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On Asymptotic Stability of Nonlinear Stochastic Systems with Delay.

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ABSTRACT

We consider the system of stochastic differential equations with delay and with non-autonomous nonlinear main part

$$dx_i(t) = \sum_{k=1}^{n} \left(p_{ki}(t)x_k^{\mu_k}(t) + f_i\left(t, [X]_t^{t-h}\right)\right) dt + \sigma_i\left(t, [X]_t^{t-h}\right) dw_t,$$

$$i = 1, \ldots, n, \quad X(s) = \phi(s), \quad s \le 0. \tag{1}$$

Here $h \ge 0$, $[X]_t^{t-h}(s) = X(s)$, when $s \in [t-h, t]$, $t > h$, $[X]_t^{t-h}(s) = \phi(s)$, when $s \in [-\infty, 0]$, $\phi(s)$ is a given initial process, $X = (x_1, x_2, \ldots, x_n)^T$, $\mu_i > 1$ are rational numbers with odd numerators and denominators, w_t is a Wiener process.

For different types of delays in coefficients $f_i\left(t, [X]_t^{t-h}\right)$ and $\sigma_i\left(t, [X]_t^{t-h}\right)$ we prove almost sure asymptotic stability of a trivial solution to the system (1) when $\phi(s) \equiv 0$.

RESUMEN

Consideramos el sistema de ecuaciones diferenciales estocásticas con retardo y con parte principal no lineal no autónomas

$$dx_i(t) = \sum_{k=1}^{n} \left(p_{ki}(t)x_k^{\mu_k}(t) + f_i\left(t, [X]_t^{t-h}\right)\right) dt + \sigma_i\left(t, [X]_t^{t-h}\right) dw_t,$$
$$i = 1, \ldots, n, \quad X(s) = \phi(s), \quad s \le 0.$$

Acá $h \ge 0$, $[X]_t^{t-h}(s) = X(s)$, donde $s \in [t-h, t]$, $t > h$, $[X]_t^{t-h}(s) = \phi(s)$, cuando $s \in [-\infty, 0]$, $\phi(s)$ es un proceso inicial dado, $X = (x_1, x_2, \ldots, x_n)^T$,

$\mu_i > 1$ son números racionales con numeradores y denominadores impares, w_t es un proceso Wiener. Para diferentes tipos de retardo en coeficientes $f_i\left(t,[X]_t^{t-h}\right)$ y $\sigma_i\left(t,[X]_t^{t-h}\right)$ probaremos estabilidad asintótica casi segura de una solución trivial del sistema (1) cuando $\phi(s)\equiv 0$.

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1 Introduction

Asymptotic stability of the trivial solution to the system of differential equations

$$dx_i(t)=\sum_{k=1}^{n}p_{ki}(t)x_k^{\mu_k}(t),\quad i=1,\dots,n, \tag{2}$$

was investigated in [2]. In this paper we prove almost sure ($a.s.$) asymptotic stability of the trivial solution to the following system of stochastic Ito equation with delays

$$dx_i(t)=\sum_{k=1}^{n}\left(p_{ki}(t)x_k^{\mu_k}(t)+f_i\left(t,[X]_t^{t-h}\right)\right)dt+\sigma_i\left(t,[X]_t^{t-h}\right)dw_t,\ i=1,\dots,n. \tag{3}$$

Here $X=(x_1,x_2,\dots,x_n)^T$, $[X]_t^{t-h}(s)=X(s)$ for $s\in[t-h,t]$, $t>h$, and $[X]_t^{t-h}(s)=\phi(s)$ for $s\in[-\infty,0]$, $h\geq 0$, $\phi(s)$ is a given initial process, w_t is a Wiener process, $\mu_i>1,\mu_1\leq\mu_2\leq\cdots\leq\mu_n$ are rational numbers with odd numerators and denominators.

Everywhere in the paper we suppose that $p_{ki}(t)$ are continuous non-random functions for $t\geq 0$ and that there exist positive constants $\lambda_1,\dots,\lambda_n$, and nonnegative function $a(t)$ such that for all $t\geq 0$ and $Y\in E^n$

$$Y^T\Lambda P(t)Y\leq -a(t)\|Y\|^2, \tag{4}$$

$$\int_0^\infty a(s)ds=\infty. \tag{5}$$

Here $P(t)=\{p_{kj}(t)\}$, $k,i=1,2,\dots,n$ and $\Lambda=diag\{\lambda_1,\dots,\lambda_n\}$ is a diagonal matrix.

The system (3) can be considered as a stochastic generalization of the system (2).

We can treat the term $\sum_{k=1}^{n}p_{ki}(t)x_k^{\mu_k}(t)$ as the main part of the system (3) and $f_i\left(t,[X]_t^{t-h}\right)$ and $\sigma_i\left(t,[X]_t^{t-h}\right)$ as deterministic and stochastic noises, respectively. We assume that for the noise part of the equations some bounds are fulfilled. We formulate our sufficiency conditions for asymptotic stability in terms of the coefficients of these bounds. It turns out that when the coefficients for X at points t and some previous points $t - h_i$ are small enough the stability results take place.

In this paper we obtain sufficient conditions on *a.s.* asymptotic stability of the trivial solution to the system of stochastic Ito equations (3) in three cases: when the noise part of system does not contain any delay, when it contains discrete delays, and in the most general case, in which it contains discrete delays as well as distributed delays.

We begin by recalling some definitions from the Theory of Functional Differential Equations (see [1], [3], [4], [5], [6]).

We say that an equation contains ***constant discrete delays*** if its right-hand part depends on $X(t)$ and $X(t-h_k)$ for some constants $h_k > 0$. We say that an equation contains ***distributed delays*** if its right-hand part depends on the whole of a previous history of $[X]_t^{t-h}(s) = X(s)$ for $t - h \le s \le t$, $h < \infty$. A sufficiently general form of such a dependence is the following:

$$\int_0^h X(t-s)dR(s) + \sum_{j=1}^{\infty} e_j X(t - \Delta_j(t)), \tag{6}$$

where R is a function of bounded variation on $[0,h]$, $0 \le \Delta_j(t) \le h$ for $t \ge 0$. We note that discrete delays, constant or non-constant, are partial cases of (6).

For the proof of asymptotic stability in the three cases mentioned above we construct a Lyapunov function or some Lyapunov-Krasovskii functionals (see [1], [3], [4], [5], [6]). As a rule, when the functions $f_i\left(t, [X]_t^{t-h}\right)$ and $\sigma_i\left(t, [X]_t^{t-h}\right)$ contain more general delay, the Lyapunov-Krasovskii functional become more complicated (see formulas (38), (53), (62) below).

For the proofs of our theorems we also apply some results from Theory of Random Processes (see, for example, [7]). Some definitions and facts from the Theory of Random Processes can be found below.

A complete probability spaces (Ω, F, P) equipped with the nondecreasing family of σ-algebras $F = \{\mathcal{F}_n\}_{n=1,2,\dots}, \mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}, s \le t$ (*filtration*), is called a ***stochastic basis*** if it satisfies the "usual" conditions:

(a) right continuity: $\mathcal{F}_t = \mathcal{F}_{t+} = \bigcap_{u>t} \mathcal{F}_u, t \ge 0$;

(b) completeness: $\mathcal{F}_0$ is augmented with the sets from $\mathcal{F}$ with P-null probability.

We say that $\{X_t\}_{t\ge 0}$, is a ***stochastic process***, if it is the family of random variables $X_t(\omega)$ defined on $(\Omega, \bar{F})$. We restrict our consideration to the processes $\{X_t\}_{t\ge 0}$ such that, for each $t \ge 0$, the random variables X_t are $\mathcal{F}_t$-measurable. We also suppose that, for all $\omega \in \Omega$, the trajectories of $X_t(\omega), t \ge 0$, are continuous as functions of t and the initial process $\phi(s)$ is $\mathcal{F}_o$-measurable. Let $\{w_t\}_{t\ge 0}$ be a 1-dimensional $\mathcal{F}_t$-measurable Wiener process.

Below we formulate the famous *Ito formula*, which can be reckoned as a generalization of the chain rule for differentiation of deterministic functions.

Lemma 1. *Let X_t be n-dimensional stochastic process having the differential $dX_t = f(t)dt + \sigma(t)dw_t$, where $f(t) = (f_1(t), \dots, f_n(t))$, $\sigma(t) = (\sigma_1(t), \dots, \sigma_n(t))$. Let also*

$V(t,\bar{x}) = V(t, x_1, \dots, x_n)$ be a differentiable function with respect to the first argument and a twice continuously differentiable function with respect to the last n variables . Then Ito formula takes place

$$V(t, X_t) = V(0, X_0) + \int_0^t \sum_{i=1}^n \frac{\partial V}{\partial x_i} f_i(t)dt + \int_0^t \frac{\partial V}{\partial t}dt +$$

$$\frac{1}{2}\int_0^t \sum_{i,j=1}^n \frac{\partial^2 V}{\partial x_i \partial x_j}\sigma_i(t)\sigma_j(t)dt + \int_0^t \sum_{i=1}^n \frac{\partial V}{\partial x_i}\sigma_i(t)dw_t. \tag{7}$$

Stochastic process $\{M_t\}_{t\geq 0}$ is said to be an *$\mathcal{F}_t$-martingale,* if $\mathbf{E}|M_t| < \infty$ and $\mathbf{E}(M_t|\mathcal{F}_s) = M_s$ for all $t > s \geq 0$. We note that a Wiener process w_t is a martingale as well as the so called Ito integral $\int_0^t g(s)dw_s$, where the process $g(s)$ is continuous and $\mathcal{F}_t$-measurable. A martingale $\{M_t\}_{t\geq 0}$ is called *square integrable* if $\sup_{t\geq 0} EX_t^2 < \infty$. A stochastic process is called a *semimartingale* if it admits the representation

$$X_t = X_0 + M_t + A_t, \tag{8}$$

where $\{M_t\}_{t\geq 0}$ is a martingale and $\{A_t\}_{t\geq 0}$ is a process of bounded variation. In particular, the process X_t, defined by (8), is a semimartingale if A_t is *a.s.* non-decreasing process.

Lemma 2. *Let $\{A_t^1\}_{t\geq 0}$, $\{A_t^2\}_{t\geq 0}$, $\{B_t^1\}_{t\geq 0}$, $\{B_t^2\}_{t\geq 0}$ be a.s. non-decreasing continuous $\mathcal{F}_t$-measurable processes with $B^1 \leq A^1$, $B^2 \geq A^2$ and $A = B^1 - B^2$. Let also $\{M_t\}_{t\geq 0}$ be a continuous $\mathcal{F}_t$-martingale. Let process $\{Z_t\}_{t\geq 0}$, $Z_t = Z_0 + M_t + A_t$, be non-negative. Then $\{\omega : A_\infty^1 < \infty\} \subseteq \{Z \to\} \cap \{\omega : A_\infty^2 < \infty\}$ a.s.*

Remark 1. *In this paper we use the designation $X(t)$ for the solution of the system (2) in contrast to X_t, which is more common in the literature on Stochastic Processes. We do it to avoid misunderstanding when we are speaking about $x_i(t)$ or $x_i(t - h_k)$.*

In the following calculations we are going to apply the inequality

$$ab \leq \frac{a^r}{r} + \frac{b^q}{q}, \quad \text{where} \quad \frac{1}{r} + \frac{1}{q} = 1, \quad a, b > 0. \tag{9}$$

2 Systems without delays

In this section we consider the case when noises in the system (3) do not contain any delays. That is the functions $f_i\left(t, [X]_t^{t-h}\right)$ and $\sigma_i\left(t, [X]_t^{t-h}\right)$ from the system (3) do not depend on the previous states of X and can be written simply as $f_i(t, X(t))$ and $\sigma_i(t, X(t))$:

$$dx_i(t) = \sum_{k=1}^n \left(p_{ki}(t)x_k^{\mu_k}(t) + f_i(t, X(t))\right)dt + \sigma_i(t, X(t))dw_t, \; i = 1, \dots, n. \tag{10}$$

We let

$$(X)_\mu = (x_1^{\mu_1}, x_2^{\mu_1}, \dots, x_n^{\mu_1})^T, \tag{11}$$

$$\sum_{\kappa=1}^{n} k_\kappa = \kappa, \quad \tilde{\lambda} = \sum_{i=1}^{n} \frac{\lambda_i(\mu_i + 1)}{2}, \tag{12}$$

$k_\kappa \geq 0, \kappa = 1, \dots, n$, and assume that for every $i = 1, \dots, n$ and for some functions $\beta_i(t)$, $\gamma_i(t)$ the following conditions hold

$$|f_i(t, X(t))| \leq \beta_i(t), \tag{13}$$

$$\int_0^\infty \beta_i^2(s) a^{-1}(s) ds < \infty, \tag{14}$$

$$|\sigma_i(t, X(t))|^2 \leq a(t) \sum_{\kappa=1}^{n} k_\kappa x_\kappa^{\mu_\kappa + \frac{\mu_\kappa}{\mu_i}}(t) + \gamma_i(t), \tag{15}$$

$$\int_0^\infty \gamma_i^{\frac{2\mu_i}{\mu_i+1}}(s) a^{\frac{1-\mu_i}{\mu_i+1}}(s) ds < \infty, \tag{16}$$

$$\max_{i=1,\dots,n} \left\{ \kappa \frac{\lambda_i(\mu_i - 1)}{4} + \frac{\tilde{\lambda} k_i}{2} \right\} = q < 1. \tag{17}$$

Remark 2. *Condition (17) holds if the coefficients k_κ in (15) are small enough. This means that the every stochastic part of the noise σ_i in the system (10) can depend on $x_\kappa^{\frac{\mu_\kappa}{2} + \frac{\mu_\kappa}{2\mu_i}}(t)$, provided coefficients are small enough.*

Theorem 1. *Let conditions (4)-(5), (13)-(17) be fulfilled. Then $P\left\{\lim_{t\to\infty} \|X(t)\|^2 = 0\right\} = 1$ for any solution $X(t)$ of system (10).*

Proof. We define Lyapunov function V by the formula:

$$V(X) = \sum_{k=1}^{n} \frac{\lambda_k x_k^{\mu_k+1}}{\mu_k + 1}.$$

It is easy to see that

$$\frac{\partial V}{\partial x_i} = \lambda_i x_i^{\mu_i}, \quad \frac{\partial^2 V}{\partial x_i^2} = \lambda_i \mu_i x_i^{\mu_i - 1}, \quad \frac{\partial^2 V}{\partial x_i \partial x_j} = 0, \quad i \neq j. \tag{18}$$

We apply Ito formula (7) to Lyapunov function $V(X(t))$ along the trajectory of solution $X(t)$ to the system (10) and obtain

$$\begin{aligned} V(X(t)) &= V(X_0) + \int_0^t \sum_{i=1}^{n} \lambda_i x_i^{\mu_i} \left(\sum_{k=1}^{n} p_{ki}(t) x_k^{\mu_k}(t) + f_i(t, X(t)) \right) dt \\ &+ \frac{1}{2} \int_0^t \sum_{i=1}^{n} \lambda_i \mu_i x_i^{\mu_i - 1} \sigma_i^2(t, X(t)) dt + m_t. \end{aligned} \tag{19}$$

Here

$$m_t = \int_0^t \sum_{i=1}^n \lambda_i x_i^{\mu_i}(t)\sigma_i(t, X(t))dw_t \tag{20}$$

is a martingale.

Now we are going to estimate three expressions in the right-hand side of (19):

$$\sum_{i=1}^n \lambda_i x_i^{\mu_i} \sum_{k=1}^n p_{ki}(t) x_k^{\mu_k}(t), \quad \sum_{i=1}^n \lambda_i x_i^{\mu_i}(t) f_i(t, X(t)) \quad \text{and} \quad \sum_{i=1}^n \lambda_i \mu_i x_i^{\mu_i - 1}(t)\sigma_i^2(t, X(t)).$$

Using (4) we obtain the estimation of the first expression

$$\sum_{i=1}^n \lambda_i x_i^{\mu_i} \sum_{k=1}^n p_{ki}(t) x_k^{\mu_k}(t) = (X)_\mu^T \Lambda P(t)(X)_\mu \le -a(t)\|(X)_\mu\|^2 = -a(t)\left(\sum_{i=1}^n x_i^{2\mu_i}\right).$$

For estimating of the second expression we note that for every $\varepsilon > 0$

$$x_i^{\mu_i}(t)\beta_i(t) = x_i^{\mu_i}(t)[\varepsilon a(t)]^{\frac{1}{2}}\beta_i(t)[a(t)\varepsilon]^{-\frac{1}{2}} \le \frac{x_i^{2\mu_i}(t)\varepsilon a(t)}{2} + \frac{\beta_i^2(t)\varepsilon^{-1}a^{-1}(t)}{2}$$

Then

$$\sum_{i=1}^n \lambda_i x_i^{\mu_i}(t) f_i(t, X(t)) \le \sum_{i=1}^n \lambda_i x_i^{\mu_i}(t)\beta_i(t) \le \sum_{i=1}^n \lambda_i \frac{x_i^{2\mu_i}(t)\varepsilon a(t)}{2} + \sum_{i=1}^n \lambda_i \frac{\beta_i^2(t)\varepsilon^{-1}a^{-1}(t)}{2}.$$

Before estimating the third expression we need to do some calculations. Applying the inequality (9) for $r = \frac{2\mu_i}{\mu_i - 1}$ and $q = \frac{2\mu_i}{\mu_i + 1}$ we have

$$\begin{aligned}
x_i^{\mu_i - 1}(t)\gamma_i(t) &= x^{\mu_i - 1}(t)[\varepsilon a(t)]^{\frac{\mu_i - 1}{2\mu_i}}\gamma_i(t)[a(t)\varepsilon]^{\frac{1-\mu_i}{2\mu_i}} \\
&\le \frac{\left(x_i^{\mu_i - 1}(t)[\varepsilon a(t)]^{\frac{\mu_i - 1}{2\mu_i}}\right)^{\frac{2\mu_i}{\mu_i - 1}}}{\frac{2\mu_i}{\mu_i - 1}} + \frac{\left(\gamma_i(t)[\varepsilon a(t)]^{\frac{1-\mu_i}{2\mu_i}}\right)^{\frac{2\mu_i}{\mu_i + 1}}}{\frac{2\mu_i}{\mu_i + 1}} \\
&= \frac{\varepsilon a(t) x_i^{2\mu_i}(t)}{\frac{2\mu_i}{\mu_i - 1}} + \frac{\gamma_i^{\frac{2\mu_i}{\mu_i + 1}}(t)\varepsilon^{\frac{1-\mu_i}{\mu_i + 1}} a^{\frac{1-\mu_i}{\mu_i + 1}}(t)}{\frac{2\mu_i}{\mu_i + 1}}
\end{aligned} \tag{21}$$

and

$$x_i^{\mu_i - 1} x_j^{\mu_j + \frac{\mu_j}{\mu_i}} \le \frac{(\mu_i - 1)}{2\mu_i} x_i^{2\mu_i} + \frac{(\mu_i + 1)}{2\mu_i} x_j^{2\mu_j}.$$

Then

$$\sum_{i=1}^{n}\lambda_i\mu_i x_i^{\mu_i-1}(t)\sigma_i^2(t,X(t))\le\sum_{i=1}^{n}\lambda_i\mu_i x_i^{\mu_i-1}(t)\left(\sum_{\kappa=1}^{n}a(t)k_\kappa x_\kappa^{\mu_\kappa+\frac{\mu_\kappa}{\mu_i}}(t)+\gamma_i(t)\right)$$

$$\le\sum_{i=1}^{n}\left[\frac{\lambda_i}{2}a(t)\sum_{\kappa=1}^{n}k_\kappa\left((\mu_i-1)x_i^{2\mu_i}(t)+(\mu_i+1)x_\kappa^{2\mu_\kappa}(t)\right)+\frac{\varepsilon\lambda_i(\mu_i-1)a(t)x^{2\mu_i}(t)}{2}\right.$$

$$\left.+\frac{\lambda_i(\mu_i+1)\gamma_i^{\frac{2\mu_i}{\mu_i+1}}(t)\varepsilon^{\frac{1-\mu_i}{\mu_i+1}}a^{\frac{1-\mu_i}{\mu_i+1}}(t)}{2}\right]$$

$$\le a(t)\left[\sum_{\kappa=1}^{n}k_\kappa\sum_{i=1}^{n}\frac{\lambda_i(\mu_i-1)}{2}x_i^{2\mu_i}(t)+\sum_{i=1}^{n}\frac{\lambda_i(\mu_i+1)}{2}\sum_{\kappa=1}^{n}k_\kappa x_\kappa^{2\mu_\kappa}(t)\right]$$

$$+\sum_{i=1}^{n}\frac{\varepsilon\lambda_i(\mu_i-1)a(t)x^{2\mu_i}(t)}{2}+\sum_{i=1}^{n}\frac{\lambda_i(\mu_i+1)\gamma_i^{\frac{2\mu_i}{\mu_i+1}}(t)\varepsilon^{\frac{1-\mu_i}{\mu_i+1}}a^{\frac{1-\mu_i}{\mu_i+1}}(t)}{2}$$

$$\le a(t)\sum_{i=1}^{n}\left((\kappa+\varepsilon)\frac{\lambda_i(\mu_i-1)}{2}+\tilde\lambda k_i\right)x_i^{2\mu_i}(t)+2S_1(t),$$

where $\tilde\lambda$ and κ are defined in (12),

$$S_1(t)=\sum_{i=1}^{n}\frac{\lambda_i(\mu_i+1)\gamma_i^{\frac{2\mu_i}{\mu_i+1}}(t)\varepsilon^{\frac{1-\mu_i}{\mu_i+1}}a^{\frac{1-\mu_i}{\mu_i+1}}(t)}{4}. \tag{22}$$

Then

$$V(X(t))=V(X(0))-\int_0^t a(t)\sum_{i=1}^{n}\left(1-\frac{\varepsilon\lambda_i}{2}-(\kappa+\varepsilon)\frac{\lambda_i(\mu_i-1)}{4}-\frac{\tilde\lambda k_i}{2}\right)x_i^{2\mu_i}(t)dt$$

$$+\int_0^t S(\theta)d\theta+m(t)\le V(X(0))-\alpha_1\int_0^t a(t)\sum_{i=1}^{n}x_i^{2\mu_i}(t)dt+\int_0^t S(\theta)d\theta+m_t$$

$$=V(X_0)-\alpha_1\int_0^t a(\theta)\|(X)_\mu(\theta)\|^2d\theta+\int_0^t S(\theta)d\theta+m_t, \tag{23}$$

where

$$S(t)=S_1(t)+S_2(t), \tag{24}$$

$$S_2(t)=\frac{\lambda_i\beta_i^2(t)\varepsilon^{-1}a^{-1}(t)}{2} \tag{25}$$

$$\alpha_1=\min_{i=1,\dots,n}\left\{1-(\kappa+\varepsilon)\frac{\lambda_i(\mu_i-1)}{4}-\frac{\tilde\lambda k_i}{2}-\frac{\varepsilon\lambda_i}{2}\right\}, \tag{26}$$

$$\varepsilon\le\min_{i=1,\dots,n}\frac{4-\kappa\lambda_i(\mu_i-1)-2\tilde\lambda k_i}{\lambda_i(\mu_i-1)-2\lambda_i}=\min_{i=1,\dots,n}\frac{4(1-q)}{\lambda_i(\mu_i+1)}. \tag{27}$$

We note that due to condition (17) the right-hand side in (27) is positive.

After integration of (23) we obtain

$$W_t \le W_0 - A_t^{(2)} + A_t^{(1)} + m_t \tag{28}$$

with

$$A_t^{(2)} = \int_0^t \alpha_1 a(\theta)\|(X)_\mu(\theta)\|^2 d\theta, \quad A_t^{(1)} = \int_0^t S(\theta)d\theta.$$

From Lemma 2 we get

$$P\{A_\infty^{(1)} < \infty\} \subset P\{V \to\} \cap P\{A_\infty^{(2)} < \infty\}. \tag{29}$$

Conditions (14) and (16) imply $P\{A_\infty^{(1)} < \infty\}$, and (29) implies $P\{V \to\} = 1$. It means in particular that $P\{|x_i(t)| \le K\} = 1$ for some *a.s.* finite $K = K(\omega) > 0$, for every $i = 1, \dots, n$ and for every $t > 0$.

To prove the theorem it is sufficient to show that $P\left\{\lim_{t\to\infty} V(t) = 0\right\} = 1$. Suppose the opposite: $P\left\{\lim_{t\to\infty} V(t) = \zeta_0(\omega) > 0\right\} = p_0 > 0$ for some $\zeta_0 = \zeta_0(\omega) > 0$. Then there exists *a.s.* finite $N = N(\omega)$ such that $P\{\Omega_1\} = p_0 > 0$, where $\Omega_1 = \{V(t) \ge \zeta_0(\omega)/2,\ t > N(\omega)\}$. We note that

$$\begin{aligned}
x_i^{\mu_i+1} &= \left(x_i^{2\mu_i}\right)^{\frac{1}{2}} x_i \le \left(x_i^{2\mu_i}\right)^{\frac{1}{2}} K, \\
V(t) &= \sum_{i=1}^n \frac{\lambda_i}{\mu_i+1} x_i^{\mu_i+1}(t) \le \sum_{i=1}^n \frac{\lambda_i}{\mu_i+1} K \left(x_i^{2\mu_i}(t)\right)^{\frac{1}{2}} \le K_1 \sum_{i=1}^n \left(x_i^{2\mu_i}(t)\right)^{\frac{1}{2}} \\
&\le K_2 \left(\sum_{i=1}^n x_i^{2\mu_i}(t)\right)^{\frac{1}{2}} = K_2\|(X)_\mu(t)\|.
\end{aligned} \tag{30}$$

Therefore

$$P\left\{\|(X)_\mu(t)\|^2 \ge \frac{\zeta_0^2}{4K_2^2}\right\} = p_0 > 0 \tag{31}$$

for $t > T(\omega)$ and for $\omega \in \Omega_1$

$$\begin{aligned}
\int_0^n a(t)\|(X)_\mu(s)\|^2 ds &= \int_0^N + \int_N^n \\
&\ge \int_N^n a(s)\|(X)_\mu(s)\|^2 ds \ge \frac{\zeta_0^2}{4K_2}\int_N^n a(s)ds \to \infty
\end{aligned}$$

as $n \to \infty$ due to condition (5). Hence $P\{A_\infty^{(2)} = \infty\} \ge p_0 > 0$. This contradicts (29) proving that $P\left\{\lim_{t\to\infty} V(t) = 0\right\} = 1$. The last implies that $P\left\{\lim_{t\to\infty} \|X(t)\| = 0\right\} = 1$. ∎

3 Systems with discrete delay

In this section we suppose that functions $f_i\left(t,[X]_t^{t-h}\right)$ and $\sigma_i\left(t,[X]_t^{t-h}\right)$ depend on the values of X at points $t-h_{jk}$, with $h_{jk}\geq 0$, $j=1,\dots m$, $k=1,\dots n$. Without loss of generality we can also suppose that the $a(t)$ from condition (5) is non-increasing and is bounded by some constant $\hat{K}$: $a(t)\leq \hat{K}$.

We assume that $g_{jk}\geq 0$, $j=1,\dots,m$, $k=1,\dots,n$ and instead of (13) and (15) for every $i=1,\dots,n$ the following conditions are fulfilled:

$$\left|f_i\left(t,[X]_t^{t-h}\right)\right| \leq a(t)\sum_{j=1}^{m}\sum_{\kappa=1}^{n} g_{j\kappa}|x_\kappa^{\mu_\kappa}(t-h_{j\kappa})|+\beta_i(t), \tag{32}$$

$$\sigma_i^2\left(t,[X]_t^{t-h}\right) \leq a(t)\sum_{\kappa=1}^{n} k_\kappa x_\kappa^{\mu_\kappa+\frac{\mu_\kappa}{\mu_i}}(t) + a(t)\sum_{j=1}^{m}\sum_{\kappa=1}^{n} g_{j\kappa}x_\kappa^{\mu_\kappa+\frac{\mu_\kappa}{\mu_i}}(t-h_{j\kappa})+\gamma_i(t). \tag{33}$$

We put

$$\bar{g}=\sum_{j=1}^{m}\sum_{\kappa=1}^{n} g_{j\kappa},\quad \hat{\lambda}=\frac{\sum_{i=1}^{n}\lambda_i}{2},\quad \bar{g}_\kappa=\sum_{j=1}^{n} g_{j\kappa}, \tag{34}$$

and suppose that

$$\frac{(\bar{\kappa}+\bar{g})\lambda_i(\mu_i-1)+2\lambda_i\bar{g}}{4}+\tilde{\lambda}k_i/2+(\tilde{\lambda}+\hat{\lambda})\bar{g}_i=q<1. \tag{35}$$

Remark 3. *Condition (35) is fulfilled if coefficients k_κ and $g_{j\kappa}, j=1,\dots,m$, $\kappa=1,\dots,n$, in (32)-(33) are small enough numbers. It means in particular that stochastic noise in the system (3) can be dependent on $x_\kappa^{\frac{\mu_\kappa}{2}+\frac{\mu_\kappa}{2\mu_i}}$, at point t as well as at previous points $t-h_{j\kappa}$, with small enough coefficients. The deterministic part of noise can also be dependent on $x_\kappa^{\frac{\mu_\kappa}{2}}(t-h_{j\kappa})$ with small enough coefficients.*

Theorem 2. *Let conditions (4), (5), (14), (16), (32), (33), (35) be fulfilled. Then $P\left\{\lim_{t\to\infty}\|X(t)\|^2=0\right\}=1$ for any solution $X(t)$ of system (3).*

Proof. The right-hand side of the system (3) depends on the previous states of solution X. In this case we have to use Lyapunov functional instead of Lyapunov function:

$$V=V_1+V_2, \tag{36}$$

$$V_1=\sum_{k=1}^{n}\frac{\lambda_k x_k^{\mu_k+1}}{\mu_k+1}, \tag{37}$$

$$V_2=\left(\tilde{\lambda}+\hat{\lambda}\right)\sum_{j=1}^{m}\sum_{\kappa=1}^{n} g_{j\kappa}\int_{t-h_{j\kappa}}^{t} a(s)x_\kappa^{2\mu_\kappa}(s)ds. \tag{38}$$

In Ito formula decomposition (17) for functional $V(X(t))$, defined by (36),

$$V(X(t)) = V(X_0) + \int_0^t \sum_{i=1}^n \lambda_i x_i^{\mu_i} \left(\sum_{k=1}^n p_{ki}(t) x_k^{\mu_k}(t) + f_i\left(t, [X]_t^{t-h}\right)\right) dt + \int_0^t \frac{\partial V_2}{\partial t} dt$$
$$+ \frac{1}{2} \int_0^t \sum_{i=1}^n \lambda_i \mu_i x_i^{\mu_i - 1} \sigma_i^2 \left(t, [X]_t^{t-h}\right) dt + m(t), \tag{39}$$

we have to estimate expressions

$$\sum_{i=1}^n \lambda_i x_i^{\mu_i}(t) f_i\left(t, [X]_t^{t-h}\right) \quad \text{and} \quad \sum_{i=1}^n \lambda_i \mu_i x_i^{\mu_i - 1}(t) \sigma_i^2\left(t, [X]_t^{t-h}\right),$$

and also to compute $\frac{\partial V_2}{\partial t}$. We note that

$$x_i^{\mu_i - 1}(t) x_\kappa^{\mu_\kappa + \frac{\mu_\kappa}{\mu_i}}(t - h_{j\kappa}) \le \frac{(\mu_i - 1)}{2\mu_i} x_i^{2\mu_i}(t) + \frac{(\mu_i + 1)}{2\mu_i} x_\kappa^{2\mu_\kappa}(t - h_{j\kappa})$$

and

$$x_i^{\mu_i}(t) x_\kappa^{\mu_\kappa}(t - h_{j\kappa}) \le \frac{1}{2}\left(x_i^{2\mu_i}(t) + x_\kappa^{2\mu_\kappa}(t - h_{j\kappa})\right).$$

For estimation of the first expression we do the following:

$$\sum_{i=1}^n \lambda_i x_i^{\mu_i} f_i\left(t, [X]_t^{t-h}\right) \le \sum_{i=1}^n \lambda_i x_i^{\mu_i}(t) \left(a(t) \sum_{j=1}^m \sum_{\kappa=1}^n g_{j\kappa} |x_\kappa^{\mu_\kappa}(t - h_{j\kappa})| + \beta_i(t) \right)$$
$$\le a(t) \sum_{i=1}^n \frac{\lambda_i}{2} \left(\sum_{j=1}^m \sum_{\kappa=1}^n g_{j\kappa} \left(x_i^{2\mu_i}(t) + x_\kappa^{2\mu_\kappa}(t - h_{j\kappa}) \right) \right) + \sum_{i=1}^n \lambda_i x_i^{\mu_i} \beta_i(t)$$
$$\le \frac{a(t)\bar{g}}{2} \sum_{i=1}^n \lambda_i x_i^{2\mu_i}(t) + a(t)\hat{\lambda} \sum_{j=1}^m \sum_{\kappa=1}^n g_{j\kappa} x_\kappa^{2\mu_\kappa}(t - h_{j\kappa})$$
$$+ \sum_{i=1}^n \frac{\lambda_i x_t^{2\mu_i} \varepsilon a(t)}{2} + \sum_{i=1}^n \frac{\lambda_i \beta_i^2(t) \varepsilon^{-1} a^{-1}(t)}{2}. \tag{40}$$

For estimation of the second expression we do the following

$$\sum_{i=1}^{n} \lambda_i \mu_i x_i^{\mu_i - 1}(t) \sigma_i^2 \left(t, [X]_t^{t-h}\right) \le$$

$$\sum_{i=1}^{n} \lambda_i \mu_i x_i^{\mu_i - 1}(t) \left(\sum_{\kappa=1}^{n} a(t) k_\kappa x_\kappa^{\mu_\kappa(t) + \frac{\mu_\kappa}{\mu_i}} + a(t) \sum_{j=1}^{m} \sum_{\kappa=1}^{n} g_{j\kappa} x_\kappa^{\mu_\kappa(t) + \frac{\mu_\kappa}{\mu_i}} (t - h_{j\kappa}) + \gamma_i(t) \right)$$

$$\le \sum_{i=1}^{n} \left[\frac{\lambda_i}{2} a(t) \sum_{\kappa=1}^{n} k_\kappa \left((\mu_i - 1) x_i^{2\mu_i}(t) + (\mu_i + 1) x_\kappa^{2\mu_\kappa}(t) \right) \right.$$

$$+ \frac{\lambda_i}{2} a(t) \sum_{j=1}^{m} \sum_{\kappa=1}^{n} g_{j\kappa} \left((\mu_i - 1) x_i^{2\mu_i}(t) + (\mu_i + 1) x_\kappa^{2\mu_\kappa}(t - h_{j\kappa}) \right)$$

$$\left. + \frac{\varepsilon \lambda_i (\mu_i - 1) a(t) x^{2\mu_i}(t)}{2} + \frac{\lambda_i (\mu_i + 1) \gamma_i^{\frac{2\mu_i}{\mu_i + 1}}(t) \varepsilon^{\frac{1-\mu_i}{\mu_i + 1}} a^{\frac{1-\mu_i}{\mu_i + 1}}(t)}{2} \right]$$

$$\le a(t) \left[\sum_{\kappa=1}^{n} k_\kappa \sum_{i=1}^{n} \frac{\lambda_i (\mu_i - 1)}{2} x_i^{2\mu_i}(t) + \sum_{i=1}^{n} \frac{\lambda_i (\mu_i + 1)}{2} \sum_{\kappa=1}^{n} k_\kappa x_\kappa^{2\mu_\kappa}(t) \right.$$

$$\left. + \sum_{i=1}^{n} \frac{\lambda_i}{2} \sum_{j=1}^{m} \sum_{\kappa=1}^{n} g_{j\kappa} (\mu_i - 1) x_i^{2\mu_i}(t) + \sum_{i=1}^{n} \frac{\varepsilon \lambda_i (\mu_i - 1) x^{2\mu_i}(t)}{2} \right]$$

$$+ \sum_{i=1}^{n} \frac{\lambda_i}{2} a(t) \sum_{j=1}^{m} \sum_{\kappa=1}^{n} g_{j\kappa} (\mu_i + 1) x_\kappa^{2\mu_\kappa}(t - h_{j\kappa})$$

$$+ \sum_{i=1}^{n} \frac{\lambda_i (\mu_i + 1) \gamma_i^{\frac{2\mu_i}{\mu_i + 1}}(t) \varepsilon^{\frac{1-\mu_i}{\mu_i + 1}} a^{\frac{1-\mu_i}{\mu_i + 1}}(t)}{2}$$

$$\le a(t) \sum_{i=1}^{n} \left((\bar{\kappa} + \tilde{g} + \varepsilon) \frac{\lambda_i (\mu_i - 1)}{2} + \tilde{\lambda} k_i \right) x_i^{2\mu_i}(t)$$

$$+ \tilde{\lambda} a(t) \sum_{j=1}^{m} \sum_{\kappa=1}^{n} g_{j\kappa} x_\kappa^{2\mu_\kappa}(t - h_{j\kappa}) + 2 S_1(t), \tag{41}$$

where $\tilde{\lambda}$ and κ are defined in (12), $\tilde{g}$ in (34), $S_1(t)$ in (22). Further,

$$\frac{\partial V_2}{\partial t} = \left(\bar{\lambda} + \hat{\lambda} \right) \sum_{j=1}^{m} \sum_{\kappa=1}^{n} g_{j\kappa} a(t) x_\kappa^{2\mu_\kappa}(t)$$

$$- \left(\bar{\lambda} + \hat{\lambda} \right) \sum_{j=1}^{m} \sum_{\kappa=1}^{n} g_{j\kappa} a(t - h_{j\kappa}) x_\kappa^{2\mu_\kappa}(t - h_{j\kappa})$$

$$\le \left(\bar{\lambda} + \hat{\lambda} \right) a(t) \sum_{\kappa=1}^{n} \bar{g}_\kappa x_\kappa^{2\mu_\kappa}(t) - \left(\bar{\lambda} + \hat{\lambda} \right) a(t) \sum_{j=1}^{m} \sum_{\kappa=1}^{n} g_{j\kappa} x_\kappa^{2\mu_\kappa}(t - h_{j\kappa}) \tag{42}$$

as $a(t-h) > a(t)$. Here $\bar g_\kappa$ and $\hat\lambda$ are defined in (34). Let

$$\begin{aligned}\varepsilon &= \min_{i=1,\dots,n}\left\{\frac{4-(\bar\kappa+\bar g)\lambda_i(\mu_i-1)-2\tilde\lambda k_i-2\lambda_i\bar g-4(\tilde\lambda+\hat\lambda)\bar g_i}{\lambda_i(\mu_i+1)}\right\}\\ &= \min_{i=1,\dots,n}\frac{4(1-q)}{\lambda_i(\mu_i+1)}, \qquad (43)\end{aligned}$$

$$\alpha_1 = \min_{i=1,\dots,n}\left\{1-(\bar\kappa+\bar g+\varepsilon)\frac{\lambda_i(\mu_i-1)}{4}-\tilde\lambda k_i/2-\frac{\lambda_i(\bar g+\varepsilon)}{2}-(\tilde\lambda+\hat\lambda)\bar g_i\right\}. \qquad (44)$$

Substituting (40)-(44) in (19) we obtain (23). From Lemma 2 we get (29), and then due to conditions (14), (16) obtain that $P\{V\to\}=1$. From the definition of functional V, (36)-(38), we see that there exists $H=H(\omega)<\infty$ *a.s.* such that $P\{\sup_{t>0}V_1(t)<H\}=1$ and $P\{\sup_{t>0}V_2(t)<H\}=1$. It implies in particular that $P\{|x_i(t)|\le K\}=1$ for some *a.s.* finite $K=K(\omega)>0$ and for every $i=1,\dots,n,\dots$.

By our assumptions $a(t)$ is non-increasing function. Then we have two opportunities:

a) $a(t)\to 0$ as $t\to\infty$,

b) $a(t)\ge c$ for some $c>0$.

In case a)

$$\begin{aligned}|V_2(t)| &\le \left(\tilde\lambda+\hat\lambda\right)\sum_{j=1}^m\sum_{\kappa=1}^n g_{j\kappa}K^{2\mu_\kappa}\int_{t-h_{j\kappa}}^t a(\tau)d\tau\\ &\le \left(\tilde\lambda+\hat\lambda\right)\sum_{j=1}^m\sum_{\kappa=1}^n g_{j\kappa}K^{2\mu_\kappa}a(t-h_{j\kappa})h_{j\kappa}\to 0 \qquad (45)\end{aligned}$$

when $t\to 0$. Then $\sum_{k=1}^n\frac{\lambda_k x_k^{\mu_k+1}}{\mu_k+1}=V_1(t)=V(t)-V_2(t)$ has to converge also, and we prove that $P\left\{\lim_{t\to\infty}\|X_t\|=0\right\}=1$ in the same way as in the proof of Theorem 1.

In case b) we note that V is *a.s.* uniformly continuous on $[0,\infty)$. It is easy to see that $V_2(t)$ is also *a.s.* uniformly continuous on $[0,\infty)$. Really, for $t\le\theta$ we have

$$\begin{aligned}&|V_2(t)-V_2(\theta)| =\\ &= \left(\tilde\lambda+\hat\lambda\right)\sum_{j=1}^m\sum_{\kappa=1}^n g_{j\kappa}\left|\int_{\theta-h_{j\kappa}}^{\theta} a(\tau)x_\kappa^{2\mu_\kappa}(\tau)d\tau-\int_{t-h_{j\kappa}}^t a(\tau)x_\kappa^{2\mu_\kappa}(\tau)d\tau\right|\\ &= \left(\tilde\lambda+\hat\lambda\right)\sum_{j=1}^m\sum_{\kappa=1}^n g_{j\kappa}\left|\int_0^{\theta}-\int_0^{\theta-h_{j\kappa}}-\int_0^t+\int_0^{t-h_{j\kappa}}\right|\\ &= \left(\tilde\lambda+\hat\lambda\right)\sum_{j=1}^m\sum_{\kappa=1}^n g_{j\kappa}\left|\int_t^{\theta} a(\tau)x_\kappa^{2\mu_\kappa}(\tau)d\tau-\int_{t-h_{j\kappa}}^{\theta-h_{j\kappa}} a(\tau)x_\kappa^{2\mu_\kappa}(\tau)d\tau\right|\\ &\le \left(\tilde\lambda+\hat\lambda\right)\sum_{j=1}^m\sum_{\kappa=1}^n g_{j\kappa}K^{2\mu_\kappa}\left(\left|\int_t^{\theta}a(\tau)d\tau\right|+\left|\int_{t-h_{jk}}^{\theta-h_{jk}}a(\tau)d\tau\right|\right)\le K_2|t-\theta|. \qquad (46)\end{aligned}$$

Then $V_1(t)$ has also to be *a.s.* uniform continuous on $[0,\infty)$. To prove that $P\left\{\limsup_{t\to\infty} V_1(t)=0\right\}=1$ we suppose the opposite: $P\left\{\limsup_{t\to\infty} V_1(t)=\zeta_0(\omega)>0\right\}$ $=p_0>0$ for some $\zeta_0=\zeta_0(\omega)>0$. Therefore there exists sequence $t_k=t_k(\omega)$ such that $P(\Omega_1)=p_0$, where $\Omega_1=\{\omega\colon V_1(t_k)(\omega)>\zeta_0(\omega)/2>0\}$. Due to *a.s.* uniformly continuity of $V_1(t)$ on $[0,\infty)$ for $\varepsilon=\varepsilon(\omega)=\zeta_0(\omega)/4$ we can find $\delta=\delta(\omega)$ such that

$$|V_1(t_k)-V_1(s)|\le\varepsilon=\zeta_0(\omega)/4$$

when $\omega\in\Omega_1$ and $|s-t_k|\le\delta$. Then for $\omega\in\Omega_1$ and $s\in[t_k-\delta,t_k+\delta]$ we have

$$|V_1(s)|\ge|V_1(t_k)|-|V_1(t_k)-V_1(s)|\ge\zeta_0(\omega)/2-\zeta_0(\omega)/4=\zeta_0(\omega)/4. \tag{47}$$

Let $k(n)$ be a number of elements of the sequence $\{t_k\}$ in the interval $[0,n]$. Applying inequality (47) and estimations (30), (31) we obtain for $\omega\in\Omega_1$

$$\int_0^n\|(X)_\mu(s)\|^2a(s)ds\ge\sum_{k:\,t_k+\delta\le n}\int_{t_k-\delta}^{t_k+\delta}\|(X)_\mu(s)\|^2a(s)ds\ge$$

$$\ge\frac{\zeta_0^2}{4K_2^2}\sum_{k:\,t_k+\delta\le n}\int_{t_k-\delta}^{t_k+\delta}cds=\frac{2c\delta\zeta_0^2}{4K_2^2}\sum_{k\le k(n)}1=\frac{k(n)c\delta\zeta_0^2}{2K_2^2}\to\infty$$

as $n\to\infty$, because $k(n)\to\infty$ as $n\to\infty$. Hence $P\{A_\infty^{(2)}=\infty\}\ge p_0>0$. This contradicts (29) proving the result. ■

4 Systems with general delays

In this section we prove two theorems on asymptotic stability of the trivial solution to the system (3), when dependence of the right-hand side of the equations on the past states of solution X_t has a general form, which includes the discrete as well as the distributed delays (see (6) in the Introduction). In the first theorem the asymptotic stability is proved when coefficients in the noise part are small. In the second one our assumptions are expressed in terms of the convergence of an integral on the infinite time interval $[0,\infty)$ of the function which is involved in the estimation. In practice, the infinite time interval is of no interest, and any function restricted to a finite time interval can be extended to $[0,\infty)$ in such a way, that the corresponding integral converges (see, for example [8, 9]). Of course the question of the speed of such convergence remains open. In this sense the second result can be regarded as more general than the first.

In the following two paragraphs we suppose that the function $R(s)$ in (6) is continuous and non-decreasing for $s\in[0,h]$, $h>0$, the function $\alpha(t)$ is positive and non-increasing for $t\in(0,\infty)$, $0\le\Delta_j(t)\le h$ and $1-\Delta_j'(t)\ge\varepsilon_j>0$ for $t\ge0$,

$j = 1, 2, \ldots$. We put

$$\tilde{R}(t) = R(t) + l(t), \quad l(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0, \end{cases}$$

$$\mathbf{e} := \sum_{j=1}^{\infty} e_j/\varepsilon_j < \infty, \quad \mathbf{R} := R(\infty) - R(0) + 1.$$

4.1 Small parameter

In this paragraph we consider the following system of stochastic equations

$$dx_i(t) = \sum_{k=1}^{n} \left(p_{ki}(t) x_k^{\mu_k} + \hat{f}_i\left(t, [X]_t^{t-h}\right)\right) dt + \hat{\sigma}_i\left(t, [X]_t^{t-h}\right) dw_t. \tag{48}$$

As in the previous section we suppose that $a(t)$ from condition (5) is non-increasing and bounded by some constant $\hat{K}$: $a(t) \leq \hat{K}$. The following conditions will be referred to below:

$$\left|\hat{f}_i\left(t, [X]_t^{t-h}\right)\right| \leq a(t)\nu \left[\int_0^h \|(X)_\mu(t-s)\|^2 dR(s) + \sum_{j=1}^{\infty} e_j \|(X)_\mu(t - \Delta_j(t))\|^2\right]^{\frac{1}{2}}, \tag{49}$$

$$\left|\hat{\sigma}_i\left(t, [X]_t^{t-h}\right)\right|^2 \leq a(t)\nu \left[\int_0^\infty \|(X)_\mu(t-s)\|^2 dR(s) + \sum_{j=1}^{h} e_j \|(X)_\mu(t - \Delta_j(t))\|^2\right]^{\frac{\mu_i+1}{2\mu_i}}, \tag{50}$$

$$(\tilde{\lambda} + \hat{\lambda})\nu(\mathbf{R} + \mathbf{e}) < 1, \tag{51}$$

where ν is some parameter, $(X)_\mu$ is defined in (11), $\tilde{\lambda}$ and $\hat{\lambda}$ are defined in (12) and (34) respectively.

Theorem 3. *Let conditions (4), (5), (49), (50), (51) be fulfilled. Then* $P\left\{\lim_{t\to\infty} \|X(t)\|^2 = 0\right\} = 1$ *for any solution* $X(t)$ *of Eqn.(48).*

Proof. Let we define Lyapunov functional V by the formula:

$$V = V_1 + V_3, \tag{52}$$

$$V_3 = (\tilde{\lambda} + \hat{\lambda})\nu \int_0^h d\tilde{R}(s) \int_{t-s}^{t} \|(X)_\mu(\tau)\|^2 a(\tau) d\tau$$

$$+ (\tilde{\lambda} + \hat{\lambda})\nu \sum_{j=1}^{\infty} e_j/\varepsilon_j \int_{t-\Delta_j(t)}^{t} \|(X)_\mu(\tau)\|^2 a(\tau) d\tau, \quad t \in (0, \infty), \tag{53}$$

where V_1 is defined in (37). Applying the Ito formula we obtain

$$V(X(t)) = V(X_0) + \int_0^t \sum_{i=1}^n \lambda_i x_i^{\mu_i} \left(\sum_{k=1}^n p_{ki}(t) x_k^{\mu_k}(t) + \hat{f}_i(t, X(t)) \right) dt + \int_0^t \frac{\partial V_3}{\partial t} dt$$
$$+ \frac{1}{2} \int_0^t \sum_{i=1}^n \lambda_i \mu_i x_i^{\mu_i - 1} \hat{\sigma}_i^2(t, X(t)) dt + m_t. \tag{54}$$

We need to estimate some terms in the right-hand side of (54). Before doing this we note that

$$x_k^{\mu_k}(t) = \left(x_k^{2\mu_k}(t)\right)^{\frac{1}{2}} \le \left(\int_0^h x_k^{2\mu_k}(t-s) dl(s) \right)^{\frac{1}{2}} \le \left(\int_0^h x^{2\mu_k}(t-s) d\tilde{R}(s) \right)^{\frac{1}{2}}$$
$$\le \left(\int_0^h \|(X)_\mu(t-s)\|^2 d\tilde{R}(s) + \sum_{j=1}^\infty e_j \|(X)_\mu(t-\Delta_j(t))\|^2 \right)^{\frac{1}{2}}. \tag{55}$$

And in a similar way,

$$x_k^{\mu_k - 1}(t) = \left(x_k^{2\mu_k}(t)\right)^{\frac{\mu_k - 1}{2\mu_k}}$$
$$\le \left(\int_0^h \|(X)_\mu(t-s)\|^2 d\tilde{R}(s) + \sum_{j=1}^\infty e_j \|(X)_\mu(t-\Delta_j(t))\|^2 \right)^{\frac{\mu_k - 1}{2\mu_k}}. \tag{56}$$

Then using (49), (50), (55) and (56) we obtain

$$\sum_{i=1}^n \lambda_i x_i^{\mu_i} \hat{f}_i\left(t, [X]_t^{t-h}\right)$$
$$\le 2\hat{\lambda}\nu a(t) \left(\int_0^\infty \|(X)_\mu(t-s)\|^2 d\tilde{R}(s) + \sum_{j=1}^\infty e_j \|(X)_\mu(t-\Delta_j(t))\|^2 \right),$$
$$\sum_{i=1}^n \lambda_i \mu_i x_i^{\mu_i - 1}(t) \hat{\sigma}_i^2(t, X_t)$$
$$\le 2(\tilde{\lambda} - \hat{\lambda})\nu a(t) \left(\int_0^h \|(X)_\mu(t-s)\|^2 d\tilde{R}(s) + \sum_{j=1}^\infty e_j \|(X)_\mu(t-\Delta_j(t))\|^2 \right).$$

Since $a(t) \le a(s)$ for $t \ge s$ we get from the above

$$\sum_{i=1}^{n} \lambda_i x_i^{\mu_i} \hat{f}_i\left(t, [X]_t^{t-h}\right) + \frac{1}{2}\sum_{i=1}^{n} \lambda_i \mu_i x_i^{\mu_i - 1}(t) \hat{\sigma}_i^2\left(t, [X]_t^{t-h}\right) + \frac{\partial V_3}{\partial t}$$

$$\le (\tilde{\lambda} + \hat{\lambda})\nu \left(\int_0^h \|(X)_\mu(t-s)\|^2 a(t-s) d\tilde{R}(s) + \sum_{j=1}^{\infty} e_j \|(X)_\mu(t - \Delta_j(t))\|^2 a(t - \Delta_j(t)) \right)$$

$$+(\tilde{\lambda} + \hat{\lambda})\nu\, (\mathbf{R} + \mathbf{e})\, a(t) \|(X)_\mu(t)\|^2$$

$$-(\tilde{\lambda} + \hat{\lambda})\nu \left(\int_0^h \|(X)_\mu(t-s)\|^2 a(t-s) d\tilde{R}(s) + \sum_{j=1}^{\infty} e_j \|(X)_\mu(t - \Delta_j(t))\|^2 a(t - \Delta_j(t)) \right)$$

$$= (\tilde{\lambda} + \hat{\lambda})\nu\, (\mathbf{R} + \mathbf{e})\, a(t) \|(X)_\mu(t)\|^2 .$$

After substituting everything in (54) we obtain (23) with $\alpha_1 = 1 - \varepsilon$, $\varepsilon = (\tilde{\lambda} + \hat{\lambda})\nu\,(\mathbf{R} + \mathbf{e})$.

Now we proceed in the same way as in Theorem 2. The only difference is that instead of functional V_2 we consider V_3 and instead of estimations (45) and (46) we have for some constants $K_1, K_2, K_3 > 0$

$$|V_3(t)| \le \left(\tilde{\lambda} + \hat{\lambda}\right) \nu\, (\mathbf{R} + \mathbf{e})\, K_1 \left[\int_{t-h}^{t} a(\theta) d\tau + \int_{t-\Delta}^{t} a(\theta) d\tau \right]$$

$$\le K_2 \max\{a(t-h), a(t-\Delta)\} \to 0$$

when $t \to \infty$, and

$$|V_3(t) - V_3(\theta)| \le \left(\tilde{\lambda} + \hat{\lambda}\right) \int_0^h dR(s) \left| \int_{\theta - s}^{\theta} a(\tau) \|(X)_\mu(\tau)\|^2 d\tau - \int_{t-s}^{t} a(\tau) \|(X)_\mu(\tau)\|^2 d\tau \right|$$

$$+ \left(\tilde{\lambda} + \hat{\lambda}\right) \sum_{j=1}^{\infty} e_j / \varepsilon_j \left| \int_{\theta - \Delta_j(\theta)}^{\theta} a(\tau) \|(X)_\mu(\tau)\|^2 d\tau - \int_{t - \Delta_j(t)}^{t} a(\tau) \|(X)_\mu(\tau)\|^2 d\tau \right|$$

$$\left(\tilde{\lambda} + \hat{\lambda}\right) \nu\, (\mathbf{R} + \mathbf{e})\, K_1 \left(\left| \int_t^{\theta} a(\tau) d\tau \right| + \left| \int_{t - \Delta_j(t)}^{\theta - \Delta_j(\theta)} a(\tau) d\tau \right| \right) \le K_3 |t - \theta| .$$

■

4.2 Integrable parameter

We consider the system of stochastic equations

$$\begin{aligned} dx_i(t) &= \sum_{k=1}^{n} \left(p_{ki}(t) x_k^{\mu_k}(t) + f_i\left(t, [X]_t^{t-h}\right) + \hat{f}_i\left(t, [X]_t^{t-h}\right) \right) dt \\ &+ \left(\sigma_i\left(t, [X]_t^{t-h}\right) + \hat{\sigma}_i\left(t, [X]_t^{t-h}\right) \right) dw_t . \end{aligned} \tag{57}$$

The following conditions will be referred to below:

$$\left|\hat{f}_i\left(t,[X]_t^{t-h}\right)\right|^{\mu_i+1} \leq \alpha^{\mu_i+1}(t)\left[\int_0^h V_1(t-s)dR(s)+\sum_{j=1}^{\infty} e_j V_1(t-\Delta_j(t))\right] \quad (58)$$

$$\left|\hat{\sigma}_i\left(t,[X]_t^{t-h}\right)\right|^{\mu_i+1} \leq \alpha^{\frac{\mu_i+1}{2}}(t)\left[\int_0^h V_1(t-s)dR(s)+\sum_{j=1}^{\infty} e_j V_1(t-\Delta_j(t))\right] \quad (59)$$

$$\int_0^{\infty} \alpha(s)ds<\infty, \quad (60)$$

where V_1 is defined in (37).

Theorem 4. *Let conditions (4), (5), (14), (16), (32), (33), (35), (58), (59), (60) be fulfilled. Then* $P\left\{\lim_{t\to\infty} \|X(t)\|^2=0\right\}=1$ *for any solution* $X(t)$ *of Eqn.(57).*

Proof. We define Lyapunov-Krasovskii functional V by the formula:

$$W=\ln(V+1), \quad V=V_1+V_2+V_3, \quad (61)$$

$$(V_3)_t = H_3\int_0^h d\tilde{R}(s)\int_{t-s}^t \alpha(\tau)V_1(\tau)d\tau$$

$$+H_3\sum_{j=1}^{\infty} e_j/\varepsilon_j \int_{t-\Delta_j(t)}^t \alpha(\tau)V_1(\tau)d\tau, \quad t\in(0,\infty), \quad (62)$$

where V_1 and V_2 are defined in (37) and (38) respectively, H_3 is some constant which will be specified below. We note that

$$\frac{\partial W}{\partial x_i}=\frac{\lambda_i x_i^{\mu_i}}{V+1}, \quad \frac{\partial^2 W}{\partial x_i^2}=\frac{\lambda_i\mu_i x_i^{\mu_i-1}(V+1)-(\lambda_i x_i^{\mu_i})^2}{(V+1)^2}\leq\frac{\lambda_i\mu_i x_i^{\mu_i-1}}{V+1},$$

$$\frac{\partial^2 W}{\partial x_i\partial x_j}=\frac{0\times(V+1)-\lambda_i x_i^{\mu_i}\lambda_j x_j^{\mu_j}}{(V+1)^2}=\frac{-\lambda_i x_i^{\mu_i}\lambda_j x_j^{\mu_j}}{(V+1)^2}, \quad i\neq j. \quad (63)$$

We apply Ito formula to the functional $W(X(t))$ and get

$$W(X(t)) = W(X_0)+\int_0^t (V+1)^{-1}\left[\sum_{i=1}^n \lambda_i x_i^{\mu_i}\left(\sum_{k=1}^n p_{ki}(t)x_k^{\mu_k}(t)+f_i+\hat{f}_i\right)\right.$$

$$\left.+\frac{\partial V_2}{\partial t}+\frac{\partial V_3}{\partial t}\right]dt+\frac{1}{2}\int_0^t\sum_{i,j=1}^n \frac{\partial^2 W}{\partial x_i\partial x_j}(\sigma_i+\hat{\sigma}_i)(\sigma_j+\hat{\sigma}_j)\,dt+m_t, \quad (64)$$

where

$$m_t=\int_0^t (V+1)^{-1}\sum_{i=1}^n \lambda_i x_i^{\mu_i}(t)(\sigma_i+\hat{\sigma}_i)\,dw_t \quad (65)$$

is a martingale. We note that

$$x_k^{\mu_k}(t) = \left(x_k^{\mu_k+1}(t)\right)^{\frac{\mu_k}{\mu_k+1}} \le \left(\frac{\mu_k+1}{\lambda_k}\int_0^h \frac{\lambda_k}{\mu_k+1} x_k^{\mu_k+1}(t-s)d\tilde{R}(s)\right)^{\frac{\mu_k}{\mu_k+1}}$$

$$\le \left(\frac{\mu_k+1}{\lambda_k}\int_0^h V_1(t-s)d\tilde{R}(s) + \sum_{j=1}^{\infty} e_j V_1(t-\Delta_j(t))\right)^{\frac{\mu_k}{\mu_k+1}},$$

$$\sum_{i=1}^n \lambda_i x_i^{\mu_i} \hat{f}_i\left(t,[X]_t^{t-h}\right)$$

$$\le \sum_{i=1}^n \lambda_i \left(\frac{\mu_k+1}{\lambda_k}\right)^{\frac{\mu_k}{\mu_k+1}} \alpha(t)\left(\int_0^h V_1(t-s)d\tilde{R}(s) + \sum_{j=1}^{\infty} e_j V_1(t-\Delta_j(t))\right) \quad (66)$$

and

$$x_i^{\mu_i-1}(t)\hat{\sigma}_i^2\left(t,[X]_t^{t-h}\right)$$

$$\le \sum_{i=1}^n \left(\frac{\mu_k+1}{\lambda_k}\right)^{\frac{\mu_k-1}{\mu_k+1}} \alpha(t)\left(\int_0^h V_1(t-s)d\tilde{R}(s) + \sum_{j=1}^{\infty} e_j V_1(t-\Delta_j(t))|\right). \quad (67)$$

From the definition of V (see (61)) we have: $V \ge \frac{\lambda_k}{\mu_k+1}K_1 x_k^{\mu_k+1}$. Then for $i \ne j$ and some constant H_1, which depends only on μ_k, λ_k, $k = 1,\ldots,n$, we have

$$\frac{x_i^{\mu_i}(t)x_j^{\mu_j}(t)\sigma_i\sigma_j}{(V+1)^2} \le H_1\left(\frac{x_i^{2\mu_i}\sigma_i^2}{x_i^{\mu_i+1}(V+1)} + \frac{x_j^{2\mu_j}(t)\sigma_j^2}{x_j^{\mu_j+1}(V+1)}\right)$$

$$\le H_1\left(\frac{x_i^{\mu_i-1}(t)\sigma_i^2}{V+1} + \frac{x_j^{\mu_j-1}(t)\sigma_j^2}{V+1}\right).$$

The similar estimates are correct for $x_i^{\mu_i}(t)x_j^{\mu_j}(t)\hat{\sigma}_i\sigma_j$ and $x_i^{\mu_i}(t)x_j^{\mu_j}(t)\hat{\sigma}_i\hat{\sigma}_j$. Then

$$\frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 W}{\partial x_i \partial x_j}(\sigma_i+\hat{\sigma}_i)(\sigma_j+\hat{\sigma}_j) \le \sum_{i=1}^n H_2(V+1)^{-1}x_i^{\mu_i-1}\left(\sigma_i^2+\hat{\sigma}_i^2\right).$$

Using the monotone property of the function α we obtain

$$\sum_{i=1}^{n} \lambda_i x_i^{\mu_i} \hat{f}_i + H_2 \sum_{i=1}^{n} x_i^{\mu_i - 1}(t)\hat{\sigma}_i^2 + \frac{\partial V_3}{\partial t} \tag{68}$$

$$\leq H_3\alpha(t)\left(\int_0^h V_1(t-s)d\tilde{R}(s) + \sum_{j=1}^{\infty} e_j V_1(t-\Delta_j(t))\right) + H_3(\mathbf{R}+\mathbf{e})\alpha(t)V_1(t)$$

$$-H_3\left(\int_0^h V_1(t-s)\alpha(t-s)d\tilde{R}(s) + \sum_{j=1}^{\infty} e_j V_1(t-\Delta_j(t))\alpha(t-s)\right)$$

$$= H_3(\mathbf{R}+\mathbf{e})\alpha(t)V_1(t) \leq H_3(\mathbf{R}+\mathbf{e})\alpha(t)V. \tag{69}$$

We note that constants H_2, H_3 depend only on μ_i, λ_k, $k = 1, \ldots, n$. Substituting (40), (41), (42) and (69) in (64) and applying the inequalities: $(V+1)^{-1} \leq 1$, $V(V+1)^{-1} \leq 1$, we obtain

$$W(X(t)) = W(X(0)) - \alpha_1 \int_0^t (V+1)^{-1} a(\theta) \sum_{i=1}^{n} x_i^{2\mu_i}(\theta)d\theta$$

$$+ \int_0^t (V+1)^{-1} H_3(\mathbf{R}+\mathbf{e})\alpha(\theta)Vd\theta + \int_0^t (V+1)^{-1}\bar{S}(\theta)d\theta + m(t)$$

$$= W(X_0) - \alpha_1 \int_0^t a(\theta)(V+1)^{-1}\|(X)_\mu(\theta)\|^2 d\theta + H_3(\mathbf{R}+\mathbf{e})\int_0^t \alpha(\theta)d\theta$$

$$+ \int_0^t \bar{S}(\theta)d\theta + m_t, \tag{70}$$

where $\bar{S}(\theta) = H_4 S(\theta)$ and α_1 is defined similar to (44).

Now we proceed in the same way as in the proof of Theorem 2, noting that

$$|V_3(t)| \leq \left(\tilde{\lambda} + \hat{\lambda}\right)\nu(\mathbf{R}+\mathbf{e})K_3\left[\int_{t-h}^{t} \alpha(\theta)d\theta + \int_{t-\Delta}^{t} \alpha(\theta)d\theta\right]$$

$$\leq \left(\tilde{\lambda} + \hat{\lambda}\right)\nu(\mathbf{R}+\mathbf{e})K_3\left[\int_{t-h}^{\infty} \alpha(\theta)d\theta + \int_{t-\Delta}^{\infty} \alpha(\theta)d\theta\right] \to 0$$

when $t \to \infty$, because an integral $\int_0^\infty \alpha(\theta)d\theta$ converges. We consider two cases about the function $a(t)$: a) $a(t) \to 0$ as $t \to \infty$, b) $a(t) \geq c$ for some $c > 0$. In the case a), $V_2(t) \to 0$ as $t \to \infty$ (see (45)), and we prove that $P\left\{\lim_{t\to\infty} \|X(t)\| = 0\right\} = 1$ similar to the proof of Theorem 1.

In case b) we note that V, V_2, V_3 are *a.s.* uniformly continuous on $[0, \infty)$. Then $V_1(t)$ is also *a.s.* uniformly continuous on $[0, \infty)$. We complete the proof in the same way as in the Theorem 2.

Remark 4. Theorem 4 can be also proved in case of unbounded delay, $h = \infty$.

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