})$,

$$< a^w(x,\epsilon_k D)u_{n_k}, u_{n_k} >_{L^2}=< U_{\epsilon_k,n_k}, a >_{\mathcal{S}',\mathcal{S}}.$$

It follows that

$$< a^w(x,\epsilon_k D)u_{n_k}, u_{n_k} >_{L^2}\to \int_{\mathbb{R}^{2d}} a(x,\xi)\mu(dx,d\xi), \quad k\to\infty.$$

It remains to prove that μ is a measure. For any $a \in C_0^\infty(\mathbb{R}^{2d})$, take $\phi \in C_0^\infty$ with $0 \leq \phi \leq 1$ and $\phi a = a$. For $\eta > 0$, put $b_\eta = \phi\sqrt{a+\eta}$. Then, $b_\eta \in C_0^\infty$ and $b_\eta^2 = a + \eta\phi^2$. Making use of symbolic calculus of semi-classical pseudo-differential operators, we have

$$a^w(x, \epsilon D) = b_\eta^w(x, \epsilon D)^2 - \eta\phi^w(x, \epsilon D)^2 + O_\eta(\epsilon), \quad \text{in} \quad \mathcal{L}(L^2).$$

From this decomposition and the boundedness of $\{u_n\}$, one obtains that there exists $C > 0$ independent of η such that

$$\liminf_{\epsilon \to 0} < a^w(x, \epsilon D)u_n, u_n >\geq -C\eta.$$

Since $\eta > 0$ is arbitrary, we get

$$\int a(x,\xi)\mu(dx, d\xi) = \lim_{k\to\infty} < a^w(x, \epsilon_k D)u_{n_k}, u_{n_k} >\geq 0.$$

Therefore, μ is a positive distribution, thus a measure on $\mathbb{R}^{2d}$. See [13]. It is clear that $\mu(\mathbb{R}^{2d}) \leq \sup_k \|u_{n_k}\|^2 < \infty$. ∎

When $\{u_n\}$ is only bounded in L^2_{loc}, one can still show that there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and a positive Radon measure μ on $\mathbb{R}^{2d}$ such that

$$\lim_{k\to\infty} < a^w(x, \epsilon_k D)u_{n_k}, u_{n_k} >= \int\int a(x,\xi)\mu(dx, d\xi), \forall a \in C_0^\infty.$$

Let $\{u_\epsilon\}$ be bounded in L^2 with $\epsilon \in I$ where I is a countable set with 0 as the only accumulating point. Let $U_\epsilon = W_\epsilon(u_\epsilon)$. Let $\{\tilde{U}_\epsilon\}$ be defined as above. By extracting successively subsequences, we can assume, by an abus of notation, that

$$\begin{array}{rcl} u_\epsilon & \rightharpoonup & u \in L^2 \\ U_\epsilon & \overset{*}{\rightharpoonup} & \mu \in \mathbf{X}^* \\ \tilde{U}_\epsilon & \overset{*}{\rightharpoonup} & \tilde{\mu} \in \mathbf{X}^*. \end{array}$$

A sequence $\{v_\epsilon\} \subset L^2(\mathbb{R}^d)$ will be said compact at infinity if

$$\sup_\epsilon \int_{|x|>R} |v_\epsilon(x)|^2 \, dx \to 0, \quad \text{as} \quad R \to \infty. \tag{2.21}$$

The main properties of semi-classical measures can be resumed in the following

Theorem 2.4 ([21]) *(a). One has* $\mu = \tilde{\mu}$.
(b). $\mu \geq |u(x)|^2\delta_0(\xi)$ *and*

$$\|u\|^2 \leq \mu(\mathbb{R}^{2d}) \leq \liminf_{\epsilon\to 0} \|u_\epsilon\|^2.$$

(c). $|u_\epsilon(x)|^2$ *converges weakly in sense of measures to* $\int_{\mathbb{R}^d_\xi} d\mu(\cdot, \xi)$ *if and only if the family* $\{\epsilon^{-d}|\hat{u}(\xi/\epsilon)|^2\}$ *is compact at infinity.*

(d). The equality $\mu(\mathbb{R}^{2d}) = \lim_{\epsilon\to 0} \|u_\epsilon\|^2$ holds if and only if both $\{u_\epsilon(x)\}$ and $\{\epsilon^{-d}|\hat{u}(\xi/\epsilon)|^2\}$ are compact at infinity. In this case, $\{u_\epsilon\}$ converges strongly to u in L^2 if and only if $\mu = |u(x)|^2\delta_0(\xi)$.

(e). Let μ be a positive finite Radon measure. Let $u \in L^2$ such that $\mu \geq |u(x)|^2\delta_0(\xi)$. Then there exists a sequence $\{u_\epsilon\}$ in L^2 such that $u_\epsilon \rightharpoonup u$ in L^2, $U_\epsilon \overset{*}{\rightharpoonup} \mu$ in $\mathbf{X}^*$ and $\mu(\mathbb{R}^{2d}) = \lim_{\epsilon\to 0} \|u_\epsilon\|^2$.

This result shows that semi-classical measures contain information about the lack of compactness for a bounded sequence in L^2. For the proof of this theorem, we refer to [21].

Example Let $u \in L^2(\mathbb{R}^d)$, $\alpha > 0$. Let $u_\epsilon(x) = \epsilon^{-d\alpha/2}u(x/\epsilon^\alpha)$. Then the Wigner transform U_ϵ of u_ϵ is given by

$$U_\epsilon = (2\pi)^{-d}\epsilon^{-d\alpha/2}\mathcal{F}_{z\to\xi}(u(\epsilon^{-\alpha}x + \epsilon^{1-\alpha}\frac{z}{2})\overline{u(\epsilon^{-\alpha}x - \epsilon^{1-\alpha}\frac{z}{2})}).$$

If $\alpha < 1$, a change of scale in x variables shows that the associated semi-classical measure is $\mu = \|u\|^2\delta_0(x)\delta_0(\xi)$.

If $\alpha > 1$, the same method shows that $\mu = 0$.

If $\alpha = 1$, then $U_\epsilon(x,\xi) = \epsilon^{-d}U(x/\epsilon,\xi)$ where U is the Wigner transform of u. In this case,

$$U_\epsilon \to \delta_0(x)\int U(y,\xi)dy, \quad \text{in} \quad \mathcal{S}'.$$

Since $\int U(x,\xi)d\xi = \frac{1}{(2\pi)^d}|\hat{u}(\xi)|^2$, one obtains

$$\mu = \frac{1}{(2\pi)^d}\delta_0(x)|\hat{u}(\xi)|^2.$$

3 Uniform resolvent estimates

In order to study the semi-classical measure related to the high frequency Helmholtz equation

$$(\Delta + \epsilon^{-2}n(x)^2 + i\epsilon^{-1}\alpha_\epsilon)u_\epsilon(x) = -S_\epsilon(x), \tag{3.1}$$

the first step is to show that the solution sequence $\{u_\epsilon\}$ is bounded in L^2_{loc}. In [3], this kind of estimates was obtained from Morrey-Compananto estimate obtained in [25]. Equation (3.1) can be rewritten as a semi-classical Schrödinger equation

$$(P(h) - (E + i\kappa))u_h = S^h(x), \tag{3.2}$$

where $h = \epsilon, u_h = u_\epsilon, S^h = -\epsilon^2 S_\epsilon, \kappa = \epsilon\alpha_\epsilon$, and

$$P(h) = -h^2\Delta + V(x), \quad V(x) = E - n^2(x), \quad E > 0.$$

We want to give an estimate uniform in $h, \kappa \in]0, 1]$ for the resolvent

$$R(z, h) = (P(h) - z)^{-1}, \quad z = E + i\kappa,$$

in Besov spaces. When the refraction index $n(x)$ satisfies $n(x) = n_0 + O(\langle x\rangle^{-\epsilon})$, as $x \to \infty$, one has $V(x) = O(\langle x\rangle^{-\epsilon})$. $P(h)$ is a two-body Schrödinger operator. If $n(x) = n_1(x_1) + n_2(x_2)$ with $x = (x_1, x_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, $d_1 + d_2 = d$, such that $n_j(x_j) = n_{0,j} + O(\langle x_j\rangle^{-\epsilon})$ with $E = n_{0,1}^2 + n_{0,2}^2 > 0$, then $P(h)$ is a three-body Schrödinger operator. Here we show how this can be done by methods from semi-classical resolvent estimates. Due to the particular form of the right hand side, we need semi-classical resolvent estimates in Besov spaces.

3.1 Mourre's method depending on a parameter

We first state a parameter dependent version of Mourre's method which is an important tool in quantum scattering theory. Given two families $\{P_\epsilon\}$, $\{A_\epsilon\}$, $\epsilon \in]0, 1]$, in some Hilbert space, we shall say A_ϵ is uniformly conjugate operator of P_ϵ on an interval $I \subset \mathbb{R}$ if the following properties are satisfied:

1. Domains of P_ϵ and A_ϵ are independent of ϵ: $D(P_\epsilon) = D_1, D(A_\epsilon) = D_2$. For each ϵ, $D = D_1 \cap D_2$ is dense in D_1 in the graph norm

$$||u||_{\Gamma_\epsilon} = ||P_\epsilon u|| + ||u||.$$

2. The unitary group $e^{i\theta A_\epsilon}$, $\theta \in \mathbb{R}$ is bounded from D_1 into itself and

$$\sup_{\epsilon \in]0,1], |\theta| \leq 1} ||e^{i\theta A_\epsilon} u||_{\Gamma_\epsilon} < \infty, \quad \forall u \in D_1.$$

3. The quadratic form $i[P_\epsilon, A_\epsilon]$ defined on D is bounded from below and extends to a self-adjoint operator B_ϵ with $D(B_\epsilon) \supset D_1$ and B_ϵ is uniformly bounded from D_1 to H, *i.e.* $\exists C > 0$ such that

$$||B_\epsilon u|| \leq C||u||_{\Gamma_\epsilon}, \quad u \in D_1$$

uniformly in ϵ.

4. The form defined by $i[B_\epsilon, A_\epsilon]$ on D extend to a uniformly bounded operator from D_1 to H.

5. (uniform Mourre's estimate) There is $m_\epsilon > 0$ such that

$$E_I(P_\epsilon)i[P_\epsilon, A_\epsilon]E_I(P_\epsilon) \geq m_\epsilon E_I(P_\epsilon) \tag{3.3}$$

Remark that the usual Mourre's estimate is of the form

$$E_I(P)i[P, A]E_I(P) \geq E_I(P)(c_0 + K)E_I(P), \tag{3.4}$$

for some $c_0 > 0$ and K a compact operator. If $E \notin \sigma_p(P)$, $E_I(P)$ tends to 0 strongly. So, one can take $\delta > 0$ small enough so that $E_I(P)i[P, A]E_I(P) \geq c_1 E_I(P)$ for $I = [E-\delta, E+\delta]$ and for some $c_1 > 0$. For Mourre's method independent of parameter, see [23, 24, 16] and also [2] for more information. The following parameter-dependent version is useful in many situations.

Theorem 3.1 *Assume that A_ϵ is a uniform conjugate operator of P_ϵ on $I =]a, b[$. Let $R_\epsilon(z) = (P_\epsilon - z)^{-1}$ and $E \in I$.*

(i) *For any $s > 1/2$, and $\delta > 0$, there exists $C > 0$ such that*

$$\|\langle A_\epsilon\rangle^{-s} R_\epsilon(E \pm i\kappa)\langle A_\epsilon\rangle^{-s}\| \leq C m_\epsilon^{-1} \tag{3.5}$$

Assume in addition that $(P_\epsilon + i)^{-1}[[B_\epsilon, A_\epsilon], A_\epsilon](P_\epsilon + i)^{-1}$ extends to uniformly bounded operator on H. One has the following

(ii). *Let $c_\pm \in \mathbb{R}$ and let $\chi_\pm$ denote the characteristic functions of $]-\infty, c_-[$ and $]c_+, +\infty[$, respectively. For any $1/2 < s < 2$, there exists $C > 0$ such that*

$$\|\langle A_\epsilon\rangle^{s-1} \chi_\mp(A_\epsilon) R_\epsilon(E \pm i\kappa)\langle A_\epsilon\rangle^{-s}\| \leq C m_\epsilon^{-1}. \tag{3.6}$$

(iii). *For any $r, s \in \mathbb{R}$, with $(r)_+ + (s)_+ < 1$, there is $C > 0$*

$$\|\langle A_\epsilon\rangle^{r} \chi_\mp(A_\epsilon) R_\epsilon(E \pm i\kappa)\chi_\pm(A_\epsilon)\langle A_\epsilon\rangle^{s}\| \leq C m_\epsilon^{-1}. \tag{3.7}$$

The above estimates are uniform in $\epsilon, \kappa \in]0, 1]$ and locally uniform for $E \in I$.

(i) of Theorem 3.1 implies the point spectrum of P_ϵ is absent in I and the spectrum of P_ϵ is absolutely continuous. Theorem 3.1 can be derived by following the Mourre's original functional differential inequality method [23] and its subsequent improvement [10, 16, 33, 34]. The conditions in parts (ii) and (iii) imply that for each ϵ, P_ϵ is 2-smooth with respect to A_ϵ in sense of [16]. For the proof of Theorem 3.1, one can see [16] for fixed $\epsilon > 0$ and [33, 34] for parameter-dependent version.

3.2 Uniform resolvent estimates in Besov spaces

We want to show that Mourre's method can be used to obtain uniform resolvent estimates in Besov spaces for operators depending on a small parameter (See [35]). This idea goes back to Mourre [23, 24] and was used in [18, 37] for operators without small parameter. We use the ideas from [24, 18] in taking care of the dependence on the small parameter.

Let F be a self-adjoint operator in H. Let F_j, $j \in \mathbb{N}$, denote the spectral projector of F onto the set Ω_j, where $\Omega_j = \{\lambda \in \mathbb{R}; 2^{j-1} \leq |\lambda| < 2^j\}$ for $j \geq 1$ and $\Omega_0 = \{\lambda \in \mathbb{R}; |\lambda| < 1\}$. Introduce the abstract Besov spaces, $B_s(F)$, defined in terms of the operator F:

$$B_s(F) = \{u \in H; \sum_{k=0}^{\infty} 2^{ks}\|F_k u\| < \infty\}, \quad s \geq 0.$$

Its dual space $(B_s^F)^*$ w. r. t. the scalar product on H is a Banach space with the norm given by

$$||u||_{B_s(F)^*} = \sup_{j\in\mathbb{N}} 2^{-js}||F_j u||.$$

When F is replaced by $|x|$, one recovers the usual Besov spaces B_s and B_s^*.

Theorem 3.2 *Let P_ϵ and A_ϵ be two families of self-adjoint operators in H. Assume that A_ϵ is uniformly conjugate to P_ϵ on an interval $I =]a,b[$ and that $(P_\epsilon + i)^{-1}[[B_\epsilon, A_\epsilon], A_\epsilon](P_\epsilon + i)^{-1}$ extends to uniformly bounded operator on H. Let $E \in I$ and $s \geq \frac{1}{2}$. One has:*

$$||R_\epsilon(E \pm i\kappa)||_{\mathcal{L}(B_s(A_\epsilon), B_s(A_\epsilon)^*)} \leq Cm_\epsilon^{-1} \tag{3.8}$$

uniformly in $0 < \epsilon, \kappa < 1$. Here m_ϵ is the constant in the uniform Mourre estimate (5) and $R_\epsilon(z) = (P_\epsilon - z)^{-1}$.

Let $l^{2,\infty}$ denote the space of measurable functions $g(t)$ on $\mathbb{R}$ such that

$$||g||_{2,\infty} = \{\sum_{k\in\mathbb{Z}} |g|_k^2\}^{\frac{1}{2}}$$

where $|g|_k =$ ess sup $\{|g(t)|; k \leq t < k+1\}$, $k \in \mathbb{Z}$.

Proposition 3.3 *Let $f_1, f_2 \in l^{2,\infty}$.*

$$||f_1(A_\epsilon)R_\epsilon(E \pm i\kappa)f_2(A_\epsilon)|| \leq Cm_\epsilon^{-1}||f_1||_{2,\infty}||f_2||_{2,\infty}, \tag{3.9}$$

uniformly in $0 < \kappa < 1$.

Proof. We follow the Mourre's argument used in the proof of (III) of Theorem 1.2 in [24] (see also [18, 37]) in checking the ϵ-dependence. Let χ_n ($\chi_\pm$, resp.) denote the characteristic function of $[n, n+1[$, $n \in \mathbb{Z}$, ($[0, +\infty[,] - \infty, 0[$, resp.). Then for $u, v \in H$,

$$\begin{aligned} &|(f_1(A_\epsilon)R_\epsilon(E \pm i\kappa)f_2(A_\epsilon)u,\ v)| \\ \leq\ & \sum_{n,m\in\mathbb{Z}} |f_1|_n|f_2|_m||\chi_n(A_\epsilon)v||\ ||\chi_m(A_\epsilon)u||\ ||\chi_n(A_\epsilon)R_\epsilon(E \pm i\kappa)\chi_m(A_\epsilon)|| \\ \leq\ & 4||u||\ ||v||\ ||f_1||_{2,\infty}||f_2||_{2,\infty} \sup_{n,m\in\mathbb{Z}} ||\chi_n(A_\epsilon)R_\epsilon(E \pm i\kappa)\chi_m(A_\epsilon)||. \end{aligned}$$

It remains to prove

$$\sup_{n,m} ||\chi_n(A_\epsilon)R_\epsilon(E \pm i\kappa)\chi_m(A_\epsilon)|| \leq Cm_\epsilon^{-1} \tag{3.10}$$

uniformly in $\kappa \in]0,1]$. Note that $A_\epsilon - n$ is still a conjugate operator of P_ϵ satisfying the uniform Mourre's estimate with the same lower bound. Theorem 3.1 gives that

$$||\chi_n(A_\epsilon)R_\epsilon(E \pm i\kappa)\chi_n(A_\epsilon)|| \leq Cm_\epsilon^{-1}$$

uniformly in n and κ. Decompose $\chi_n(A_\epsilon)R_\epsilon(E+i\kappa)\chi_m(A_\epsilon)$ as

$$\begin{aligned}&\chi_n(A_\epsilon)R_\epsilon(E+i\kappa)\chi_m(A_\epsilon)\\&= \chi_n(A_\epsilon)\{\chi_+(A_\epsilon-m)R(E+i\kappa,h)+\chi_-(A_\epsilon-m)R_\epsilon(E-i\kappa)\\&\quad +2i\kappa\chi_-(A_\epsilon-m)R_\epsilon(E-i\kappa)R_\epsilon(E+i\kappa)\}\chi_m(A_\epsilon)\end{aligned}$$

The first two terms can be bounded by Cm_ϵ^{-1} according to (3.6). For the third term, remark that

$$\begin{aligned}&2\kappa\|\chi_n(A_\epsilon)R_\epsilon(E-i\kappa)R_\epsilon(E+i\kappa)\chi_m(A_\epsilon)\|\\&\le 4\|\chi_n(A_\epsilon)R_\epsilon(E+i\kappa)\chi_n(A_\epsilon)\|^{\frac12}\|\chi_m(A_\epsilon)R_\epsilon(E+i\kappa)\chi_m(A_\epsilon)\|^{\frac12}\\&\le Cm_\epsilon^{-1}\end{aligned}$$

uniformly in n, m and κ. (3.10) is proved. ■

Proof of Theorem 3.2. Let $f\in B_s(A_\epsilon)$. By Proposition 3.3, one has for $s\ge\frac12$

$$\begin{aligned}&2^{-js}\|F_jR(E\pm i\kappa)f\|\\&\le \sum_{k=0}^{\infty}2^{-js}\|F_jR(E\pm i\kappa)F_k\|\|F_kf\|\\&\le Cm_\epsilon^{-1}\sum_{k=0}^{\infty}2^{-j(s-\frac12)}2^{k/2}\|F_kf\|\le Cm_\epsilon^{-1}\|f\|_{B_s(A_\epsilon)},\end{aligned}$$

uniformly in ϵ,κ and j. This proves Theorem 3.2. ■

3.3 Applications to Schrödinger operators

3.3.1 Semi-classical resolvent estimates

An interesting application of the above abstract results is the resolvent estimate of semi-classical Schrödinger operators $P(h)=-h^2\Delta+V(x)$ near a non-trapping energy. Recall that the energy $E>0$ is called non-trapping for the classical Hamiltonian $p(x,\xi)=|\xi|^2+V(x)$ if

$$\lim_{|t|\to\infty}|x(t;y,\eta)|=\infty,\quad \forall\ (y,\eta)\in p^{-1}(E). \tag{3.11}$$

Here, $(x(t;y,\eta),\xi(t;y,\eta))$ is the solution of the classical Hamiltonian system associated with $p(x,\xi)$:

$$\left\{\begin{array}{lll}\frac{\partial x}{\partial t} &=& \partial_\xi p(x,\xi),\quad x(0;y,\eta)=y,\\ \frac{\partial \xi}{\partial t} &=& -\partial_x p(x,\xi),\quad \xi(0;y,\eta)=\eta.\end{array}\right. \tag{3.12}$$

The set of non-trapping energy is open in $\mathbb{R}_+$.

The Mourre's method depending on a parameter can also be applied generalized N-body Schrödinger operators. For the Helmholtz equation, this allows to include the case where the refraction index $n(x)$ is a sum of functions of the form:

$$n(x) = \sum_{j=1}^{M} n_j(x_j), \quad n_j(x_j) = n_{j,0} + o(1), \quad |x_j| \to \infty, \tag{3.13}$$

where $n_{j,0} \geq 0$ with $E = \sum_j n_{j,0}^2 > 0$, and $x_j \in \mathbb{R}^{d_j} \subseteq \mathbb{R}^d$.

Let us recall some notation for generalized N-body Schrödinger operators. Let $\mathbf{X}$ be a d-dimensional Euclidean space equipped with a quadratic form $q(\cdot)$. To simplify notation, we assume that $q(\cdot)$ is the canonical form on $\mathbf{X} = \mathbb{R}^d$. Let $\mathcal{A}$ denote the set of all cluster decompositions of an N-body system labelled by $\{1, 2, \cdots, N\}$. To each $a \in \mathcal{A}$, it is assigned a subspace $\mathbf{X}_a$ of $\mathbf{X}$ with $\mathbf{X}_{a_{min}} = \mathbf{X}$ for some $a_{min} \in \mathcal{A}$ and $\cap_{a\in\mathcal{A}} \mathbf{X}_a = \{0\}$. Let $\mathcal{A}$ be partially ordered by

$$a \subset b \text{ iff } \mathbf{X}_b \subset \mathbf{X}_a.$$

Assume also that for $a, b \in \mathcal{A}$, the union of a and b, $a \cup b$, belongs to $\mathcal{A}$, and is defined so that

$$\mathbf{X}_a \cap \mathbf{X}_b = \mathbf{X}_{a\cup b}.$$

For the definition of $a \cup b$ in physical N-body Schrödinger operators, we refer to [34]. For each a, we denote $\mathbf{X}^a$ the orthogonal complement of $\mathbf{X}_a$ in $\mathbf{X}$. We write the corresponding orthogonal decomposition of coordinates x as:

$$x = x^a + x_a.$$

With these notation, the N-body Schrödinger operators we are interested in are of the form:

$$P(h) = -h^2\Delta + \sum_{a\in\mathcal{A}} V_a(x^a), \tag{3.14}$$

where $h > 0$ is a small parameter, Δ is the Laplacian on $(\mathbf{X}, q(\cdot))$. We assume that V_a satisfies

$$|\partial_y^\alpha V_a(y)| \leq C_\alpha r(y)\langle y\rangle^{-|\alpha|}, \quad y \in \mathbf{X}^a, \quad \forall\ \alpha \in \mathbb{N}^{d_a}. \tag{3.15}$$

Here $r(y) \to 0$ as $y \to \infty$.

For each $a \in \mathcal{A}$, we denote $\#a$ the number of clusters in a, $P^a(h)$ the cluster Hamiltonian:

$$P^a(h) = -h^2\Delta^a + \sum_{b\subseteq a} V_b(x^b),$$

where Δ^a is the Laplacian in x^a-variables. Put

$$I_a(x) = \sum_{b\not\subseteq a} V_b(x^b), \quad P_a(h) = P^a(h) - h^2\Delta_a,$$

where Δ_a is the Laplacian in x_a-variables. Then one has: $P(h) = P_a(h) + I_a(x)$ for any cluster decomposition a. Let p^a denote the semi-classical symbol of $P^a(h)$. Assume that

$$\forall\ a, \quad p^a \text{ is non-trapping at the energy } E. \tag{3.16}$$

Note that when $\#a = 1$, $p^a = p$ is the classical hamiltonian of $P(h)$.

Under the assumptions (3.15) and (3.16), one can construct a uniform conjugate operator, $F(h)$, of $P(h)$ near E in the form

$$F(h) = h(x \cdot D + D \cdot x)/2 + r^w(x, hD)$$

where $r^w(x, hD)$ is a self-adjoint bounded smoothing semi-classical pseudo-differential operator and one has

$$i\chi(P(h))[P(h), F(h)]\chi(P(h)) \geq c_0 h \chi(P(h))^2, \quad h \in]0, 1], \tag{3.17}$$

where $c_0 > 0$ is independent of h and χ is a smooth real function on $\mathbb{R}$ supported sufficiently near E. The following results are proved in [33, 34].

Theorem 3.4 (i). *Assume the conditions (3.15). The resolvent estimate*

$$\|\langle x\rangle^{-s} R(\lambda \pm i\kappa, h)\langle x\rangle^{-s}\| \leq Ch^{-1}, \tag{3.18}$$

holds for $\lambda \in [E - \delta, E + \delta]$ uniformly in h, κ if and only if E is non-trapping for all p^a and $s > 1/2$.

(ii). *Assume (3.15 and (3.16). Let χ_+ and χ_- denote the characteristic functions for $[c_+, +\infty[$ and $]-\infty, c_-]$, $c_\pm \in \mathbb{R}$. Then for any $s > 1/2$, there exists $C > 0$ such that*

$$\|\langle F(h)\rangle^{s-l}\chi_\mp(F(h))R(E \pm i\kappa, h)\langle x\rangle^{-s}\| \leq Ch^{-1} \tag{3.19}$$

For any any $s, r \in \mathbb{R}$, one has

$$\|\langle F(h)\rangle^s \chi_\mp(F(h))R(E \pm i\kappa, h)\chi_\pm(F(h))\langle F(h)\rangle^r\| \leq Ch^{-1}. \tag{3.20}$$

For two-body Schrödinger operators, under the non-trapping condition, the semi-classical resolvent estimate (3.18) was firstly proved in [28] by method of global parametrix. The necessity of non-trapping condition to obtain (3.18) was proved in [31]. Its proof based on Mourre's method was given in [10]. Since then, there are many extensions and new proofs, among which we mention an interesting proof using method of semi-classical measures (see [5, 15]). The sufficient part in (3.18) for $N = 3$ is due to C. Gérard [8]. The general case $N \geq 3$ is proved in [33]. In [33], the necessity of the non-trapping condition (3.16) is also proved. Remark that (3.19) and (3.20) are some kind of microlocal resolvent estimates. For microlocal resolvent estimates, see [14, 16, 19, 9, 34] for the case $h > 0$ is fixed. For semi-classical resolvent estimates with microlocalization, see [31, 32] in two-body case and also [34] in general N-body case.

One can apply Theorems 3.2 and 3.4 to $P(h)$ to obtain Besov space semi-classical resolvent estimates.

Theorem 3.5 *(Besov space estimate) Let $s \geq \frac{1}{2}$. Under the assumptions (3.15) and (3.16), one has:*

$$\|R(E \pm i\kappa, h)\|_{\mathcal{L}(B_s, B_s^*)} \leq Ch^{-1} \tag{3.21}$$

uniformly in $0 < h, \kappa < 1$.

Proof. Let $F(h)$ denote the uniform conjugate operator of $P(h)$ satisfying (3.17). Theorem 3.2 is true with A_ϵ replaced by $F(h)$. Let $\chi \in C_0^\infty(\mathbb{R})$ with $\chi(t) = 1$ for t near E. $(1 - \chi(P(h))^2)R(E \pm i\kappa, h)$ is uniformly bounded in $\mathcal{L}(L^2, L^2)$, therefore also in $\mathcal{L}(B_s, B_s^*)$. Note that $F(h)$ is a semi-classical pseudo-differential operator with Weyl symbol $x \cdot \xi + r(x, \xi)$ where r is a bounded symbol (cf [33]). We can show that for $s \geq 0$,

$$\|\langle F(h)\rangle^s \chi(P(h))\langle x\rangle^{-s}\| \leq C \tag{3.22}$$

uniformly in h. An argument of interpolation (cf [1, 13]) gives then

$$\|\chi(P(h))\|_{\mathcal{L}(B_s, B_s(F(h)))} \leq C$$

uniformly in h. By duality, the same is true for $\chi(P(h))$ as operator from $(B_s^F)^*$ to B_s^*. It follows that

$$\|\chi(P(h))^2 R(E \pm i\kappa, h)\|_{\mathcal{L}(B_s, B_s^*)} \leq Ch^{-1},$$

which completes the proof of (3.21). ■

The regularity on potentials is only needed to make use of theory of pseudo-differential operators in the construction of uniform conjugate operator. As one can see from the next subsection that if we make the virial assumption

$$2E - 2V(x) - x\nabla V(x) \geq c_0 > 0, \quad \forall\ x \tag{3.23}$$

which implies (3.16), the condition $(x\nabla)^j V(x) \in L^\infty$ for $0 \leq j \leq 3$ is sufficient to have the result of Theorems 3.4 and 3.5.

3.3.2 Potentials depending on a parameter

Consider the Schrödinger operator $P_\epsilon = -\Delta + V_\epsilon(x)$ on $\mathbb{R}^d$ with potential depending on a parameter $\epsilon \in]0, 1]$. Assume that $V_\epsilon(x)$ is uniformly bounded on $\mathbb{R}^d$ and the multiplication operators $(x \cdot \nabla_x)^j V_\epsilon$, $j = 1, 2, 3$, are $-\Delta$-bounded uniformly in ϵ. Let $E > 0$. Assume further that there exist $c_0 > 0$ such that

$$2E - 2V_\epsilon(x) - x \cdot \nabla V_\epsilon(x) \geq c_0, \quad x \in \mathbb{R}^d, \tag{3.24}$$

uniformly in ϵ.

For $\mu \in \mathbb{R}$, we denote by $S_\pm(\mu)$ the class of μ-dependent bounded symbols $a_\pm$ on $\mathbb{R}^{2d}$ satisfying

$$\text{supp} a_\pm \subset \{(x, \xi); \pm x \cdot \xi \geq \pm\mu\langle x\rangle\}, \tag{3.25}$$

$$a_\pm \in C^\infty(\mathbb{R}^{2d}),\ |\partial_x^\alpha \partial_\xi^\beta a_\pm(x,\xi)| \le C_{\alpha\beta}\langle x\rangle^{-|\alpha|}\langle \xi\rangle^{-|\beta|}.$$

Denote by $a(x,D)$ the pseudo-differential operator with symbol a defined by

$$a(x,D)u(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d}} e^{i(x-y)\cdot\xi}a(x,\xi)u(y)dyd\xi, \quad u \in \mathcal{S}(\mathbb{R}^d)$$

Theorem 3.6 *Let $R_\epsilon(z) = (P_\epsilon - z)^{-1}$. Under the above assumptions, there exists $\mu_0 > 0$ such that the following estimates hold uniformly in $\epsilon, \kappa \in]0,1]$.*
(i) *For $s \ge 1/2$, one has*

$$||R_\epsilon(E \pm i\kappa)||_{\mathcal{L}(B_s,B_s^*)} \le C. \tag{3.26}$$

(ii). *Let $1/2 < s < 2$ and $b_\pm \in S_\pm(\mp\mu_0)$, there exists $C > 0$ such that*

$$||\langle x\rangle^{s-l}b_\mp(x,D)R_\epsilon(E\pm i\kappa)\langle x\rangle^{-s}|| \le C \tag{3.27}$$

(iii). *For $s \in \mathbb{R}$, put $s_+ = \max\{s,0\}$. Then, for $s,r \in \mathbb{R}$ with $s_+ + r_+ < 1$ and $b_\pm \in S_\pm(\mp\mu_\pm)$ with $\mu_- < \mu_+$, one has*

$$||\langle x\rangle^s b_\mp(x,D)R_\epsilon(E\pm i\kappa)b_\pm(x,D)\langle x\rangle^r|| \le C. \tag{3.28}$$

Proof. Let $F_0 = (x\cdot D_x + D_x\cdot x)/2$. Under the condition (3.24), one can see that F_0 is a uniform conjugate operator on the interval $I = [E-\delta, E+\delta]$ for $\delta > 0$ small enough and

$$E_I(P_\epsilon)i[P_\epsilon, F_0]E_I(P_\epsilon) \ge c_0/2E_I(P_\epsilon).$$

for all ϵ.

By Theorem 3.2, one has

$$||R_\epsilon(E\pm i\kappa)||_{\mathcal{L}(B_s(F_0),B_s(F_0)^*)} \le C, \quad s \ge 1/2. \tag{3.29}$$

(3.26) follows then from the argument of Theorem 3.5.

The proof of (ii) and (iii) follows the method of Theorem 2.1 of [34]. Consider the operator

$$F_\nu = F_0 + \mu s(x), \qquad s(x) = \frac{x^2}{\langle x\rangle}.$$

Since $i[-\Delta, \mu s(x)] = \mu(\nabla s(x)\cdot D_x + D_x\cdot\nabla s(x)) \ge -|\mu|(|\nabla s(x)|^2 - \Delta)$, one can show that there exists c_1, $\mu_1 > 0$ such that

$$E_I(P_\epsilon)i[P_\epsilon, F_\mu]E_I(P_\epsilon) \ge c_1 E_I(P_\epsilon).$$

uniformly in ϵ and μ with $|\mu| \le \mu_1$. Therefore, Theorem 3.2 holds with A_ϵ replaced by F_μ. By methods of [34], one can replace $\chi_\pm(F_\mu)$, $\langle F_\mu\rangle$ by $b_\pm(x,D)$ and $\langle x\rangle$, respectively, with a uniform control over parameters. ■

In the case $\epsilon = 1$, Theorem 3.6 for $b_\pm \in S_\pm(\mu)$, $\forall \mu > 0$, was due to A. Jensen [16]. Remark that in applications, in order to construct a partition of unity in phase space $\mathbb{R}^{2d}_{x,\xi}$, it is important to have Theorem 3.6 (ii) and (iii) for $b_\pm \in S_\pm(\mp\mu_0)$ for some $\mu_0 > 0$. See the proof of Theorem 4.1 below.

3.3.3 Remark on Morrey-Campanato estimates

In [25], the authors mentioned that it is interesting in itself to study Morrey-Campanato estimates for Schödinger operators. We indicate here how this kind of estimates can be deduced from Besov space estimates.

Denote the Morrey-Companato norm

$$|||u|||^2 = \sup_{R>0} \frac{1}{R} \int_{|x|<R} |u|^2 dx$$

and $N(f)$ the dual norm

$$N(f) = \sum_{j\in\mathbb{Z}} \left(2^{j+1} \int_{C(j)} |f|^2 dx \right)^{\frac{1}{2}}$$

where $C_j = \{x \in \mathbb{R}^d; 2^j \le |x| \le 2^{j+1}\}$.

Proposition 3.7 *Let $P_\epsilon = -\Delta + V_\epsilon(x)$ be a Schrödinger operator on $\mathbb{R}^d$, $d \ge 3$. Under the assumptions of Theorem 3.6, one has*

$$|||(P_\epsilon - (E \pm i\kappa))^{-1} u||| \le CN(u), \tag{3.30}$$

for all $u \in L^2_{\text{loc}}$ with $N(u) < \infty$, uniformly in ϵ, κ.

Proof. Recall the Hardy inequality for $d \ge 3$:

$$\||x|^{-1} f\|^2_{L^2(\mathbb{R}^d)} \le \frac{4}{(d-2)^2} \|\nabla f\|^2_{L^2(\mathbb{R}^d)}, \quad f \in C_0^\infty(\mathbb{R}^d).$$

It follows that $|x|^{-1}(1-\Delta)^{-1/2}$ and $(1-\Delta)^{-1/2}|x|^{-1}$ are bounded as operators on L^2. By a complex interpolation, we obtain that for any $0 \le s \le 1$,

$$|x|^{-s}(1-\Delta)^{-s/2} \quad \text{and} \quad (1-\Delta)^{-s/2}|x|^{-s} \in \mathcal{L}(L^2). \tag{3.31}$$

Let χ_1 be a cut-off on $\mathbb{R}^d$ with $\chi_1(x) = 1$ for $|x| \le 1$, and 0 for $|x| \ge 2$. Set $\chi_2 = 1 - \chi_1$. On supp χ_2, $B_{\frac{1}{2}}$ (resp., $B^*_{\frac{1}{2}}$) norm is equivalent with $N(\cdot)$ (resp., $|||\cdot|||$). Since $(-\Delta+1)^{-\frac{1}{2}}$ is bounded from $B_{\frac{1}{2}}$ to $B_{\frac{1}{2}}$ and from $B^*_{\frac{1}{2}}$ to $B^*_{\frac{1}{2}}$, splitting u as $u = \chi_1 u + \chi_2 u$ and applying (3.31) to $\chi_1 u$ with appropriate $\frac{1}{2} \le s \le 1$, one has

$$\|(-\Delta+1)^{-\frac{1}{2}} u\|_{B_{\frac{1}{2}}} \le CN(u) \tag{3.32}$$

$$|||(-\Delta+1)^{-\frac{1}{2}} u||| \le C\|u\|_{B^*_{\frac{1}{2}}} \tag{3.33}$$

Let χ be the same cut-off function around E with support sufficiently near E. Then for all $u \in C_0^\infty(\mathbb{R}^d)$, one has

$$|||(1-\chi(P_\epsilon)^2)(P_\epsilon - (E \pm i\kappa))^{-1} u||| \le C\|(-\Delta+1)^{-\frac{1}{2}} u\|,$$

uniformly in ϵ, κ. By (3.32),

$$|||(1-\chi(P_\epsilon)^2)(P_\epsilon-(E\pm i\kappa))^{-1}u||| \le C'N(u).$$

On the other hand, by (3.26), (3.32), (3.33) and the argument used above, one has

$$\begin{aligned} |||\chi(P_\epsilon)^2(P_\epsilon-(E\pm i\kappa))^{-1}u||| &\le C||(P_\epsilon-(E\pm i\kappa))^{-1}\chi(P_\epsilon)u||_{B^*_{\frac{1}{2}}} \\ &\le C_1||\chi(P_\epsilon)u||_{B_{\frac{1}{2}}} \le C_2N(u) \end{aligned}$$

for all $u \in C_0^\infty$, uniformly in ϵ, κ. Combining the above two estimates, we obtain the desired estimate for $u \in C_0^\infty$. An argument of density completes the proof of Proposition 3.7. ■

Note that condition (3.24) is satisfied for $V_\epsilon = E - n^2(\epsilon x)$ if n^2 verifies

$$2\sum_{j\in\mathbb{Z}} \sup_{2^j<|x|\le 2^{j+1}} \frac{(x\cdot\nabla_x n^2(x))_-}{n^2(x)} < 1$$

and $n^2(x) \ge n_0 > 0$. Thus, Proposition 3.7 can be regarded as an alternative approach to prove the Morrey-Campanato estimate (see [25]).

4 The high frequency Helmholtz equation

Now we study the Helmholtz equation with a source term concentrated near one point

$$(\Delta+\epsilon^{-2}n(x)^2+i\epsilon^{-1}\alpha_\epsilon)u_\epsilon(x) = -S_\epsilon(x) \tag{4.1}$$

in $\mathbb{R}^d$, $d \ge 1$. Here $\epsilon > 0$ is regarded as a small parameter, $\alpha_\epsilon > 0$ is a regularizing constant and $\alpha_\epsilon \to \alpha \ge 0$ when $\epsilon \to 0$, $n(x)$ is the refraction index and

$$S_\epsilon(x) = \epsilon^{-\frac{3+d}{2}}S(\frac{x}{\epsilon}). \tag{4.2}$$

We assume that $S(x)$ is smooth and decays sufficiently rapidly at the infinity. For $n(x)^2$, we assume that there exists $E > 0$ such that $V(x) = E - n(x)^2$ satisfies

$$(x\cdot\nabla)^jV(x) \text{ is bounded on } \mathbb{R}^d \text{ for } 0\le j\le 3. \tag{4.3}$$

Assume also that there exists $c_0 > 0$ such that

$$2E-2V(x)-x\cdot\nabla V(x) \ge c_0, \quad \forall \quad x\in\mathbb{R}^d. \tag{4.4}$$

It is convenient to write (4.1) in the form

$$(-h^2\Delta+V(x)-E-i\kappa)u_h(x) = h^{\frac{1-d}{2}}S(\frac{x}{h}) \tag{4.5}$$

where

$$h = \epsilon \to 0, \quad \kappa = \kappa(h) = h\alpha_h.$$

Put $w_\epsilon(x) = \epsilon^{d/2}u_\epsilon(\epsilon x)$. Then w_ϵ is the solution of

$$(-\Delta + V(\epsilon x) - E - i\kappa)w_\epsilon(x) = \epsilon^{1/2}S(x) \tag{4.6}$$

Theorem 4.1 *Assume (4.3) and (4.4).*

(a). Let $S \in B_{\frac{1}{2}}$. One has $w_\epsilon \in B^*_{\frac{1}{2}}$ and

$$\|w_\epsilon\|_{B^*_{\frac{1}{2}}} \leq C\epsilon^{1/2}\|S\|_{B_{\frac{1}{2}}} \tag{4.7}$$

(b). Assume that $\langle x\rangle^r S \in L^2$ for some $r > 3/2$ and that

$$V(x) \text{ is smooth with bounded derivatives and } \alpha_\epsilon \geq \epsilon^\gamma \text{ for some } \gamma \in \mathbb{R}_+. \tag{4.8}$$

Suppose that $E - V(0) > 0$. Then $\epsilon^{-1/2}w_\epsilon$ converges $*$-weakly to w_0 in $B^*_{\frac{1}{2}}$ where w_0 is the outgoing solution of the equation

$$(-\Delta + V(0) - E - i0)w_0(x) = S(x) \tag{4.9}$$

Remark *The $*$-weak convergence of $\epsilon^{-1/2}w_\epsilon$ to w_0 is conjectured in [3]. It is proved in [6] for $d \geq 3$ under an assumption on the dimension of self intersection set near zero of the Hamilton flow. Under some additional decay assumptions, the results of [35], when simplified to the case of point source, shows that there exists a subsequence of $\{\epsilon^{-1/2}w_\epsilon\}$ converging $*$-weakly to w_0 in $B^*_{\frac{1}{2}}$. The approach presented here is new.*

Proof. (4.3) and (4.4) show that the conditions of Theorem 3.6 are satisfied with $V_\epsilon(x) = V(\epsilon x)$. So, Theorem 3.6 gives

$$\|(P_\epsilon - E - i\kappa)^{-1}\|_{\mathcal{L}(B_s, B_s^*)} \leq C \tag{4.10}$$

uniformly in ϵ and κ. (4.7) follows.

To prove (b), put $R_\epsilon(E + i\kappa) = (-\Delta + V(\epsilon x) - E - i\kappa)^{-1}$ and $R_0(E + i\kappa) = (-\Delta + V(0) - E - i\kappa)^{-1}$. Write $\epsilon^{-1/2}w_\epsilon - R_0(E + i\kappa)S$ as

$$v_\epsilon = R_\epsilon(E + i\kappa)(V(0) - V(\epsilon x))R_0(E + i\kappa)S$$

Let $\rho \in C_0^\infty(]E - 2, E + 2[)$ with $\rho(\lambda) = 1$ on $[E - 1, E + 1]$ and

$$r_\epsilon = R_\epsilon(E + i\kappa)(V(0) - V(\epsilon x))R_0(E + i\kappa)\rho(-\Delta)S$$

Then it is easy to check that

$$\lim_{\epsilon \to 0} < v_\epsilon - r_\epsilon, f >= 0 \tag{4.11}$$

for any $f \in C_0^\infty(\mathbb{R}^d)$. Let $\chi_\pm \in C^\infty(\mathbb{R})$ such that $\chi_+ + \chi_- = 1$ on $\mathbb{R}$, $\chi_+ = 1$ on $[\frac{1}{2}, \infty[$, and 0 on $]\infty, -\frac{1}{2}]$. Let

$$b_\pm(x,\xi) = \chi_\pm\left(\frac{x\cdot\xi}{\mu_0\langle x\rangle}\right)\rho_1(\xi/R),$$

where $\mu_0 > 0$ is small and $\rho_1 \in C_0^\infty(\mathbb{R})$ and is equal to 1 near 0. Then $b_\pm \in S_\pm(\mp\mu_0)$ and we can apply Theorem 3.6. Note that $b_+(x,D) + b_-(x,D) = \rho_1(-\Delta/R)$ so for $R > 1$ large enough,

$$(b_+(x,D) + b_-(x,D))\rho(-\Delta)R_0(E+i\kappa) = \rho(-\Delta)R_0(E+i\kappa)$$

Inserting this decomposition into r_ϵ and applying Theorem 3.6, we obtain for $\frac{1}{2} < s < s' < 1$

$$\begin{aligned}|< r_\epsilon, f >| \;\le\; & C\delta(\epsilon)(\|\langle x\rangle^{s'} b_-(x,D)R_0(E+i\kappa)S\|\|\langle x\rangle^s f\| \\ & + \|\langle x\rangle^s S\|\|\langle x\rangle^{s'} b_+(x,D)R_\epsilon(E-i\kappa)f\|) \\ & + |< \rho(-\Delta)R_0(E+i\kappa)S, [b_+(x,D), V(\epsilon x)]R_\epsilon(E-i\kappa)f >|\end{aligned} \tag{4.12}$$

where C' is independent of ϵ and κ, and $\delta(\epsilon) = \|\langle x\rangle^{s-s'}(V(0) - V(\epsilon x))\|_\infty \to 0$ as $\epsilon \to 0$ for $s' > s$. By Theorem 3.6,

$$\|\langle x\rangle^{s'} b_-(x,D)R_0(E+i\kappa)S\| \text{ and } \|\langle x\rangle^{s'} b_+(x,D)R_\epsilon(E-i\kappa)f\|$$

are uniformly bounded respectively by $C\|\langle x\rangle^{1+s'}S\|$ and $C\|\langle x\rangle^{1+s'}f\|$. To estimate the last term in (4.12), we remark that by symbolic calculus of pseudo-differential operators, $[b_+(x,D), V(\epsilon x)]$ can be decomposed as

$$[b_+(x,D), V(\epsilon x)] = \epsilon\rho_2(D)r_1(x,D;\epsilon) + \epsilon r_2(x,D,\epsilon) + O(\epsilon^N) \tag{4.13}$$

where $N > 2\gamma + 2$, $\rho_2(\xi)\rho_0(\xi^2) = 0$, $r_1(x,\xi;\epsilon)$ is a family of bounded symbols and $r_2(\cdot,\cdot;\epsilon)$ is bounded in $S_+(-\mu_0) \cap S_-(\mu_0)$. By (4.8), since $\kappa = \epsilon\alpha_\epsilon \ge \epsilon^{\gamma+1}$,

$$|< \rho(-\Delta)R_0(E+i\kappa)S, O(\epsilon^N)R_\epsilon(E-i\kappa)f >| = O(\epsilon^{N-2-2\gamma}) \to 0,$$

as $\epsilon \to 0$. Due to the support properties of ρ and ρ_2, one has

$$< \rho(-\Delta)R_0(E+i\kappa)S, \rho_2(D)r_1(x,D;\epsilon)R_\epsilon(E-i\kappa)f >= 0.$$

By Theorem 3.6,

$$|< \rho(-\Delta)R_0(E+i\kappa)S, \epsilon r_2(x,D;\epsilon)R_\epsilon(E-i\kappa)f >| \le C\epsilon\|\langle x\rangle S\|\|\langle x\rangle f\|$$

This proves that $\lim_{\epsilon\to 0} < v_\epsilon, f >= 0$ for $f \in C_0^\infty$. Since v_ϵ is uniformly bounded in $B_{\frac{1}{2}}$, by an argument of density, one deduces that v_ϵ converges $*$-weakly to 0 in $B^*_{\frac{1}{2}}$. Noticing that $R_0(E+i\kappa)S$ converges to w_0 in $B^*_{\frac{1}{2}}$, (b) is proved. ■

Note that the assumption (4.8) is only used to study the term related to $[b_+(x,D), V(\epsilon x)]$ and the decay of $V(x)$ is not needed.

Corollary 4.2 *Let $S \in B_{\frac{1}{2}}$. Then $u_\epsilon \in B^*_{\frac{1}{2}}$ and there exists $C > 0$ such that*

$$\|u_\epsilon\|_{B^*_{\frac{1}{2}}} \le C\|S\|_{B_{\frac{1}{2}}} \tag{4.14}$$

uniformly in ϵ.

Proof. By Theorem 4.1,

$$\|w_\epsilon\|_{B^*_{\frac{1}{2}}} \le C\epsilon^{\frac{1}{2}}\|S\|_{B_{\frac{1}{2}}}.$$

For $0 < \epsilon < 1$, one has

$$\begin{aligned}
\|w_\epsilon\|_{B^*_{\frac{1}{2}}} &= \sup_{R>1} \frac{1}{R^{\frac{1}{2}}}\Big(\int_{|x|_\epsilon<R} |u_\epsilon(\epsilon x)|^2\epsilon^d dx\Big)^{\frac{1}{2}} \\
&= \epsilon^{\frac{1}{2}} \sup_{R>1} \frac{1}{(\epsilon R)^{\frac{1}{2}}}\Big(\int_{|x|<\epsilon R} |u_\epsilon(x)|^2 dx\Big)^{\frac{1}{2}} \\
&= \epsilon^{\frac{1}{2}} \sup_{R'>\epsilon} \frac{1}{(\epsilon R')^{\frac{1}{2}}}\Big(\int_{|x|<\epsilon R'} |u_\epsilon(x)|^2 dx\Big)^{\frac{1}{2}} \\
&\ge \epsilon^{\frac{1}{2}} \sup_{R'>1} \frac{1}{R'^{\frac{1}{2}}}\Big(\int_{|x|<R'} |u_\epsilon(x)|^2 dx\Big)^{\frac{1}{2}} = \epsilon^{\frac{1}{2}}\|u_\epsilon\|_{B^*_{\frac{1}{2}}}.
\end{aligned}$$

(4.14) follows. ■

$\{u_\epsilon\}$ is bounded in L^2_{loc}. By the remark following Theorem 2.3, there exists a semi-classical measure associated with a subsequence $\{u_{\epsilon_k}\}$. To give more information on the semi-classical measure, we introduce the space X_λ (see [3], [7] and [21]) as the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^{2d})$ under the norm:

$$\|\varphi\|_{X_\lambda} : \int_{\mathbb{R}^d} \sup_x \{\langle x, y\rangle^\lambda |(\mathcal{F}_{\xi\to y}\varphi)(x,y)|\}\, dy,$$

where we denote $\langle x, y\rangle = (1 + |x|^2 + |y|^2)^{\frac{1}{2}}$ and $(\mathcal{F}_{\xi\to y}\varphi)(x,y)$ the partial Fourier transform of $\varphi(x,\xi)$ with respect to ξ. The space X_λ is a Banach space with dual X^*_λ.

Applying Corollary 4.2, we immediately get the following

Proposition 4.3 *The family of Wigner transforms f_ϵ of u_ϵ is bounded in X^*_λ, for any $\lambda > 1$, and $\{f_\epsilon\}$ admits a subsequence converging $*$-weakly in X^*_λ to some nonnegative, locally bounded measure f which satisfies*

$$\sup_{R>1} \frac{1}{R}\int_{|x|\le R}\int_{\xi\in\mathbb{R}^d} f(x,\xi)\, dx\, d\xi \le C\|S\|^2_{B_{\frac{1}{2}}}. \tag{4.15}$$

Let $f_\epsilon(x,\xi)$ denote the Wigner transform of $u_\epsilon(x)$. An elementary calculation shows that

$$\alpha_\epsilon f_\epsilon + \xi\cdot\nabla_x f_\epsilon - \Theta_\epsilon(f_\epsilon) = Q_\epsilon \tag{4.16}$$

where Θ_ϵ is defined by

$$\Theta_\epsilon(f_\epsilon)(x,\xi) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{2d}_{y,\eta}} e^{-iy\cdot(\xi-\eta)}\frac{1}{2\epsilon i}(V(x+\frac{\epsilon y}{2}) - V(x-\frac{\epsilon y}{2}))f_\epsilon(x,\eta)dyd\eta$$

and

$$Q_\epsilon(x,\xi) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-iy\cdot\xi}\frac{\epsilon}{2i}[S_\epsilon(x+\frac{\epsilon y}{2})\overline{u_\epsilon(x-\frac{\epsilon y}{2})} - \overline{S_\epsilon(x-\frac{\epsilon y}{2})}u_\epsilon(x+\frac{\epsilon y}{2})]dy$$

By proposition 4.3, one has for some sequence $\epsilon_k \to 0$

$$\alpha_{\epsilon_k} f_{\epsilon_k} + \xi\cdot\nabla_x f_{\epsilon_k} - \Theta_{\epsilon_k}(f_{\epsilon_k}) \to \alpha f + \xi\cdot\nabla_x f - \nabla V(x)\cdot\nabla_\xi f \qquad (4.17)$$

in $\mathcal{D}'$, where f is given by Proposition 4.3. A subtle task is to compute the limit of the source term Q_ϵ which depends on u_ϵ.

Theorem 4.4 *Let $\alpha_\epsilon \to \alpha \geq 0$. Under the assumptions of Theorem 4.1 (b), the semi-classical measure f verifies the following Liouville equation*

$$\alpha f + \xi\cdot\nabla_x f - \frac{1}{2}\nabla_x V(x)\cdot\nabla_\xi f = Q(x,\xi), \quad \textit{in } \mathcal{D}'(\mathbb{R}^{2d}) \qquad (4.18)$$

with

$$Q(x,\xi) = \frac{\pi}{(2\pi)^d}|\hat{S}(\xi)|^2\delta(x)\delta(\xi^2 - n(0)^2) \qquad (4.19)$$

Moreover, f is given by the outgoing solution

$$f(x,\xi) = \int_0^\infty e^{-\alpha s}Q(y(-s;x,\xi),\eta(-s;x,\xi))ds \qquad (4.20)$$

in the sense of distributions, where $(y(s),\eta(s))$ is solution of the Hamiltonian system

$$\begin{cases} \frac{\partial y}{\partial s} = \eta(s), & y(0) = x, \\ \frac{\partial \eta}{\partial s} = -\frac{1}{2}V(y(s)), & \eta(0) = \xi. \end{cases} \qquad (4.21)$$

With Theorem 4.1 and Corllary 4.2, Theorem 4.4 can be proved along the line of [3] and [35]. To show how to calculate the limiting source term Q in the Liouville equation, we take $\varphi,\psi \in \mathcal{S}$ and write

$$\int Q_\epsilon(x,\xi)\varphi(x)\psi(\xi)\,dxd\xi$$
$$= -\frac{1}{(2\pi)^d 2i\epsilon^{(d+1)/2}}\int_{\mathbb{R}^{2d}}[S(\frac{x}{\epsilon}+\frac{y}{2})\overline{u_\epsilon(x-\frac{\epsilon y}{2})} - \overline{S(\frac{x}{\epsilon}-\frac{y}{2})}u_\epsilon(x+\frac{\epsilon y}{2})]\varphi(x)\hat{\psi}(y)dxdy$$
$$= -\frac{\epsilon^{(d-1)/2}}{(2\pi)^d 2i}\int_{\mathbb{R}^{2d}}[S(x')\overline{u_\epsilon(\epsilon(x'-y))}\varphi(\epsilon x'-\frac{\epsilon y}{2}) - \overline{S(x')}u_\epsilon(\epsilon(x'+y))\varphi(\epsilon x'+\frac{\epsilon y}{2})]\hat{\psi}(y)dx'dy$$

Applying Theorem 4.1 b, we obtain

$$\int Q_\epsilon(x,\xi)\varphi(x)\psi(\xi)\,dxd\xi$$

$$\underset{\epsilon\to 0}{\to} \quad -\frac{\varphi(0)}{2i(2\pi)^d}\int [S(x')\overline{w_0(x'-y)} - \overline{S(x')}w_0(x'+y)]\hat{\psi}(y)dx'dy$$

$$= \quad \frac{\varphi(0)}{(2\pi)^d}\int_{\mathbb{R}^d} \Im(\xi^2+V(0)-E-i0)^{-1}|\hat{S}(\xi)|^2\psi(\xi)d\xi$$

We finally find that $Q_\epsilon \to \frac{1}{(2\pi)^d}\delta(x)\Im(\xi^2+V(0)-E-i0)^{-1}|\hat{S}(\xi)|^2 = Q(x,\xi)$ in sense of distributions. For rigorous proof of this convergence, see [3, 35].

Remarks (1). *Let $\Omega = p^{-1}(]E-\delta, E+\delta[)$, where $p=\frac{1}{2}(\xi^2-V(x))$, $\delta>0$ small. Ω is invariant by the solutions of (4.21) and the conditions on V implies that the classical flow is non-trapping for initial data $(x,\xi)\in\Omega$. One can show that f is a well-defined as a distribution on Ω. Since supp Q is contained in $p^{-1}(\{E\})$, the same is true for supp f. The outgoing property of f can be then interpreted as*

$$\lim_{t\to-\infty} f\circ\Phi^t = 0, \quad \text{in } \mathcal{D}',$$

where $\Phi^t(x,\xi)=(y^t(x,\xi),\eta^t(x,\xi))$ is solution of (4.21). See [35] for more details.

(2). *Applying Theorem 4.4 to any subsequence of $\{u_\epsilon\}$, we conclude that if μ is the semi-classical measure associated with a subsequence $\{u_{\epsilon_k}\}$, then μ is given by (4.20). This shows the uniqueness of semi-classical measure associated with subsequences of $\{u_\epsilon\}$.*

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