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On blowing-up of solutions of Sobolev-type equation with source[1]

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ABSTRACT

We consider the initial-boundary-value problem for the three-dimensional, Sobolev-type equation with source. Under some conditions, the theorems on blowup of solutions at finite time of this problem are obtained. Twosided estimates for blowing-up are derived.

RESUMEN

Consideramos el problema de valor inicial de frontera para la ecuación tridimensional de tipo Sobolev con fuente. Bajo ciertas condiciones, los teoremas en "blowup" de soluciones en tiempo finito de este problema son obtenidos. Estimaciones bilaterales para "blowing-up" son derivadas.

Key words and phrases: *blow up, Sobolev type equation, pseudoparabolic equation, nonlinear equation, breaking of waves.*

Math. Subj. Class.: *35M20, 35B38, 35D05, 35K70, 35K55*

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1 Introduction. Statement of problem.

In this paper, we obtain sufficient conditions for the blow-up of solutions for the following initial-boundary-value problem for the following strongly nonlinear Sobolev type equation

$$\frac{\partial}{\partial t}(\Delta u - u) + \frac{\partial |u|^{q+1}}{\partial x_1} + |u|^{2q}u = 0, \quad u|_{\partial\Omega} = 0, \quad u(x,0) = u_0(x), \quad q > 0, \qquad (1.1)$$

where a bounded domain $\Omega \subset \mathbb{R}^3$ has smooth boundary of class $\partial\Omega \in \mathbb{C}^{(2,\delta)}$, $\delta \in (0,1]$, and $(x_1, x_2, x_3) \in \Omega$.

This problem has a physical meaning and its derivation is explaned in [1].

Sobolev wave-type equations were studied in [2]–[10]. In these papers, initial-value, initial-boundary-value, and periodic-value problems were considered; some results on the global solvability, the asymptotic behavior as $t \to +\infty$, the stability of travelling-wave solutions, scattering theory for the one-dimensional and N-dimensional Benjamine-Bona-Machony-Burgers equations were obtained.

For problem (1.1), we obtain the blowing-up results for strong generalizied solutions and deduce two-sided estimates for the blowing time. Moreover, for problem (1.1), we prove the breaking of "weakened" solutions and we obtain an upper estimate for the blowing-up time.

Note that our technique is a modification of the well-known Levine concavity method [11].

2 Blowing-up of strong generalizied solution of problem (1.1).

Definition 1. A strong generalized solution of problem (1.1) is a solution of problem (1.1) in the class $\mathbb{C}^{(1)}([0,T];\mathbb{H}_0^1(\Omega))$, where Eq. $(1.1)_1$ is considered in the sence $\mathbb{H}^{-1}(\Omega)$ for all $t \in [0,T]$.

It is easy to prove that in the strong sence problem (1.1) is equivalent to the following integral equation

$$u(t) = u_0 + \int_0^t ds \mathbb{A}^{-1}\mathbb{F}(u), \quad u_0 \in \mathbb{H}_0^1(\Omega), \quad \mathbb{F}(u) = \frac{\partial |u|^{q+1}}{\partial x_1} + |u|^{2q}u,$$

where $\mathbb{A}^{-1} : \mathbb{H}^{-1}(\Omega) \to \mathbb{H}_0^1(\Omega)$ is the inverse operator to the operator $-\Delta + \mathrm{I}$.

Denote by $\|\cdot\|_1$ the norm in the Hilbert space $\mathbb{H}_0^1(\Omega)$.

Using the contraction operator method in the same way as in [12] we can easily prove the following theorem

Theorem 1. *Let $q \in (0,2]$. Then for all $u_0 \in \mathbb{H}_0^1(\Omega)$, there exist maximal $\mathrm{T}_0 > 0$ such that problem (1.1) have a unique maximal solution in the class*

$$u(x,t) \in \mathbb{C}^{(1)}([0,\mathrm{T}_0);\mathbb{H}_0^1(\Omega)),$$

where either $T_0 = +\infty$ *or* $T_0 < +\infty$ *and in the case when* $T_0 < +\infty$*, the following equality holds:*

$$\lim_{t\uparrow T_0} \| u \|_1 = +\infty.$$

In this section, we prove that under the conditions of Theorem 1, the quantity $T_0 > 0$ is finite.

Lemma 1. *Assume that* $\Phi(t) \in \mathbb{C}^{(2)}[0, T_0)$*, for some maximal* $T_0 > 0$*, is maximal in the sence that either* $T_0 = +\infty$ *or* $T_0 < +\infty$ *and in the case when* $T_0 < +\infty$*, the limit equality*

$$\lim_{t\uparrow T_0} \Phi(t) = +\infty \tag{2.1}$$

holds Moreover, assume that for all $t \in [0, T_0]$*,* $\Phi(t) > 0$ *and* $\Phi'(t) > 0$*, and the ordinary differential inequality of second-order*

$$\Phi''\Phi - \alpha(\Phi')^2 + \gamma\Phi\Phi' \geq 0, \quad t \in [0, T_0), \quad \alpha > 1, \quad \gamma > 0, \tag{2.2}$$

and the inequality

$$\Phi_0 < \frac{\alpha - 1}{\gamma}\Phi_1, \quad \Phi_0 \equiv \Phi(0), \quad \Phi_1 \equiv \Phi'(0), \tag{2.3}$$

are valid. Then $T_0 \leq T_2$*, where*

$$T_2 \equiv -\frac{1}{\gamma}\ln\left(1 - \frac{\gamma}{\alpha - 1}\frac{\Phi_0}{\Phi_1}\right), \tag{2.4}$$

and the limit equality (2.1) holds.

Proof. Dividing both sides of inequality (2.2) by the function $\Phi^{1+\alpha}$ and performing simple transformations, we obtain

$$\left(\frac{\Phi'}{\Phi^\alpha}\right)' + \gamma\frac{\Phi'}{\Phi^\alpha} \geq 0, \quad \alpha > 1, \quad \gamma > 0.$$

Let us introduce the notation

$$\Gamma(t) \equiv \frac{\Phi'}{\Phi^\alpha}.$$

Then for the function $\Gamma(t)$, the following inequalities are valid:

$$\Gamma'(t) + \Gamma(t) \geq 0, \quad \Gamma(t) \geq \Gamma_0 \exp(-\gamma t), \quad \frac{\Phi'}{\Phi^\alpha} \geq \Gamma_0 \exp(-\gamma t),$$

$$\Phi \geq \frac{1}{\left[\Phi_0^{1-\alpha} - \Gamma_0\frac{\alpha-1}{\gamma}\left[1 - \exp(-\gamma t)\right]\right]^{1/(\alpha-1)}}. \tag{2.5}$$

We assumed the fulfillment of inequality (2.3); then (2.5) cannot be valid for all $t \in \mathbb{R}^1_+$. Namely, there exists $T_0 \leq T_2$ such that the limit inequality (2.1) holds, where T_2 is defined by Eq. (2.4). ∎

Now let us prove the main result of this section.

Theorem 2. *If for problem (1.1) the following conditions are valid:*

$$\| u_0 \|_{2(q+1)}^{2(q+1)} > \| \nabla u_0 \|_2^2 + \| u_0 \|_2^2, \quad q \in (0,2],$$

then for all $u_0 \in \mathbb{H}_0^1(\Omega)$, *the maximal* T_0 *of Theorem 1 is finite and, hence, the following limit equality fulfill:*

$$\lim_{t \uparrow T_0} \| u \|_1 = +\infty.$$

Moreover, for the time of blowup of the solution of problem (1.1), the upper estimate $T_0 \leq T_2$ *is valid and the lower etimate* $T_1 \leq T_0$ *under additional condition* $q \in (0,2]$ *holds, where* T_1 *and* T_2 *are defined as follows:*

$$T_2 = -\frac{1}{q+1} \ln \left(1 - \frac{\| \nabla u_0 \|_2^2 + \| u_0 \|_2^2}{\| u_0 \|_{2(q+1)}^{2(q+1)}} \right),$$

$$T_1 = \frac{1}{2q C_1^{2q+2}(\Omega)} \frac{1}{[\| \nabla u_0 \|_2^2 + \| u_0 \|_2^2]^q},$$

$C_1(\Omega)$ *is the best constant of the embedding* $\mathbb{H}_0^1(\Omega)$ *into* $\mathbb{L}^{2q+2}(\Omega)$. *This embeding is valid under additional condition* $q \in (0,2]$.

Proof. Theorem 1 implies that for problem (1.1) under the condition $u_0 \in \mathbb{H}_0^1(\Omega)$, there exist maximal $T_0 > 0$ such that for all $T \in (0, T_0)$ there exist a unique maximal solution $u(x,t)$ in the class $u(x,t) \in \mathbb{C}^{(1)}([0,T]; \mathbb{H}_0^1(\Omega))$. Therefore, we can multiply Eq. $(1.1)_1$ first by $u(x,t)$ and then by $u_t(x,t)$ with respect to the pairing between the Hilbert spaces $\mathbb{H}_0^1(\Omega)$ and $\mathbb{H}^{-1}(\Omega)$. Then integrating by parts we obtain the following energetic equalities:

$$\frac{1}{2} [\| \nabla u \|_2^2 + \| u \|_2^2] = \| u \|_{2(q+1)}^{2(q+1)}, \tag{2.6}$$

$$\| \nabla u_t \|_2^2 + \| u_t \|_2^2 = -\frac{1}{2} \int_\Omega dx \frac{\partial^2 u}{\partial x_1 \partial t} |u|^{q+1} + \frac{1}{2q+2} \frac{d}{dt} \| u \|_{2(q+1)}^{2(q+1)} . \tag{2.7}$$

Let us introduce the function

$$\Phi(t) \equiv \| \nabla u \|_2^2 + \| u \|_2^2 . \tag{2.8}$$

The following chain of inequalities holds:

$$\frac{1}{4} \left| \frac{d\Phi}{dt} \right|^2 \leq \left| \int_\Omega dx \, [(\nabla u_t, \nabla u) + (u_t, u)] \right|^2 \leq (\| \nabla u_t \|_2^2 + \| u_t \|_2^2)(\| \nabla u \|_2^2 + \| u \|_2^2)$$

$$\leq \Phi \, (\| \nabla u_t \|_2^2 + \| u_t \|_2^2) . \tag{2.9}$$

On the other hand,

$$\| \nabla u_t \|_2^2 + \| u_t \|_2^2 \le \frac{1}{4(q+1)}\Phi'' + \frac{1}{2}\varepsilon \| \nabla u' \|_2^2 + \frac{1}{2}\frac{1}{\varepsilon} \| u \|_{2(q+1)}^{2(q+1)}$$

$$\le \frac{1}{4(q+1)}\Phi'' + \frac{\varepsilon}{2}\left[\| \nabla u' \|_2^2 + \| u' \|_2^2\right] + \frac{1}{4\varepsilon}\Phi',$$

$$\left(1 - \frac{\varepsilon}{2}\right)\left(\| \nabla u_t \|_2^2 + \| u_t \|_2^2\right) \le \frac{1}{4(q+1)}\Phi'' + \frac{1}{4\varepsilon}\Phi', \tag{2.10}$$

for all $\varepsilon \in (0,2)$. From (2.9) and (2.10) it follows that function $\Phi(t)$ satisfy the differential inequality

$$\frac{2-\varepsilon}{8}\left|\Phi'\right|^2 \le \Phi\left(\frac{1}{4(q+2)}\Phi'' + \frac{1}{4\varepsilon}\Phi'\right)$$

therefore

$$\Phi\Phi'' - \alpha\left(\Phi'\right)^2 + \gamma\Phi\Phi' \ge 0, \quad \alpha = \frac{(2-\varepsilon)(q+1)}{2}, \quad \gamma = \frac{q+1}{\varepsilon}, \quad \varepsilon \in (0,2). \tag{2.11}$$

We see that the function $\Phi(t)$ under the additional condition

$$\Phi_0 < \frac{\alpha - 1}{\gamma}\Phi_1, \quad \Phi_0 \equiv \Phi(0), \quad \Phi_1 \equiv \Phi'(0), \tag{2.12}$$

satisfy all conditions of Lemma 2. Note that inequality (2.12) contain the variable $\varepsilon \in (0,2)$. Thus, our aim is to obtain optimal condition of the form (2.12). For this aim, we must find the maximum of the function

$$f_1(\varepsilon) \equiv \frac{\alpha - 1}{\gamma} = \frac{(2-\varepsilon)\varepsilon}{2}.$$

This function has the maximum at the point $\varepsilon_0 = 1$ and its value is $f_1(\varepsilon_0) = 1/2$ at this point. Thus, condition (2.12) takes the form

$$\Phi_0 < \frac{1}{2}\Phi_1; \tag{2.13}$$

by definition of the function $\Phi(t)$, this implies the first part of the Theorem 2.

Now, let us obtain estimates for the blowing-up time of problem (1.1).

For problem (1.1) the following energy inequality holds:

$$\frac{1}{2}\frac{d}{dt}\left(\| \nabla u \|_2^2 + \| u \|_2^2\right) = \| u \|_{2q+2}^{2q+2}.$$

From this for the function

$$\Phi(t) \equiv \| \nabla u \|_2^2 + \| u \|_2^2;$$

it follows that the the inequality

$$\Phi(t) \leq \Phi(0) + 2C_1^{2q+2}(\Omega) \int_0^t ds \Phi^{q+1}(s),$$

is valid. Here C_1 is the constant of the best embedding $\| v \|_{2q+2} \leq C_1(\Omega) \| \nabla v \|_2$. This embedding is valid under the additional condition $q \in (0, 2]$. From the last inequality by Gronwall-Bellman theorem we have

$$\Phi(t) \leq \frac{\Phi(0)}{\left[1 - q2C_1^{2q+2}(\Omega)\Phi^q(0)t\right]^{1/q}}$$

and, therefore,

$$T_1 = \frac{1}{2qC_1^{2q+2}(\Omega)} \frac{1}{[\| \nabla u_0 \|_2^2 + \| u_0 \|_2^2]^q},$$

which is what was required. ∎

3 Breaking of the classical solution of the problem (1.1).

To prove the breaking of the classical solution of problem (1.1) we must show that there exist maximal $T_0^* > 0$ such that there exist a unique maximal classical solution of problem (1.1) in the class $\mathbb{C}^{(1)}([0, T]; \mathbb{C}^{(1)}(\overline{\Omega}))$ for all $T \in (0, T_0^*)$. Then by blowing-up results of Theorem 2 we obtain $0 < T_0^* \leq T_0 \leq T_2 < +\infty$, and, hence,

$$\lim_{T \uparrow T_0^*} \sup_{t \in [0,T], x \in \Omega} |\nabla u(x,t)| = +\infty. \tag{3.1}$$

Thus, at some moment of time $T_0^* > 0$, the quantity

$$\max_{x \in \overline{\Omega}} |\nabla u(x,t)|$$

becoms infinite.

Let us give the definition of the "weakened" solution of problem (1.1):

Definition 2. A classical solution of problem (1.1) is a solution $u(x,t)$ of the related with the problem (1.1) doubly nonlinear integral equation

$$u(x,t) = u_0(x) + \int_0^t ds \int_\Omega dy G(x,y) \left[\frac{\partial |u|^{q+1}(y,s)}{\partial y_1} + |u|^{2q}u(y,s)\right], \quad \forall\, t \in [0, T], \tag{3.2}$$

in the class $u(x,t) \in \mathbb{C}^{(1)}([0, T]; \mathbb{C}^{(1)}(\overline{\Omega}) \bigcap \mathbb{C}_0(\overline{\Omega}))$ for some $T > 0$, where $G(x,y)$ is the Green's function of the first bounary value problem for the operator $-\Delta + \mathbb{I}$ in the domain Ω.

It is obviosly that from definition, for the existence of the classical solution the validity of the following condition on the initial function is necessary: $u_0(x) \in \mathbb{C}^{(1)}(\Omega)\bigcap \mathbb{C}_0(\overline{\Omega})$.

Now let us explain the connection between strong generalized and classical solutions of problem (1.1). As we have shown above, the problem (1.1) is equivalent in the strong sence to the following integral-operator equation:

$$u(t) = u_0 + \int\limits_0^t ds \mathbb{A}^{-1}\mathbb{F}(u), \quad u_0 \in \mathbb{H}_0^1,$$

where the operator $\mathbb{A}^{-1} : \mathbb{H}^{-1}(\Omega) \to \mathbb{H}_0^1(\Omega)$ is the inverse to the operator $-\Delta u + u : \mathbb{H}_0^1(\Omega) \to \mathbb{H}^{-1}(\Omega)$. It is clear that the restriction of the operator $\mathbb{A}^{-1}$ to the Banach space $\mathbb{C}(\overline{\Omega})$ coincids with the integral operator

$$\overline{\mathbb{A}} = \int\limits_\Omega dy G(x,y)\cdot,$$

where Ω have smooth boundary $\partial\Omega \in \mathbb{C}^{(2,\delta)}$, $\delta \in (0,1]$ Thus, we conclude that the classical solution of problem (1.1) is a strong generalized solution of problem (1.1).

Theorem 3. *For all $u_0 \in \mathbb{C}^{(1)}(\Omega) \cap \mathbb{C}_0(\overline{\Omega})$, there exist maximal $\mathrm{T}_0^* > 0$ such that there exist a unique maximal classical solution $u(x,t)$ of problem (1.1) in the class $u(x,t) \in \mathbb{C}^{(1)}([0,\mathrm{T}]; \mathbb{C}^{(1)}(\overline{\Omega})\bigcap \mathbb{C}_0(\overline{\Omega}))$ for all $\mathrm{T} \in (0, \mathrm{T}_0^*)$. Moreover, either $\mathrm{T}_0^* = +\infty$ or $\mathrm{T}_0^* < +\infty$, and in the last case, relation (3.1) holds.*

Proof. By the explicit form of the Green function for the operator $-\Delta + \mathbb{I}$, we have

$$G(x,y) = \psi(x,y) + \frac{1}{4\pi}\frac{\exp(-|x-y|)}{|x-y|}, \tag{3.3}$$

where $\psi(x,y) \in \mathbb{C}^{(1)}(\overline{\Omega} \times \overline{\Omega})$. Then the following relation holds:

$$\sup_{x\in\Omega} \int\limits_\Omega |\nabla_y G(x,y)|\, dy = \mathrm{C} < +\infty. \tag{3.4}$$

Thus, we may integrate by parts in the integral equation (3.2) and obtain relation

$$u(x,t) = u_0(x) + \int\limits_0^t ds \int\limits_\Omega dy G(x,y)|u|^{2q}u(y,s) - \int\limits_0^t ds \int\limits_\Omega dy G'_{y_1}(x,y)|u|^{q+1}. \tag{3.5}$$

Let us introduce the operator

$$\mathbb{U}(u) \equiv u_0 + \int\limits_0^t ds \int\limits_\Omega dy G(x,y)|u|^{2q}(y,s)u(y,s) - \int\limits_0^t ds \int\limits_\Omega dy G'_{y_1}(x,y)|u|^{q+1}. \tag{3.6}$$

We shall prove that operator (3.6) acts from $\mathbb{C}(\overline{Q}_T)$ into $\mathbb{C}(\overline{Q}_T)$, where $Q_T = (0,T)\times\Omega$ at some $T>0$ and

$$\| v \|_{T,0} = \sup_{t\in[0,T],x\in\Omega} |v|$$

is the norm in the Banach space $\mathbb{C}(\overline{Q}_T)$.

Let us introduce the notation

$$\mathbb{U}_1(u) = \int_0^t ds \int_\Omega dy G'_{y_1}(x,y)|u|^{q+1}, \quad \mathbb{U}_2(u) = \int_0^t ds \int_\Omega dy G(x,y)|u|^{2q}u(y,s),$$

Let us prove that each of the operators $\mathbb{U}_i(u)$ act form $\mathbb{C}(\overline{Q}_T)$ to $\mathbb{C}(\overline{Q}_T)$.

Consider the operator $\mathbb{U}_1(u)$. We have

$$\psi'_{y_1}(x,y) \in \mathbb{C}(\Omega\times\Omega),$$

$$\frac{1}{4\pi}\frac{\partial}{\partial y_1}\left(\frac{\exp(-|x-y|)}{|x-y|}\right) = \frac{1}{4\pi}\frac{x_1-y_1}{|x-y|^3}\exp(-|x-y|) + \frac{1}{4\pi}\frac{x_1-y_1}{|x-y|^2}\exp(-|x-y|).$$

Then we can represent the operator $\mathbb{U}_1(u)$ in the form

$$\mathbb{U}_1(u) = \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3,$$

where

$$\mathrm{I}_1 = \int_0^t ds \int_\Omega dy \psi'_{y_1}(x,y)|u|^{q+1}(y,s),$$

$$\mathrm{I}_2 = \frac{1}{4\pi}\int_0^t ds \int_\Omega dy \frac{\exp(-|x-y|)}{|x-y|^2}\frac{x_1-y_1}{|x-y|}|u|^{q+1}(y,s),$$

$$\mathrm{I}_3 = \frac{1}{4\pi}\int_0^t ds \int_\Omega dy \frac{\exp(-|x-y|)}{|x-y|}\frac{x_1-y_1}{|x-y|}|u|^{q+1}(y,s).$$

Let us prove that $\mathrm{I}_1(x,t) \in \mathbb{C}(\overline{Q}_T)$. Indeed, for all (x,t) and (x_0,t_0) from Q_T the following inequality holds:

$$|\mathrm{I}_1(x,t) - \mathrm{I}_2(x_0,t_0)| \le \left| \int_{t_0}^t ds \int_\Omega dy |\psi'_{y_1}(x,y)||u|^{q+1} \right| +$$

$$+ \int_0^{t_0} ds \left| \int_\Omega dy \left[\psi'_{y_1}(x,y) - \psi'_{y_1}(x_0,y)\right] |u|^{q+1} \right|,$$

which, by the continuity of $\psi'_{y_1}(x,y) \in C(\overline{\Omega} \times \overline{\Omega})$, we immediately implies that each of the terms in the last inequality is less than $\varepsilon/2$ under the additional condition $|x-x_0| + |t-t_0| \leq \delta(\varepsilon)$ for sufficiently small $\delta > 0$. Therefore, $I_1(x,t) \in C(\overline{Q}_T)$.

Let us prove that $I_2(x,t) \in C(\overline{Q}_T)$. Indeed, for all (x,t) and (x_0,t_0) from $\overline{Q}_T$, the following inequalities hold:

$$|I_2(x,t) - I_2(x_0,t_0)| \leq \frac{1}{4\pi}\left|\int\limits_{t_0}^{t} ds \int\limits_{\Omega} dy \frac{\exp(-|x-y|)}{|x-y|^2}|u|^{q+1}(y,s)\right|$$

$$+\frac{1}{4\pi}\int\limits_{0}^{t_0} ds \left|\int\limits_{\Omega} dy \left[\frac{\exp(-|x_0-y|)}{|x_0-y|^2}\frac{x_{10}-y_1}{|x_0-y|} - \frac{\exp(-|x-y|)}{|x-y|^2}\frac{x_1-y_1}{|x-y|}\right]|u|^{q+1}(y,s)\right|$$

$$= I_{21} + I_{22}.$$

Let us consider separately each of the terms of the relations for the quantity I_{21} and I_{22}. I_{21} is less than $\varepsilon/3$ under the condition $|x-x_0| + |t-t_0| \leq \delta(\varepsilon)$. Now we consider I_{22}. Assume that $u(x,t) \in C(\overline{Q}_T)$ and $|u(x,t)| \leq M(T)$. By virtue of the standard scheme of proffing the continuity of potential-type integrals we have

$$|I_{22}(x,x_0,t,t_0)| \leq M(T)\int\limits_{0}^{t_0} ds \int\limits_{\Omega_\eta(x_0)} dy \left[\frac{\exp(-|x_0-y|)}{|x_0-y|^2} + \frac{\exp(-|x-y|)}{|x-y|^2}\right]$$

$$+M(T)\int\limits_{0}^{t_0} ds \int\limits_{\Omega\setminus\Omega_\eta(x_0)} \left|\frac{\exp(-|x_0-y|)}{|x_0-y|^2}\frac{x_{10}-y_1}{|x_0-y|} - \frac{\exp(-|x-y|)}{|x-y|^2}\frac{x_1-y_1}{|x-y|}\right|,$$

The first term is less than $\varepsilon/3$ for sufficiently small η. In the second term, the intergrand is uniformly continuous in the closed domain $|x-x_0| \leq \eta/2$, $|y-x_0| \geq \eta$, $y \in \overline{\Omega}$ and the integrand vaniches at the point $x = x_0$. Thus, the integral is less than $\varepsilon/3$ for all $|x-x_0| + |t-t_0| \leq \delta(\varepsilon)$ for sufficiently small $\delta(\varepsilon)$. Hence, $I_2(x,t) \in C(\overline{Q}_T)$.

It is clear that we can similarly prove the inclusion $I_3(x,t) \in C(\overline{Q}_T)$. Therefore, we prove that

$$U_1(u) : C(\overline{Q}_T) \to C(\overline{Q}_T).$$

Let us consider the operator

$$U_2(u) = \int\limits_{0}^{t} ds \int\limits_{\Omega} dy G(x,y)|u|^{2q}u(y,s).$$

Using the properies of the Green function, we obtain

$$U_2 : C(\overline{Q}_T) \to C(\overline{Q}_T).$$

Hence, the operator $\mathbb{U}(u)(x,t)$ defined by formula (3.6), acts from $\mathbb{C}(\overline{Q}_T)$ to $\mathbb{C}(\overline{Q}_T)$.

Let $\mathbb{B}_R$ be a closed, bounded, convex set in the Banach space $\mathbb{C}(\overline{Q}_T)$:

$$\mathbb{B}_R \equiv \left\{ v(x,t) \in \mathbb{C}(\overline{Q}_T) : \| v \|_T \equiv \max_{(x,t)\in\overline{Q}_T} |v(x,t)| \leq R \right\}. \tag{3.7}$$

Let us prove that for sufficiently small $T > 0$ and under sufficiently large $R > 0$ the operator $\mathbb{U}$ acts from $\mathbb{B}_R$ to $\mathbb{B}_R$ and this operator is contractive on $\mathbb{B}_R$. Indeed,

$$\| \mathbb{U}(u) \|_T \leq \| u_0 \|_T + CT \| u \|_T^{2q+1} + CT \| u \|_T^{q+1} \leq \| u_0 \|_T + CT(R^{2q+1} + R^{q+1}) \leq R,$$

under the condition $\| u_0 \|_T \leq R/2$ and $0 < T \leq 2^{-1}(CR^{2q} + CR^q)^{-1}$. Hence, for some sufficiently small $T > 0$ and sufficiently large $R > 0$, the operator $\mathbb{U}$ acts from $\mathbb{B}_R$ to $\mathbb{B}_R$.

Let us prove the contractivity of the operator $\mathbb{U}(u)$ on $\mathbb{B}_R$ for sufficiently small $T > 0$ and sufficiently large $R > 0$. Indeed,

$$\| \mathbb{U}(u_1) - \mathbb{U}(u_2) \|_T \leq T\left(CR^{2q} + CR^q\right) \| u_1 - u_2 \|_T \leq \frac{1}{2} \| u_1 - u_2 \|_T,$$

under the condition $0 < T \leq 2^{-1}(CR^{2q} + CR^q)^{-1}$. Thus, the operator $\mathbb{U}(u)$ is contractive on $\mathbb{B}_R$.

Therefore, under the condition $u_0 \in \mathbb{C}_0(\overline{\Omega})$, there exist a unique solution of the intehral equation (3.5) in the class $\mathbb{C}([0,T];\mathbb{C}_0(\overline{\Omega}))$.

Assume that $u_0 \in \mathbb{C}^{(1)}(\overline{\Omega}) \bigcap \mathbb{C}_0(\overline{\Omega})$. We shall prove that there exist a unique solution of the integral equation (3.5) in the class $\mathbb{C}([0,T];\mathbb{C}^{(1)}(\overline{\Omega}) \bigcap \mathbb{C}_0(\overline{\Omega}))$.

For this, we consider the Banach space $\mathbb{C}([0,T];\mathbb{C}^{(1)}(\overline{\Omega}))$ with norm

$$\| v \|_{T,1} \equiv \| v \|_{T,0} + \sum_{i=1}^{3} \| \frac{\partial v}{\partial x_i} \|_{T,0}, \quad \| v \|_{T,0} \equiv \max_{(x,t)\in\overline{Q}_T} |v(x,t)|. \tag{3.8}$$

Let us prove that the operator

$$\mathbb{U}(u) \equiv u_0 + \int_0^t ds \int_\Omega dy G(x,y) \left[\frac{\partial |u|^{q+1}(y,s)}{\partial y_1} + |u|^{2q} u(y,s) \right],$$

act from $\mathbb{C}([0,T];\mathbb{C}^{(1)}(\overline{\Omega}))$ to $\mathbb{C}([0,T];\mathbb{C}^{(1)}(\overline{\Omega}))$. For this, we use the fact that

$$G(x,y) = \psi(x,y) + \frac{1}{4\pi} \frac{\exp(-|x-y|)}{|x-y|},$$

where $\psi'_{x_1}(x,y) \in \mathbb{C}(\overline{\Omega} \times \overline{\Omega})$. Let us represent the operator $\mathbb{U}(u)$ in the form

$$\mathbb{U}(u) = u_0 + \mathbb{U}_1(u) + \mathbb{U}_2(u), \tag{3.9}$$

where

$$\mathbb{U}_1(u) = \int_0^t \int_\Omega dy \psi(x,y) \left[\frac{\partial |u|^{q+1}}{\partial y_1} + |u|^{2q}u\right],$$

$$\mathbb{U}_2(u) = \frac{1}{4\pi} \int_0^t \int_\Omega dy \frac{\exp(-|x-y|)}{|x-y|} \left[\frac{\partial |u|^{q+1}}{\partial y_1} + |u|^{2q}u\right].$$

Since $\psi(x,y) \in \mathbb{C}^{(1)}(\overline{\Omega} \times \overline{\Omega})$, the operator $\mathbb{U}_1(u)$ acts from $\mathbb{C}([0,T];\mathbb{C}^{(1)}(\overline{\Omega}))$ to $\mathbb{C}([0,T];\mathbb{C}^{(1)}(\overline{\Omega}))$.

Now let us consider the operator $\mathbb{U}_2(u)$. Similarly to the above arguments, we conclude that the operator $\mathbb{U}_2(u)$ acts from $\mathbb{C}([0,T];\mathbb{C}^{(1)}(\overline{\Omega}))$ to $\mathbb{C}([0,T];\mathbb{C}^{(1)}(\overline{\Omega}))$.

Let us introduce the following closed, bounded, convex set in the Banach space $\mathbb{C}([0,T];\mathbb{C}^{(1)}(\overline{\Omega}))$:

$$\mathbb{B}_{1R} \equiv \left\{ v \in \mathbb{C}([0,T];\mathbb{C}^{(1)}(\overline{\Omega})) : \| v \|_{1,T} \leq R \right\},$$

where the norm is defined by (3.8). It is easily to prove that the operator $\mathbb{U}(u)$ acts from $\mathbb{B}_{1R}$ to $\mathbb{B}_{1R}$ and it is contractive on $\mathbb{B}_{1R}$ for sufficiently small $T > 0$ and at some sufficiently large $R > 0$. Indeed,

$$\| \mathbb{U}(u) \|_{1,T} \leq \| u_0 \|_{1,T} + CT \| u \|_{1,T}^{q+1} + CT \| u \|_{1,T}^{2q+1}$$

$$\leq \| u_0 \|_{1,T} + CTR^{q+1} + CTR^{2q+1} \leq R,$$

under the condition $\| u_0 \|_{1,T} \leq R/2$, $T \leq 2^{-1}(CR^q + CR^{2q})^{-1}$.

Let us prove the contractivity of the operator $\mathbb{U} : \mathbb{B}_{1R} \to \mathbb{B}_{1R}$. We note that,

$$\| \mathbb{U}(u_1) - \mathbb{U}(u_2) \|_{1,T} \leq CR^q T \| u_1 - u_2 \|_{1,T} + CR^{2q} T \| u_1 - u_2 \|_{1,T} \leq \frac{1}{2} \| u_1 - u_2 \|_{1,T},$$

under the condition $T \leq 2^{-1}(CR^q + CR^{2q})^{-1}$. Hence, the operator $\mathbb{U} : \mathbb{B}_{1R} \to \mathbb{B}_{1R}$ is contractive.

Using the standard algorithm of continuation in time of solutions of the integral equation (3.2), we see that there exist maximal T_0^* such that there exist a unique solution of Eq. (3.2) in the class $\mathbb{C}([0,T];\mathbb{C}^{(1)}(\overline{\Omega}) \bigcap \mathbb{C}_0(\overline{\Omega}))$ for all $T \in (0, T_0^*)$. Moreover, either $T_0^* = +\infty$ or $T_0^* < +\infty$, and in the last case, the limit equality (3.1) holds.

Thus, there exist a unique solution of the integral equation (3.2) in the class $\mathbb{C}([0,T];\mathbb{C}^{(1)}(\overline{\Omega}) \bigcap \mathbb{C}_0(\overline{\Omega}))$. From the explicit form of Eq. (3.2) it easily follows that the solution belong to the class $\mathbb{C}^{(1)}([0,T];\mathbb{C}^{(1)}(\overline{\Omega}) \bigcap \mathbb{C}_0(\overline{\Omega}))$.

Thus, we have proved the unique solvability of the "weakened" solution of problem (1.1) in the sence of the definition 2.

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Note, that the initial norm on the Banach space $\mathbb{C}([0,T];\mathbb{C}^{(1)}(\overline{\Omega}) \bigcap \mathbb{C}_0(\overline{\Omega}))$ for bounded domains is equivalent to the norm

$$\| v \|'_T \equiv \max_{(x,t) \in \overline{Q}_T} |\nabla v(x,t)|, \quad Q_T \equiv (0,T) \times \Omega. \tag{3.10}$$

As since a classical solution of problem (1.1) is a strong generalized solution of problem (1.1), Theorem 2 with the additional condition

$$\| u_0 \|_{2q+2}^{2q+2} > \| \nabla u_0 \|_2^2 + \| u_0 \|_2^2$$

implies that there exist time interval $0 < T_0 \leq T_2$ such that

$$\lim_{t \uparrow T_0} \| \nabla u \|_2^2 = +\infty,$$

where

$$T_2 = -\frac{1}{q+1} \ln \left(1 - \frac{\| \nabla u_0 \|_2^2 + \| u_0 \|_2^2}{\| u_0 \|_{2q+2}^{2q+2}} \right).$$

Thus, for some moment of time T_0^* the inequalities $0 < T_0^* \leq T_0 \leq T_2$ hold. Therefore, for some finite time T_0^*, the classical solution of problem (1.1) breakes:

$$\lim_{T \uparrow T_0^*} \| u \|_T' = +\infty.$$

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