

Symplectic Geometry Applied to Boundary Problems on Hamiltonian Difference Systems ¹

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ABSTRACT

In this work, we consider the boundary problem for Hamiltonian difference system

$$\begin{cases} \Delta x(t) = A(t)x(t+1) + B(t)u(t) + \lambda W_2 u(t) \\ \Delta u(t) = (C(t) - \lambda W_1(t))x(t+1) - A^*(t)u(t), \end{cases}$$

on an discrete interval I . Applying the concept of symplectic geometry, we give a complete account to the form of all possible symmetric boundary conditions with respect to separation or coupling at the endpoints for the complete Lagrangian space, following the development of the GKN-theory.

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RESUMEN

En este trabajo consideramos el problema de frontera para el sistema de diferencia Hamiltoniano

$$\begin{cases} \Delta x(t) = A(t)x(t+1) + B(t)u(t) + \lambda W_2 u(t) \\ \Delta u(t) = (C(t) - \lambda W_1(t))x(t+1) - A^*(t)u(t), \end{cases}$$

en un intervalo discreto I . Aplicando el concepto de geometría simpléctica damos una descripción completa de la forma de todas las condiciones de fronteras simétricas posibles con respecto a la separación o unión en los puntos finales para el espacio Lagrangiano completo, siguiendo el desarrollo de la GKN-teoría.

Key words and phrases: *Hamiltonian difference system; boundary space; symplectic invariant; boundary condition.*
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1 Introduction

The GKN-theory for the Hamiltonian difference system (1.1) has been developed by authors in [13]. In the paper [13], we considered the self-adjoint extension problem for a class of singular discrete linear Hamiltonian system defined on finite or infinite interval. We gave the complex symplectic geometric characterization of all self-adjoint extensions of the minimal difference operator generated by the formally Hamiltonian difference operator. The result shows that there is a one to one correspondence between the set of all self-adjoint extensions and the set of all complete Lagrangian subspaces of the boundary space S . In this paper, following the development of the GKN theory we have given in [13], we relate the concept of symplectic geometric to the boundary space in the classification of boundary conditions for self-adjoint extension, and we give out a complete account to the form of all possible symmetric boundary conditions with respect to separation or coupling at the endpoints for the complete Lagrangian space corresponding to the self-adjoint extension.

2 Main results

Consider the following linear Hamiltonian difference system

$$J\Delta y(t) := (\lambda W(t) + P(t))R(y)(t), \quad (2.1)$$

for $t \in \mathbf{I}$, where \mathbf{I} is a discrete interval $[a, b] := \{a, a + 1, \dots, b\}$, $a, b \in \mathbf{Z}$, $b \leq +\infty$.

$$R(y)(t) = \begin{pmatrix} x(t+1) \\ u(t) \end{pmatrix}, \text{ with } y(t) = \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}.$$

$$P = \begin{pmatrix} -C & A^* \\ A & B \end{pmatrix}, W = \begin{pmatrix} W_1 & 0 \\ 0 & W_2(t) \end{pmatrix}_{2n \times 2n}, J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

From now on we shall always make the following hypotheses (H): $A(t), B(t), C(t), W_1(t), W_2(t)$ are $n \times n$ complex-valued matrices, and $A^*(t)$ is the complex conjugate transpose of $A(t)$. $B(t), C(t), W_1(t), W_2(t)$ are Hermitian matrices on \mathbf{I} , $W_1(t)$ is a $n \times n$ positive definite matrix, $W_2(t)$ is a $n \times n$ non-negative definite matrix, and $I_n - A(t)$ is nonsingular on \mathbf{I} .

In the context, we always assume the system is definite over \mathbf{I} , that is if y is a solution to the equation above, then

$$\sum_{s=a}^t R(y)^*(s)W(s)R(y)(s) = 0, \forall t \in \mathbf{I}$$

if and only if $y \equiv 0, t \in \mathbf{I}^*$, where $\mathbf{I}^* = [a, b + 1]$, if $b < +\infty$; $\mathbf{I}^* = [a, \infty)$, if $b = +\infty$.

Define the formally Hamiltonian difference operator for the system (2.1)

$$l(y)(t) := J\Delta y(t) - P(t)R(y)(t)$$

for $y \in D(l)$, where $D(l) := \{y : \mathbf{I} \rightarrow \mathbf{C}^{2n}\}$.

Let $l_W^2(\mathbf{I})$ be

$$l_W^2(\mathbf{I}) = \{y \in D(l) \mid \sum_{t=a}^b R(y)^*(t)W(t)R(y)(t) < \infty\}.$$

We introduce the following equivalence relation in $l_W^2(\mathbf{I})$.

$$y_1 \approx y_2 \iff \|y_1 - y_2\|_W = 0.$$

Then the induced space with inner product $(y, z)_W = \sum_{t=a}^b R(z)^*(t)W(t)R(y)(t)$ is a Hilbert space under the above equivalence relation.

Definition 2.1 Denote $D(T_1) := \{y \in l_W^2(\mathbf{I}) \mid \exists f, f \in l_W^2(\mathbf{I}), \text{ such that } ly = J\Delta y(t) - P(t)R(y)(t) = \{W(t)R(f)(t), t \in \mathbf{I}\}$.

Define an operator T_1 by setting $T_1y = f$ if and only if $ly = WR(f)$, for all $y \in D(T_1)$.

T_1 is said to be the maximal Hamiltonian difference operator associated l .

Lemma 2.1^[13] (Green's formula) Let $f, g \in D(T_1), \alpha < \beta \in \mathbf{I}$, then

$$\sum_{t=\alpha}^{\beta} \{R(g)^*(t)W(t)R(T_1f)(t) - R(T_1g)^*(t)W(t)R(f)(t)\} = [g^*(t)Jf(t)]_{\alpha}^{\beta+1}.$$

Define the symplectic form (or symplectic product) for $f, g \in D(T_1)$ as follows:

$$[f : g] := \sum_{t=a}^b \{R(g)^*(t)W(t)R(T_1f)(t) - R(T_1g)^*(t)W(t)R(f)(t)\} \\ = (T_1f, g) - (f, T_1g). \quad (2.2)$$

for $\mathbf{I} = [a, b]$,

$$[f : g] := \sum_{t=a}^{\infty} \{R(g)^*(t)W(t)R(T_1f)(t) - R(T_1g)^*(t)W(t)R(f)(t)\} \\ = (T_1f, g) - (f, T_1g). \quad (2.2)'$$

for $\mathbf{I} = [a, \infty)$.

Note that the symplectic form is well defined for $b = \infty$, since

$$\sum_{t=a}^{\infty} \{R(g)^*(t)W(t)R(T_1f)(t) - R(T_1g)^*(t)W(t)R(f)(t)\} < \infty$$

for all $f, g \in D(T_1)$ by the Cauchy-Schwarz inequality.

From now on, we denote $\bar{b} = b + 1$ for $b < \infty$, $\bar{b} = +\infty$ for $b = +\infty$.

Clearly, $[\cdot : \cdot]$ is a quasi-bilinear form on $D(T_1) \times D(T_1) \rightarrow C$, and from Green's formula, it's easy to see that

$$[f : g] = g^* Jf(t)|_a^{\bar{b}}$$

for any $f, g \in D(T_1)$.

Definition 2.2 Define the operator T_0 as follows:

$$T_0 : D(T_0) \rightarrow l^2_W(\mathbf{I}), \mathbf{y} \mapsto \mathbf{T}_0\mathbf{y} = \mathbf{T}_1\mathbf{y},$$

where

$$D(T_0) = \{y \in D(T_1) | [y : D(T_1)] = 0\}, \quad (2.3)$$

T_0 is said to be the minimal Hamiltonian difference operator.

$\forall f, g \in D(T_1)$, define $(f, g)_1 = (f, g) + (T_1f, T_1g)$, then $D(T_1)$ is a Hilbert space with inner product $(\cdot, \cdot)_1$, and moreover,

$$D(T_1) = D(T_0) \oplus D^+ \oplus D^-,$$

where D^+ and D^- are deficiency spaces of T_0 .

The theory of Von Neumann asserts that there exist self-adjoint extensions T of T_0 if and only if $d^+ = d^-$. In the context, we shall always assume that $d^+ = d^- = d$, so that T_0 has self-adjoint extension.

From the Theorem in [12], $n \leq d \leq 2n$.

Define the boundary space (or endpoint space) by

$$S = D(T_1)/D(T_0).$$

Further denote the natural projection of $D(T_1)$ onto S

$$\psi : D(T_1) \rightarrow S, f \mapsto \psi f = \{f + D(T_0)\}. \tag{2.4}$$

For convenience, we denote $\hat{f} = \psi f$, for each $f \in D(T_1)$.

Lemma 2.2^[13] Let $S = D(T_1)/D(T_0)$ be the boundary space for the system (2.1). Then S together with the symplectic form $[\hat{f} : \hat{g}] := [f : g], \forall f, g \in S$, constitutes a complex symplectic space of dimension $2d, 2n \leq 2d \leq 4n$.

Lemma 2.3 Let $S = D(T_1)/D(T_0)$ with the symplectic form $[\hat{f} : \hat{g}]$ be complex symplectic space of dimension $2d = 2d^\pm$, then S is complex symplectic isomorphism to C^{2d} , where the symplectic form of C^{2d} is defined as $[u : v]_1 = u^* K v$, with

$$K = \begin{pmatrix} iI_d & 0 \\ 0 & -iI_d \end{pmatrix}.$$

Proof From the Corollary 2.1 in [2], we have know that all complex symplectic spaces of dimension $2d$ are symplectically isomorphic to C^{2d} with symplectic form $[\cdot]$ defined by $[u : v]_1 = u^* \Omega v$, with

$$\Omega = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}.$$

Thus, note that if K is congruent to Ω , the result can be obtained immediately.

Definition 2.3 Let $S = D(T_1)/D(T_0)$ together with $[\cdot]$, be the boundary complex symplectic space. Set

$$\begin{aligned} D_-(T_1) &= \{f \in D(T_1) | \exists \alpha \in I^*, s.t. f(t) = 0, \alpha \leq t \leq b+1\} \\ D_+(T_1) &= \{f \in D(T_1) | \exists \beta \geq a, s.t. f(t) = 0, a \leq t \leq \beta\} \end{aligned}$$

in case of $I = [a, b]$;

$$\begin{aligned} D_-(T_1) &= \{f \in D(T_1) | \lim_{t \rightarrow \infty} f(t) = 0\} \\ D_+(T_1) &= \{f \in D(T_1) | \exists \beta \geq a, s.t. f(t) = 0, a \leq t \leq \beta\} \end{aligned}$$

in case of $I = [a, \infty)$

Define $S_- := \psi D_-(T_1), S_+ := \psi D_+(T_1)$, and call them left-boundary space and right boundary space, respectively.

Clearly, $[S_- : S_+] = 0$ by Green's formula. We shall show that S_- and S_+ are symplectic subspaces which provide a direct sum decomposition of S . For this purpose, we first give the following Lemma.

Definition 2.4 Consider the linear control system

$$J\Delta y(t) = P(t)R(y)(t) + W(t)R(\varphi)(t) \tag{2.5}$$

with controllers $\varphi \in l_{W'}^2(\mathbf{I})$ on the interval \mathbf{I} . For a prescribed $t_0 \in \mathbf{I}$, we say that the system (2.5) is fully controllable at $t_0 \in \mathbf{I}$ in case for each pair of $y_0, y_1 \in C^{2n}$ and for each $t_1 > t_0, t_1 \in I^*$, there exists a controller $\varphi \in l_{W'}^2([t_0, t_1])$, so that the response

$$y(t) = \Phi(t)y_0 - \Phi(t) \sum_{s=t_0}^{t-1} JR(\Phi)^*(s)W(s)R(\varphi)(s)$$

is steered from $y(t_0) = y_0$ to $y(t_1) = y_1$, where $\Phi(t)$ is the fundamental matrix for

$$J\Delta y(t) = P(t)R(y)(t), t \in \mathbf{I}$$

satisfying $\Phi(t_0) = I_{2n}$. Further, we say that the system (2.5) is fully controllable on \mathbf{I} in case (2.5) is fully controllable at each $t_0 \in \mathbf{I}$.

Lemma 2.4 The linear control system (2.5) is fully controllable at \mathbf{I} .

Proof For any $t_0 \in \mathbf{I}$, we shall show that the system (2.5) is fully controllable at t_0 .

Suppose to the contrary that (2.5) is not fully controllable at t_0 , so there exists $t_1 > t_0, t_1 \in I^*$ with the corresponding attainable set

$$\begin{aligned} K(t_1) &:= \{y(t_1)\} \\ &= \{\Phi(t_1)y(t_0) - \Phi(t_1) \sum_{s=t_0}^{t_1-1} JR(\Phi)^*(s)W(s)R(\varphi)(s) \mid \text{all controllers } \varphi \in l_{W'}^2(I)\} \neq C^{2n}. \end{aligned}$$

and consequently

$$K_0(t_1) := \{-\Phi(t_1) \sum_{s=t_0}^{t_1-1} JR(\Phi)^*(s)W(s)R(\varphi)(s) \mid \text{all controllers } \varphi \in l_{W'}^2(I)\} \neq C^{2n}.$$

In this case, there exists a constant vector $\eta_1 \neq 0$ with $(\eta_1, K_0(t_1))_2 = 0$. Here the inner product is defined in term of that in C^{2n} . So

$$\eta_1^* \Phi(t_1) \sum_{s=t_0}^{t_1-1} JR(\Phi)^*(s)W(s)R(\varphi)(s) = 0$$

for all controllers $\varphi(s)$ on $[t_0, t_1]$.

Now we define $\eta_0^* = \eta_1^* \Phi(t_1)J$. Then

$$\eta_0^* \sum_{s=t_0}^{t_1-1} R(\Phi)^*(s)W(s)R(\varphi)(s) = 0$$

for all controllers $\varphi(s)$ on $[t_0, t_1]$. This implies that $\Phi(s)\eta_0 = 0, s \in [t_0, t_1]$ and consequently $\eta_0 = 0$ by the hypothesis, which is a contradiction.

Therefore we conclude that (2.5) is fully controllable at $t_0 \in \mathbf{I}$, and so the system is fully controllable on \mathbf{I} . ■

Corollary 2.1 Let T_1 be maximal Hamiltonian difference operator generated by l . Then for any $\gamma_1 < \gamma_2 \in I^*, \forall \xi, \eta \in C^{2n}$, there exists $y(t), t \in [\gamma_1, \gamma_2]$, such that $y(\gamma_1) = \xi, y(\gamma_2) = \eta$.

Furthermore, $y \in D(T_1)$.

Proof From Lemma 2.4, we have that for any $\gamma_1 < \gamma_2, \gamma_1 \in I, \gamma_2 \in I^*, \forall \xi, \eta \in C^{2n}$, there exists $f \in l_W^2(I)$ such that the solution $y(t)$ satisfying the equation

$$J\Delta y(t) = P(t)R(y)(t) + W(t)R(f)(t), t \in [\gamma_1, \gamma_2 - 1]$$

with $y(\gamma_1) = \xi, y(\gamma_2) = \eta$.

Further, from Lemma 2.4, we can extend $y(t)$ to the whole interval I with $y(t) \equiv 0$ for t outside $[\gamma_1, \gamma_2]$ satisfying

$$J\Delta y(t) = P(t)R(y)(t) + W(t)R(g)(t)$$

for some $g \in l_W^2(I)$. Clearly, $y \in l_W^2(I)$ and $y \in D(T_1)$. The proof is complete. ■

Lemma 2.5 For each $f \in D(T_1)$, there exists a decomposition $f = f_- + f_+ + z$. Here $f_- \in D_-(T_1), f_+ \in D_+(T_1)$ and $z \in D(T_0)$.

Proof Take $f \in D(T_1)$ and construct functions f_- and f_+ as follows:

$$f_+(t) := \begin{cases} 0, & t = a \\ f(t), & t \geq a + 1, t \in I^* \end{cases}, f_-(t) := \begin{cases} 0, & t \geq c \\ f(t), & a \leq t \leq c - 1 \end{cases},$$

for some $c \in I^*$.

Then from Lemma 2.4, we have that $f_{\pm} \in D(T_1)$.

Furthermore, we have that $f_- \in D_-(T_1), f_+ \in D_+(T_1)$ and $f - (f_- + f_+) \in D(T_0)$.

Set $z = f - f_- - f_+$, then $f = f_- + f_+ + z$, as required. ■

Theorem 2.1 Let $S = D(T_1)/D(T_0)$ be the boundary space. Then both S_{\pm} are symplectic subspaces of S , where the symplectic forms defined in S_{\pm} are the same as that defined in S . Moreover, $S = S_- \oplus S_+$ with $[S_- : S_+] = 0$.

Proof For any given $\hat{f} \in S$. Let $\hat{f} = \{f + D(T_0)\}$, then $f = f_- + f_+ + z$, with $f_- \in D_-(T_1), f_+ \in D_+(T_1)$ and $z \in D(T_0)$. So $\hat{f} = \hat{f}_- + \hat{f}_+$ with $\hat{f}_{\pm} \in D_{\pm}(T_1)$. Thus $S = S_- + S_+$.

On the other hand, if $\hat{f} \in S_- \cap S_+$, then $[\hat{f} : S] = 0$, since $[S_- : S_+] = 0$. Hence $\hat{f} = 0$. Thus $S = S_- \oplus S_+$.

It is easy to see that both S_{\pm} are linear subspaces of S . Now we show that both S_{\pm} are symplectic subspaces of S . Define the symplectic forms in S_{\pm} the same as that defined in S . Thus we need only to prove that the symplectic forms in S_{\pm} are non-degenerate.

In fact, if $[\hat{f} : S_{\pm}] = 0$, then $[\hat{f} : S] = 0$, since $S = S_- \oplus S_+$. So $\hat{f} = 0$, since S is non-degenerate. The proof is complete. ■

Definition 2.5 Let $S = D(T_1)/D(T_0)$ with the symplectic form $[\hat{f} : \hat{g}]$ be complex symplectic space of dimension $2d = 2d^\pm$. Define the following symplectic invariants of S :

$$\begin{aligned} p &:= \max \{ \text{complex dimension of linear subspaces whereon } \text{Im} [v : v] > 0 \text{ for all } v \neq 0 \} \\ q &:= \max \{ \text{complex dimension of linear subspaces whereon } \text{Im} [v : v] < 0 \text{ for all } v \neq 0 \} \\ \Delta &:= \max \{ \text{complex dimension of Lagrangian subspaces of } S \}, \end{aligned}$$

p, q are called positivity index and negativity index of S , respectively, Δ is called the Lagrangian index. Further define the excess of $S : Ex(S) := p - q$.

Lemma 2.6 Let $S = D(T_1)/D(T_0)$ with the symplectic form $[\hat{f} : \hat{g}]$ be complex symplectic space of dimension $2d = 2d^\pm$, then

$$p = q = d, \text{ so } Ex = 0.$$

Proof From Lemma 2.3, S is symplectic isomorphic to C^{2d} with symplectic form $[\cdot : \cdot]_1$. Again we know that

$$\begin{aligned} p &= \text{number of } (+i) \text{ terms on the diagonal of matrix } K, \\ q &= \text{number of } (-i) \text{ terms on the diagonal of matrix } K \end{aligned}$$

by Theorem 1 in [10]. This complete the proof. ■

Definition 2.6 Let $S = D(T_1)/D(T_0)$ be the boundary symplectic space for l as above.

A vector $v \in S$ is separated at a in case $v \in S_-$. Similarly, a vector $v \in S$ is separated at \bar{b} in case $v \in S_+$. If $v \in S$ is neither separated at a , nor separated at \bar{b} , then v is called coupled.

Theorem 2.2 Let $S = D(T_1)/D(T_0) = S_- \oplus S_+$ be the boundary symplectic $2d$ -space for l as above, then

(1) The vector $v \in S$ is separated at the left endpoint a of I , that is $v \in S_-$ if and only if $v = \{f_- + D(T_0)\}$ has a representative function $f_- \in D_-(T_1)$, so $f_-(t) \equiv 0, \alpha \leq t \leq b+1$ for some $\alpha \in \mathbf{I}^*$ in case of $I = [a, b]$; $\lim_{t \rightarrow \infty} f_-(t) = 0$ in case of $I = [a, \infty)$.

The vector $v \in S$ is separated at the right endpoint \bar{b} of \mathbf{I} , that is $v \in S_+$ if and only if $v = \{f_+ + D(T_0)\}$ has a representative function $f_+ \in D_+(T_1)$, so $f_+(t) \equiv 0, a \leq t \leq \beta$ for some $\beta \in \mathbf{I}$.

v is coupled in case for each of representative functions $f, \forall a \leq \alpha, \beta \leq b+1, \exists t_0, a \leq t_0 \leq \alpha$ or $\beta \leq t_0 \leq b+1$, such that $f(t_0) \neq 0$ in case of $\mathbf{I} = [a, b]$; $\forall a \leq \alpha < \infty, \exists t_0, a \leq t_0 < \infty$ such that $f(t_0) \neq 0$ or $\lim_{t \rightarrow \infty} f_-(t) = 0$ in case of $\mathbf{I} = [a, \infty)$.

(2) Each function $f \in D(T_0)$ if and only if $f(a) = f(\bar{b}) = 0$. Here $f(\bar{b}) = 0$ means the limit $\lim_{t \rightarrow \infty} f(t) = 0$ in case of $I = [a, \infty)$.

(3) If $\tilde{v} \in S$ have one representative function h with $h(\tilde{b}) = 0$, then every representative function u of \tilde{v} satisfies $u(\tilde{b}) = 0$, and moreover, $v \in S_-$. Here the meaning of $u(\tilde{b}) = 0$ is the same as (2) above.

Similar results hold for the endpoint a .

Proof (1) The vector $v \in S_-$ just in case $v = \hat{f}_- = \{f_- + D(T_0)\}$ for some function $f_- \in D_-(T_1)$. This means that there exists $\alpha \in I^*$, such that $f(t) \equiv 0, \alpha \leq t \leq b + 1$ in case of $I = [a, b], \lim_{t \rightarrow \infty} f_-(t) = 0$ in case of $I = [a, \infty)$.

The conclusions for $v \in S_+$ and $v \in S_- \cup S_+$ are similar.

(2) If $f \in D(T_0)$, then $[g : f] = 0$, for all $g \in D(T_1)$. From Lemma 2.4, choose $g_i \in D(T_1)$, s.t. $g_{ii}(a) = 1, g_{ij}(a) = 0, 1 \leq j \leq 2n, j \neq i, g_i(t) = 0, t \geq c$ for some $c \in I^*, i = 1, 2, \dots, 2n$. Then from $[f : g_i] = g_i^* J f|_a^b = 0$, we obtain $f(a) = 0$. Similarly, we can conclude that $f(b + 1) = 0$ in case of $I = [a, b]; \lim_{t \rightarrow \infty} f_-(t) = 0$ in case of $I = [a, \infty)$. Thus $f(\tilde{b}) = 0$.

The converse is evident from Green's formula and the definition of $D(T_0)$.

(3) If $\tilde{v} \in S$ have one representative function h with $h(\tilde{b}) = 0$, then each function in $\tilde{v} = \hat{h} = \{h + D(T_0)\}$ satisfies $v(\tilde{b}) = 0$ from (2). In this case, it is clear that $h \in D_-(T_1)$, so $v = \hat{h} \in S_-$.

The case at a is similar, and the proof is complete. ■

From the second result of Theorem 2.2, we can conclude that $\{f + D(T_0)\} = \{h + D(T_0)\} \in S$ if and only if $f(a) = h(a)$ and $f(\tilde{b}) = h(\tilde{b})$.

Corollary 2.2 S_{\pm} can be rewritten as

$$S_- = \{\hat{f} \in S | f(\tilde{b}) = 0\} \tag{2.6}$$

and

$$S_+ = \{\hat{f} \in S | f(a) = 0\} \tag{2.7}$$

Proof Set

$$S_1 = \{\hat{f} \in S | f(\tilde{b}) = 0\} \text{ and } S_2 = \{\hat{f} \in S | f(a) = 0\}.$$

From Theorem 2.2, we know that the descriptions of S_1, S_2 are meaningful. Now we prove that $S_- = S_1, S_+ = S_2$.

Note that $S_- = \Psi D_-(T_1)$, it is evident that each function in $D_-(T_1)$ must satisfies $f(\tilde{b}) = 0$, so each function in S_- must belong to S_1 , that is $S_- \subset S_1$. Thus we need verify only the converse. Take any function $\hat{f} \in S_1$. So $f \in D(T_1)$ with $f(\tilde{b}) = 0$. Then from Corollary 2.1, there exists some $f_- \in D_-(T_1)$ with $f_-(t) \equiv f(t), a \leq t \leq \alpha$ for some $\alpha \in I$. Thus $f - f_- \in D(T_0)$, so $\hat{f}_- = \hat{f}$ and $\hat{f} \in \Psi D_-(T_1)$, this implies that $S_1 \subset S_-$ and hence $S_- = S_1$ as required.

An analogous argument holds for S_+ . ■

Definition 2.7 Let $S = D(T_1)/D(T_0)$ be the boundary symplectic space for l as above. Define the coupling grade of Lagrangian space L :

$$\text{grade } L = \Delta_- - \dim L \cap S_- = \Delta_+ - \dim L \cap S_+.$$

Define the necessary coupling of L :

$$\text{Nec-coupling } L = \Delta - \dim L \cap S_- - \dim L \cap S_+.$$

A Lagrangian d -space $L \subset S$ is called strictly separated if Nec-coupling $L = 0$; $L \subset S$ is called totally coupled if Nec-coupling $L = \Delta$.

A basis of L is called minimally coupled if it contains exactly (Nec-coupling L) vectors, each of which is coupled.

The next Theorem describe all possible bases for a given Lagrangian and the range of coupled grade for all Lagrangian spaces. It is a special case of Corollary 3 and Theorem 4 in [10].

Theorem 2.3 Consider the Hamiltonian system (2.1). By the GKN-Theorem in [13], there is a one to one correspondence between the self-adjoint operators T generated by l with domain $D(T)$ and the Lagrangian n -spaces L in S . Namely, for each $L \subset S$, take any basis of $2n$ -vectors $\hat{f}_1, \hat{f}_2, \dots, \hat{f}_d$ and any representative functions $f_1, f_2, \dots, f_d \in D(T_1)$, then $D(T) = c_1 f_1 + c_2 f_2 + \dots + c_d f_d + D(T_0)$ for arbitrary complex constants $c_1, \dots, c_d \in C$. Thus from Theorem 2.2, we have that all $f \in D(T)$ can be determined (modulo $D(T_0)$) by the homogeneous linear boundary conditions

$$\begin{aligned} f(a) &= c_1 f_1(a) + c_2 f_2(a) + \dots + c_d f_d(a), \\ f(\hat{b}) &= c_1 f_1(\hat{b}) + c_2 f_2(\hat{b}) + \dots + c_d f_d(\hat{b}), \end{aligned}$$

for choices of c_1, \dots, c_d . Furthermore,

(i) each base of Lagrangian d -space contains at most $\dim L \cap S_-$ vectors in S_- , at most $\dim L \cap S_+$ vectors in S_+ , and at least (Nec-coupling L) vectors neither in S_- nor in S_+ .

(ii) For each integer $k = 0, 1, \dots, \min\{\Delta_-, \Delta_+\}$, there exists a Lagrangian d -space L_k with grade $L_k = k$.

(iii) For each Lagrangian d -space $L \subset S$, and each triple $\{\alpha, \beta, \gamma\}$ of non-negative integers satisfying $\alpha + \beta + \gamma = d$, and

$$\alpha \leq \dim L \cap S_-, \beta \leq \dim L \cap S_+, \gamma \geq \text{Nec-coupling } L,$$

there exists a basis for L consisting of α vectors in S_- , β vectors in S_+ , γ vectors neither in S_- nor in S_+ .

(iv) For each Lagrangian d -space L of S , there exists a minimally coupled basis for L with exactly (Nec-coupling L) basis vectors each coupled on \mathbf{I} , and consequentially exactly $(\dim L \cap S_-)$ vectors each separated at the left, and $(\dim L \cap S_+)$ vectors each separated at the right of \mathbf{I}^* .

Proof Note that the boundary space S is a complex symplectic space and has a direct sum decomposition, and furthermore $\dim S = 2d = 2\dim L$, the result can be concluded immediately from the Corollary 3 and Theorem 4 in [10]. ■

3 Classification of all boundary conditions for self-adjoint extension in the limit circle case

In this section, we shall describe explicitly the kinds of boundary conditions for self-adjoint operators T generated by l when l is in limit circle case that is the deficiency indices $d^\pm = 2n$.

Definition 3.1 Consider the Hamiltonian system (2.1). Assume that l is in limit circle case. Let T_1 and T_0 be the maximal and minimal operators generated by l in the complex Hilbert space $l^2_W(I)$. Define the evaluation map

$$\nu : D(T_1) \rightarrow \mathbb{C}^{4n}$$

$$f = (f_1, f_2, \dots, f_{2n})^T \mapsto \nu f = (f_1(a), f_2(a), \dots, f_{2n}(a), f_1(\bar{b}), f_2(\bar{b}), \dots, f_{2n}(\bar{b}))^T.$$

and also the evaluation map (which is still denote by ν) on the boundary space $S = D(T_1)/D(T_0)$,

$$\nu : S \rightarrow \mathbb{C}^{4n}$$

$$\hat{f} = \{f + D(T_0)\} \mapsto \nu \hat{f} = (f_1(a), f_2(a), \dots, f_{2n}(a), f_1(\bar{b}), f_2(\bar{b}), \dots, f_{2n}(\bar{b}))^T,$$

which is well-defined since $\nu D(T_0) = 0$.

Lemma 3.1 Consider the complex vector space \mathbb{C}^{4n} , and define the quasi-bilinear

form $[\cdot]$ on \mathbb{C}^{4n} by $[u : v] = u^* F v$ for $u, v \in \mathbb{C}^{4n}$ with $F = \begin{pmatrix} 0 & I_n & 0 & 0 \\ -I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_n \\ 0 & 0 & I_n & 0 \end{pmatrix}$.

Then \mathbb{C}^{4n} with the form $[\cdot]$ is a complex symplectic space of dimension $4n$ and excess $Ex = 0$.

Further, the linear subspaces of \mathbb{C}^{4n} ,

$$C_-^{2n} = \{u \in \mathbb{C}^{4n} | u_{2n+1} = \dots = u_{4n} = 0\}$$

and

$$C_+^{2n} = \{u \in \mathbb{C}^{4n} | u_1 = \dots = u_{2n} = 0\}$$

determine a direct sum decomposition of \mathbb{C}^{4n} :

$$\mathbb{C}^{4n} = C_-^{2n} \oplus C_+^{2n}, \text{ with } [C_-^{2n} : C_+^{2n}] = 0.$$

Hence both C_\pm^{2n} are complex symplectic $2n$ -subspaces, and their symplectic invariants are $Ex_- = Ex_+ = 0$ and $\Delta_- = \Delta_+ = n$.

Proof Note that F is skew-Hermitian and nonsingular, then we have that \mathbb{C}^{4n} with $[\cdot]$ is a complex symplectic space of dimension $4n$.

Again F is congruent with $G = \begin{pmatrix} iI_n & 0 & 0 & 0 \\ 0 & -iI_n & 0 & 0 \\ 0 & 0 & -iI_n & 0 \\ 0 & 0 & 0 & iI_n \end{pmatrix}$. This implies that

$Ex = 0$.

It is obvious that C_-^{2n} and C_+^{2n} are symplectic $2n$ -subspaces of C^{4n} , corresponding to the $2n \times 2n$ skew-Hermitian matrices $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, respectively. Furthermore, it is easy to see that

$$C^{4n} = C_-^{2n} \oplus C_+^{2n}, \text{ with } [C_-^{2n} : C_+^{2n}] = 0.$$

Now we compute the symplectic invariants for C_-^{2n} and C_+^{2n} .

Since F is a real skew-symmetric matrix with eigenvalues of $(+i)2n$ -fold and $(-i)2n$ -fold. Hence, the symplectic invariants of C_{\pm}^{2n} are $\dim C_{\pm}^{2n} = 2n, Ex_{\pm} = 0, \Delta_{\pm} = n$, respectively. ■

Theorem 3.1 Consider the Hamiltonian system (2.1). Assume that l is in the limit circle case. Let $S = D(T_1)/D(T_0) = S_- \oplus S_+$ be the boundary symplectic $2n$ -space for l as above, then the evaluation

$$\nu : S \rightarrow C^{4n}$$

is a symplectic isomorphism of S with the form $[\cdot]$ onto C^{4n} with the form $[u : v] = u^* F v$.

Moreover, $\nu S_- = C_-^{2n}$, and $\nu S_+ = C_+^{2n}$. So the symplectic invariants for S are

$$\dim S = 4n, Ex = 0, \Delta = 2n,$$

and further the symplectic invariants for S_{\pm} are

$$Ex_{\pm} = 0, \Delta_{\pm} = n.$$

Proof Clearly ν is a linear map on domain $D(T_1)$ and hence on S , and is surjective onto C^{4n} . Furthermore, from Lemma 2.4, we obtain that ν is injective on S , and hence ν define a linear isomorphism of the complex vector space S onto C^{4n} .

By a simple calculation, we can find that

$$[\hat{f} : \hat{g}] = g^* J f|_a^b = (\nu \hat{g})^* F (\nu \hat{f}) = [\nu \hat{f} : \nu \hat{g}].$$

So the symplectic form is preserved under the map ν of S onto C^{4n} .

Hence ν is a symplectic isomorphism of S onto C^{4n} , and it follows that the symplectic of S and S_{\pm} are the same as those of C^{4n} and C_{\pm}^{2n} . Thus the results can be obtained from Lemma 3.1. ■

Theorem 3.2 Consider the Hamiltonian system (2.1). Assume that l is in the limit circle case. Then by the GKN-Theorem in [13], there is a one to one correspondence between the self-adjoint operators T generated by l with domain $D(T)$

and the Lagrangian $2n$ -spaces L in S . Namely, for each $L \subset S$, take any basis of $2n$ -vectors f_1, f_2, \dots, f_{2n} and any representative functions $f_1, f_2, \dots, f_{2n} \in D(T_1)$, then $D(T) = c_1 f_1 + c_2 f_2 + \dots + c_{2n} f_{2n} + D(T_0)$ for arbitrary complex constants $c_1, \dots, c_{2n} \in C$. Thus from Theorem 2.2, we have that all $f \in D(T)$ can be determined (modulo $D(T_0)$) by the homogeneous linear boundary conditions

$$\begin{aligned} f(a) &= c_1 f_1(a) + c_2 f_2(a) + \dots + c_{2n} f_{2n}(a), \\ f(\bar{b}) &= c_1 f_1(\bar{b}) + c_2 f_2(\bar{b}) + \dots + c_{2n} f_{2n}(\bar{b}), \end{aligned}$$

for choices of c_1, \dots, c_{2n} . Furthermore

(i) each base of Lagrangian $2n$ -space contains at most $(n - \text{grade} L)$ vectors in S_- , at most $(n - \text{grade} L)$ vectors in S_+ , and at least $(\text{Nec-coupling} L)$ vectors neither in S_- nor in S_+ . Here the coupling grade of L is defined by

$$\text{grade } L = n - \dim L \cap S_- = n - \dim L \cap S_+,$$

so

$$0 \leq \text{grade } L \leq n.$$

(ii) For each integer $k = 0, 1, \dots, n$, there exists a Lagrangian $2n$ -space L_k with $\text{grade} L_k = k$.

(iii) For each Lagrangian d -space $L \subset S$, and each triple $\{\alpha, \beta, \gamma\}$ of non-negative integers satisfying $\alpha + \beta + \gamma = 2n$, and

$$\alpha \leq n - \text{grade } L, \beta \leq n - \text{grade } L, \gamma \geq \text{Nec-coupling } L,$$

there exists a basis for L consisting of α vectors in S_- , β vectors in S_+ , γ vectors neither in S_- nor in S_+ .

(iii) The necessary coupling of L is given by $\text{Nec-coupling} = 2 \text{grade} L$.

(iv) For each Lagrangian $2n$ -space L of S , there exists a minimally coupled basis for L with exactly $(\text{Nec-coupling } L)$ basis vectors each coupled on \mathbf{I}^* , and consequentially exactly $(n - \text{grade } L)$ vectors each separated at the left, and $(n - \text{grade } L)$ vectors each separated at the right of \mathbf{I}^* .

Proof According the Definition 2.7, the results (i),(ii) and (iv) can be obtained immediately by applying Theorem 2.3 and Theorem 3.1. (iii) can be concluded if we note that $\Delta = \Delta_- + \Delta_+ + |Ex_{\pm}|$, so $\Delta = \Delta_- + \Delta_+$ in this case. ■

From Theorem 3.2, we can tabulate the structure of minimally coupled bases for Lagrangian $2n$ -spaces L of every possible grade, with special attention to the cases of separated boundary conditions at the left and right endpoints, and coupled boundary conditions.

References

- [1] V. I. ARNOLD, *Mathematical methods of classical mechanics*, Springer-Verlag, New York, 1983.
- [2] K. R. MEYER AND G. R. HALL, *Introduction to Hamiltonian dynamical Systems and the n-body problem*, Applied Mathematical Sciences, vol.90, Springer-Verlag, New York, 1992.
- [3] W. KRATZ, *Quadratic Functionals in Variational Analysis and Control Theory*, Akademik Verlag GmbH, Berlin, 1995.
- [4] F. V. ATKINSON, *Discrete and Continuous Boundary Value Problems*, Academic Press, New York, 1964.
- [5] C. D. AHLBRANDT, *Equivalence of discrete Euler equations and discrete Hamiltonian systems*, J. Math. Anal. Appl. **180**(1993), 498-517.
- [6] C. AHLBRANDT AND A. PETERSON, *Discrete Hamiltonian Systems: Difference Equations, Continued Fractions, and Riccati Equations*, Kluwer Academic Publishers, The Netherlands, 1996.
- [7] YUMING SHI, *Symplectic Structure of Discrete Hamiltonian Systems*, J. Math. Anal. Appl. **266**(2002), 472-478.
- [8] YUMING SHI, *Spectral theory of discrete linear Hamiltonian systems*, J. Math. Anal. Appl. **289**(2004), 554-570.
- [9] W. N. EVERITT AND L. MARKUS, *Complex symplectic geometry with applications to ordinary differential operators*, Trans. of the Amer. Math. Soc. **351**(1999), 4905-4945.
- [10] W. N. EVERITT AND L. MARKUS, *Boundary value problems and symplectic algebra for ordinary differential and quasi-differential operators*, Math. Surveys and Monographs, Vol. **61**, Amer. Math. Soc., Providence, RI, 1999.
- [11] W.N.EVERITT AND A.ZETTL, *Differential operators generated by a countable number of quasi-differential expressions on the real line*, Proc. London Math. Soc. **64**(1992), 524-544.
- [12] Y.SHI, *Classification for a class of singular discrete linear Hamiltonian systems with one singular endpoint*. To appear.
- [13] SHURONG SUN, SHAOZHU CHEN AND ZHEMLAI HAN, *The Complex Symplectic Geometric Characterization of Self-adjoint Extensions for Singular Discrete Hamiltonian systems*. Submitted.