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Global Weak Solutions to the Landau-Lifshitz System in 3D[1]

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ABSTRACT

By considering a general form of the Landau-Lifshitz equation under the influence of a homogeneous external magnetic field, we prove that for a ferromagnetic body which occupies a bounded domain Ω in $\mathbb{R}^3$ there exists a global weak solution either for the Dirichlet problem or for the Neumann problem. Although there is, in general, non-uniqueness result for the Landau-Lifshitz equation, the uniqueness result for the dynamic equation with constant initial data, which connects with the ground state of the magnetization in physical meanings, is pointed out.

RESUMEN

Mediante la consideración de una forma general de la ecuación de Landau-Lifshitz, bajo la influencia de un campo magnético externo homogéneo, probamos que para un cuerpo ferromagnético que ocupa un dominio acotado Ω en $\mathbb{R}^3$ existe una solución débil global, ya sea para el problema de Dirichlet o bien para el problema de Neumann. Aún cuando hay, en general, resultados de no unicidad para la ecuación de Landau-Lifshitz, se muestra el resultado de unicidad para

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la ecuación dinámica con condición inicial constante, que conecta con el estado fundamental de la magnetización en el sentido físico.

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Math. Subj. Class.: *82D40, 35K55, 35D05*

1 Introduction

The magnetization dynamics is an interesting field researched in mathematical physics. The macroscopic theory of ferromagnetism says that the state of a magnetic material can be described by the magnetization vector u; and thus, the dynamics and kinetics of a ferromagnet is dictated by variations in its magnetization. The magnetization in a continuum ferromagnet as a function of time and space, $u \equiv (u_1(t,x), u_2(t,x), u_3(t,x))^T \in \mathbb{R}^3$, is a solution of the nonlinear Landau-Lifshitz equation [16]:

$$\partial_t u = -\rho u \times \tilde{H}_{eff} - \lambda u \times (u \times \tilde{H}_{eff}), \tag{1.1}$$

where $\rho \in \mathbb{R}\backslash\{0\}$ and $\times$ denotes the cross product in $\mathbb{R}^3$. The parameter $\lambda > 0$, called the Gilbert damping constant [11, 17], represents the relaxation constant determining the motional damping of the vector u. The magnetic energy $\tilde{E}$ is assumed to be a functional of u and its spatial derivatives,

$$\tilde{E} = \int h(u, \nabla u) \mathrm{d}x, \tag{1.2}$$

where $h(u, \nabla u)$ is the magnetic energy density and ∇u the gradient of u. The effective magnetic field $\tilde{H}_{eff}$ is equal to the variational derivative of the magnetic energy $\tilde{E}$ with respect to the vector u,

$$\tilde{H}_{eff} = -\delta \tilde{E}/\delta u. \tag{1.3}$$

In order to give (1.1) a definite meaning one has to specify the energy density. In physics settings [14, 21], the density often takes the form of

$$h(u, \nabla u) = \frac{\alpha}{2} \nabla u \cdot \nabla u + h_{an} - u \cdot H, \tag{1.4}$$

where the first term is the nonuniform exchange energy density with constant $\alpha > 0$, the third term is the contribution from the external field H and h_{an} is the anisotropy energy density,

$$h_{an} = -\frac{1}{2}\beta_1 u_1^2 - \frac{1}{2}\beta_3 u_3^2, \tag{1.5}$$

where $\beta_1, \beta_3 \in \mathbb{R}$ are the anisotropy constants. Since there is a special feature of (1.1) that it preserves in the time-direction the modulus of u,$\partial_t |u|^2 = 2\partial_t u \cdot u \equiv 0$, one can assume $|u|$ to be constant provided that $|u(0,x)|$ is constant for all x (physicists say that this constant relies on the material and its temperature), so the term with u_2^2 in h_{an} could be omitted. Moreover, one may set this constant to be the unit, i.e. $u(t,x) \in \mathbb{S}^2 \subset \mathbb{R}^3$. Let $H_{eff} = \tilde{H}_{eff}/\alpha$, then we have

$$H_{eff} = \Delta u + \frac{\beta_1}{\alpha} u_1 \hat{k}_1 + \frac{\beta_3}{\alpha} u_3 \hat{k}_3 + \frac{H}{\alpha}, \tag{1.6}$$

where $\hat{k}_i (i = 1, 2, 3)$ are the three standard orthogonal axes in $\mathbb{R}^3$.

When the anisotropy energy is of the form of (1.5),

- if $\beta_1 \neq 0, \beta_3 \neq 0$, it is a case of biaxial ferromagnet;
- if $\beta_1 = 0, \beta_3 \neq 0$, (1.5) is corresponding to a situation of uniaxial ferromagnet, with the anisotropy axis coincident with the $\hat{k}_3$-axis;
- if $\beta_1 = 0, \beta_3 > 0$, the anisotropy is of the easy-axis type;
- if $\beta_1 = 0, \beta_3 < 0$, the anisotropy is of the easy-plane type;
- if $\beta_1 = \beta_3 = 0$, this is the case of isotropic ferromagnet.

The Landau-Lifshitz equation (1.1) bears a fundamental role in the understanding of non-equilibrium magnetism [14, 20], just as the Navier-Stokes equation does in that of fluid dynamics. Many physicists and mathematicians do a lot of work on it. With the effective field to be simplified by $H_{eff} = \Delta u$, they found that the equation

$$\partial_t u = -\rho\alpha u \times \Delta u - \lambda\alpha u \times (u \times \Delta u) \tag{1.7}$$

could be thought of as a linear combination of two parts, one is the Heisenberg system [18], also called the Schrödinger map [5]:

$$\partial_t u = u \times \Delta u; \tag{1.8}$$

the other is the heat flow into spheres:

$$\partial_t u = \Delta u - (\Delta u \cdot u)u. \tag{1.9}$$

The former is a generalized nonlinear Schrödinger equation, which has complete integrability in the spherically symmetric case [6, 19]. Although some authors have studied (1.7) or (1.8), many basic mathematical questions remain open. For example, the global existence of smooth solutions is not known when the space dimension is greater than one [5, 9]. The global existence of smooth solutions of (1.8), up to now, has been established only for small initial data [2, 5]. In 1986, C. Bardos, C. Sulem and P.L. Sulem [2] studied the Cauchy problem of (1.8) in general dimensions and obtained both the global existence of weak solutions and that of smooth solutions partly. Lately researchers in [8, 10] constructed some exact nontrivial global solutions and some blow-up solutions for (1.8) in 2D cylindrical symmetric case. For the Cauchy Problem of (1.7) in 3D, the global existence of weak solutions has been proved by F. Alouges and A. Soyeur [1]. In the same paper, the global existence and non-uniqueness of weak solutions to the homogeneous Neumann problem of (1.7) have also been proved provided that λ is different from zero. On the other hand, the heat flow (1.9) is a parabolic equation according to the static Landau-Lifshitz equation:

$$\Delta u - (\Delta u \cdot u)u = 0, \tag{1.10}$$

which is just an Euler-Lagrange equation with respect to the Landau-Lifshitz functional. The equation (1.10) can be treated as a generalized harmonic map [4, 7]. Due to this, the interesting properties, like maximum principles, existence and uniqueness results for the small solutions, and non-uniqueness (in 2D) for the large solutions to the Dirichlet problem of (1.10), were studied by Q. Chen [3]. He also obtained similar properties of (1.9) with Dirichlet condition by the standard heat-flow method. Compared with (1.10), the static Landau-Lifshitz equation with the external field is

$$\Delta u - (\Delta u \cdot u)u + H - (H \cdot u)u = 0. \tag{1.11}$$

In [12, 13], M.-C. Hong and L. Lemaire studied the properties of the smooth solutions to (1.11) with constant boundary value. From this, they showed us the important effect of H for the isotropic ferromagnet.

The main aim of this paper is to offer a global existence result of weak solutions to (1.1) in 3D when the general form of (1.4) is considered. In Section 2, we describe the magnetization models for a bounded ferromagnetic body either with the nonhomogeneous Dirichlet boundary condition or with the homogeneous Neumann boundary condition. The main global existence results for these problems are stated with that. We then proceed to prove these results in Section 3 and 4. We use the Gilbert damping term to realize the global weak solutions with finite energy by Galerkin's method and penalty function method. Finally, in Section 5 we point out that some constant solutions for the dynamic Landau-Lifshitz equation are unique. From this, we show partly the properties of the ground states in the motion of the magnetization.

2 The Models and Main Theorems

We are interested in the magnetization phenomenon when one puts a ferromagnetic body into a $\hat{h}$-direction external uniform field, $H = H_0\hat{h}$, where $H_0 \in \mathbb{R}$ is a constant and $\hat{h} = (h_1, h_2, h_3)^T$ a unit vector in $\mathbb{R}^3$. In this case, the effective field reads

$$H_{eff} = \Delta u + \frac{\beta_1 u_1}{\alpha}\hat{k}_1 + \frac{\beta_3 u_3}{\alpha}\hat{k}_3 + \frac{H_0}{\alpha}\hat{h}, \tag{2.1}$$

and the magnetization is supposed to be equal to H outside the ferromagnetic body. In mathematics, we explain this physical model by the Landau-Lifshitz system with the nonhomogeneous Dirichlet boundary condition,

$$\begin{cases} \partial_t u = -\rho\alpha u \times H_{eff} - \lambda\alpha u \times (u \times H_{eff}) & \text{on } \Omega \\ u(0,x) = u_0(x) \quad \text{on } \Omega \\ u(t,x) = \hat{g} \quad \text{on } \partial\Omega \\ |u| = 1 \quad \text{on } \Omega, \end{cases} \tag{2.2}$$

where the magnetic body Ω ($\subset \mathbb{R}^3$) is a bounded domain with smooth boundary $\partial\Omega$, and $u_0(x) \equiv (u_{10}(x), u_{20}(x), u_{30}(x))^T$ is the initial data for the magnetization vector $u = (u_1, u_2, u_3)^T : [0, +\infty) \times \Omega \to \mathbb{S}^2 \subset \mathbb{R}^3$. The constraint Dirichlet boundary value, $\hat{g} = (g_1, g_2, g_3)^T$ a unit vector in $\mathbb{R}^3$, may come either from the case of the strong external field considered or from the case of the ground states considered (another case is that when the non-topological magnetic soliton is considered in the whole space, one may suppose that it has constant value on the boundary corresponding to $|x| = \infty$) [12, 13, 14]. See the examples for the ground states in section 5.

Another mathematical description can be given with the homogeneous Neumann boundary condition [1],

$$\begin{cases} \partial_t u = -\rho\alpha u \times H_{eff} - \lambda\alpha u \times (u \times H_{eff}) & \text{on } \Omega \\ u(0, x) = u_0(x) & \text{on } \Omega \\ \frac{\partial u}{\partial \vec{n}}(t, x) = 0 & \text{on } \partial\Omega \\ |u| = 1 & \text{on } \Omega, \end{cases} \tag{2.3}$$

where $\vec{n}$ is the outward normal on the boundary of Ω.

Make an important assumption in this paper that $\lambda > 0$. For u smooth enough the Gilbert damping term plays an important role for us to obtain

$$\partial_t u - \frac{\lambda}{\rho} u \times \partial_t u + \frac{\alpha(\rho^2 + \lambda^2)}{\rho} u \times H_{eff} = 0, \tag{2.4}$$

$$\frac{\lambda}{\rho}\partial_t u + u \times \partial_t u + \frac{\alpha(\rho^2 + \lambda^2)}{\rho}((u \cdot H_{eff})u - H_{eff}) = 0, \tag{2.5}$$

where H_{eff} is defined in (2.1) and

$$u \times H_{eff} = \sum_{i=1}^{3} \partial_i(u \times \partial_i u) + \frac{\beta_1 u_1}{\alpha} u \times \hat{k}_1 + \frac{\beta_3 u_3}{\alpha} u \times \hat{k}_3 + \frac{H_0}{\alpha} u \times \hat{h}. \tag{2.6}$$

Definition 2.1. *We define the space*

$$W \equiv \{w | w \in L^\infty(\mathbb{R}^+; \mathbb{H}^1(\Omega)) \text{ and } \partial_t w \in L^2(\mathbb{R}^+; \mathbb{L}^2(\Omega))\}, \tag{2.7}$$

where $\mathbb{R}^+ \equiv [0, \infty)$, $\mathbb{H}^1(\Omega) = (H^1(\Omega))^3$ *and* $\mathbb{L}^q(\Omega) = (L^q(\Omega))^3$, *and define*

$$W_0 \equiv \{w | w \in L^\infty(\mathbb{R}^+; \mathbb{H}_0^1(\Omega)) \text{ and } \partial_t w \in L^2(\mathbb{R}^+; \mathbb{L}^2(\Omega))\} \subset W, \tag{2.8}$$

where $\mathbb{H}_0^1(\Omega) = (H_0^1(\Omega))^3$.

Theorem 2.2. *Suppose* $u_0 \in \mathbb{H}^1(\Omega)$ *satisfies* $u_0 - \hat{g} \in \mathbb{H}_0^1(\Omega)$ *and* $|u_0(x)| = 1$ *a.e. on* Ω. *Then there exists a global weak solution* $u(t, x)$ *to (2.2), such that (in the sense of)*

(i) $u \in W$ *and* $|u(t, \cdot)| = 1$ *a.e. on* Ω*;*

(ii) for all $T > 0$ and $\varphi \in L^2(0,T;\mathbb{H}_0^1(\Omega))$,

$$\int_{V_T} \partial_t u \cdot \varphi - \frac{\lambda}{\rho}(u \times \partial_t u)\cdot\varphi - \frac{\alpha(\rho^2+\lambda^2)}{\rho}\sum_{i=1}^{3}(u\times\partial_i u)\cdot\partial_i\varphi$$

$$+\frac{(\rho^2+\lambda^2)}{\rho}[\beta_1 u_1(u\times\hat{k}_1)\cdot\varphi + \beta_3 u_3(u\times\hat{k}_3)\cdot\varphi + H_0(u\times\hat{h})\cdot\varphi]dxdt = 0, \qquad (2.9)$$

where $\cdot$ denotes the inner product in $\mathbb{R}^3$ and $V_T = [0,T]\times\Omega$;

(iii) $u(0,x) = u_0(x)$ in the sense of traces;

(iv) $u(t,x) = \hat{g}$ on $\partial\Omega$; and

(v) the energy inequality holds:

$$E(T) + \frac{\lambda}{\alpha(\rho^2+\lambda^2)}\int_{V_T}|\partial_t u|^2 dxds \le E(0) \qquad \text{for all } T>0, \qquad (2.10)$$

where $E(T)$ is defined in (3.69).

Remark 2.3. *It is an immediate consequence of the Definition 2.1 and Theorem 2.2 that the solution u is in $\mathbb{H}^1(V_T)$ for all $T > 0$. We also note that $u \in C(\mathbb{R}^+;\mathbb{L}^2(\Omega))$ since we can get $u \in C(0,T;\mathbb{L}^2(\Omega))$, for each $T > 0$, from the facts that $u \in L^2(0,T;\mathbb{H}^1(\Omega))$ and $\partial_t u \in L^2(0,T;\mathbb{L}^2(\Omega))$.*

Theorem 2.4. *Suppose $u_0 \in \mathbb{H}^1(\Omega)$ satisfies $|u_0(x)| = 1$ a.e. on Ω. Then there exists a global weak solution $u(t,x)$ to (2.3), such that (in the sense of)*

(i) $u \in W$ with $u(0,x) = u_0(x)$ in the sense of traces and $|u(t,\cdot)| = 1$ a.e. on Ω;

(ii) for all $T > 0$ and $\varphi \in L^2(0,T;\mathbb{H}^1(\Omega))$, (2.9) and (2.10) hold.

3 Proof of Theorem 2.2

The idea of the proof is mainly based upon both the method in Alouges and Soyeur [1] and a domination in (3.25—3.26). We replace the Landau-Lifshitz equation in system (2.2) by (2.5),

$$\begin{cases} \frac{\lambda}{\rho}\partial_t u + u\times\partial_t u + \frac{\alpha(\rho^2+\lambda^2)}{\rho}((u\cdot H_{eff})u - H_{eff}) = 0 & \text{on } \Omega \\ u(0,x) = u_0(x) & \text{on } \Omega \\ u(t,x) = \hat{g} & \text{on } \partial\Omega \\ |u| = 1 & \text{on } \Omega, \end{cases} \qquad (3.1)$$

where H_{eff} is as in (2.1). Using penalty function method we remove the constraint condition $|u| = 1$ in (3.1). Consider a family of problems

$$\begin{cases} \frac{\lambda}{\rho}\partial_t u + u\times\partial_t u = \frac{\alpha(\rho^2+\lambda^2)}{\rho}\left(\Delta u + \frac{\beta_1 u_1}{\alpha}\hat{k}_1 + \frac{\beta_3 u_3}{\alpha}\hat{k}_3 + \frac{H_0}{\alpha}\hat{h} - k(|u|^2-1)u\right) \\ u(0,x) = u_0(x) \qquad \text{on } \Omega \\ u(t,x) = \hat{g} \qquad \text{on } \partial\Omega, \end{cases} \qquad (3.2)$$

where k is a positive integer and $u : [0,+\infty)\times\Omega \to \mathbb{R}^3$.

3.1 Solve (3.2) by Galerkin method

We hope $u(t,\cdot)$ is in $\mathbb{H}^1(\Omega)$ when u solves the problem (3.2). As well known, the operator $-\Delta$ maps $H_0^1(\Omega)$ to its dual space $H^{-1}(\Omega)$. Let $\{e_i(x)\}_{i=1}^{\infty}$ be the eigenvectors of operator $-\Delta$ in $H_0^1(\Omega)$, and $\lambda_i \geq 0$ be the eigenvalue corresponding to $e_i(x)$. Thus $\{e_i(x)\}_{i=1}^{\infty}$ construct the orthogonal basis of $H_0^1(\Omega)$, and they are supposed to be standard under L^2 inner product. Indeed, one can treat $\{\frac{e_i(x)}{(1+\lambda_i)^{1/2}}\}_{i=1}^{\infty}$ as the standard orthogonal basis of $H_0^1(\Omega)$. That is, for all $i,j = 1,2,\ldots$

$$\begin{cases} e_i(x) \in C_0^{\infty}(\Omega) \subset H_0^1(\Omega), & \text{(elliptic property)} \\ \langle -\Delta e_i, \omega\rangle_{L^2(\Omega)} = \lambda_i \langle e_i, \omega\rangle_{L^2(\Omega)} & \text{for all } \omega \in H_0^1(\Omega) \\ \langle e_i, e_j\rangle_{L^2(\Omega)} = \delta_{ij} \\ \langle e_i, e_j\rangle_{H^1(\Omega)} = (1+\lambda_i)^{1/2}(1+\lambda_j)^{1/2}\delta_{ij}. \end{cases} \tag{3.3}$$

Let

$$\begin{aligned} v_n^k &\equiv (v_{1,n}^k(t,x), v_{2,n}^k(t,x), v_{3,n}^k(t,x))^T \\ &= \Sigma_{i=1}^n (1+\lambda_i)^{1/2}\varphi_i^k(t)\frac{e_i(x)}{(1+\lambda_i)^{1/2}} \qquad n = 1,2,\ldots, \end{aligned} \tag{3.4}$$

and

$$u_n^k \equiv (u_{1,n}^k(t,x), u_{2,n}^k(t,x), u_{3,n}^k(t,x))^T = v_n^k + \hat{g}, \tag{3.5}$$

where $\varphi_i^k(t) = (\varphi_{1,i}^k(t), \varphi_{2,i}^k(t), \varphi_{3,i}^k(t))^T \in \mathbb{R}^3$. We search u_n^k that verifies the following inner product problem. For all $l = 1,2,\ldots,n$,

$$\begin{cases} \langle \frac{\lambda}{\rho}\partial_t u_n^k + u_n^k \times \partial_t u_n^k, e_l(x)\rangle_{L^2(\Omega)} = \\ \frac{\alpha(\rho^2+\lambda^2)}{\rho}\left(\langle \Delta u_n^k + \frac{\beta_1 u_{1,n}^k}{\alpha}\hat{k}_1 + \frac{\beta_3 u_{3,n}^k}{\alpha}\hat{k}_3 + \frac{H_0}{\alpha}\hat{h} - k(|u_n^k|^2 - 1)u_n^k, e_l(x)\rangle_{L^2(\Omega)}\right) \\ \langle u_n^k(0,x) - u_0(x), e_l(x)\rangle_{H^1(\Omega)} = 0 \qquad \text{on } \Omega \\ u_n^k(t,x) = \hat{g} \qquad \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

The identities (3.4—3.5) lead us to having $v_n^k|_{\partial\Omega}(t,x) = 0$ and $u_n^k|_{\partial\Omega}(t,x) = \hat{g}$ trivially.

Proposition 3.1. *For all k and n, the problem (3.6) has a unique local smooth solution.*

Proof. Using (3.4—3.5), (3.6) becomes

$$\begin{cases} \tilde{\lambda}\langle \Sigma_1^n \dot{\varphi}_i^k e_i(x), e_l(x)\rangle_{L^2(\Omega)} + \tilde{\rho}\langle (\Sigma_1^n \varphi_i^k e_i(x) + \hat{g}) \times \Sigma_1^n \dot{\varphi}_i^k e_i(x), e_l(x)\rangle_{L^2(\Omega)} \\ \qquad = \langle \Sigma_1^n \varphi_i^k \Delta e_i(x), e_l(x)\rangle_{L^2(\Omega)} + \frac{\beta_1}{\alpha}\langle (\Sigma_1^n \varphi_{1,i}^k e_i(x) + g_1)\hat{k}_1, e_l(x)\rangle_{L^2(\Omega)} \\ \qquad + \frac{\beta_3}{\alpha}\langle (\Sigma_1^n \varphi_{3,i}^k e_i(x) + g_3)\hat{k}_3, e_l(x)\rangle_{L^2(\Omega)} + \frac{H_0}{\alpha}\langle \hat{h}, e_l(x)\rangle_{L^2(\Omega)} \\ \qquad - k\langle (|\Sigma_1^n \varphi_i^k e_i(x) + \hat{g}|^2 - 1)(\Sigma_1^n \varphi_i^k e_i(x) + \hat{g}), e_l(x)\rangle_{L^2(\Omega)} \\ \langle \Sigma_1^n \varphi_i^k(0) e_i(x) - v_0(x), e_l(x)\rangle_{H^1(\Omega)} = 0, \end{cases} \tag{3.7}$$

where $\tilde{\lambda} = \frac{\lambda}{\alpha(\rho^2+\lambda^2)}$, $\tilde{\rho} = \frac{\rho}{\alpha(\rho^2+\lambda^2)}$ and $v_0(x) \equiv u_0(x) - \hat{g} \in \mathbb{H}_0^1(\Omega)$. The formula,

$$\langle \Sigma_1^n \varphi_i^k(0) e_i(x) - v_0(x), e_l(x) \rangle_{H^1(\Omega)} = \varphi_l^k(0)(1+\lambda_l) - \langle v_0(x), e_l(x) \rangle_{H^1(\Omega)}, \tag{3.8}$$

implies that the second equation in (3.7) is

$$\varphi_l^k(0) = \frac{\langle v_0(x), e_l(x) \rangle_{H^1(\Omega)}}{(1+\lambda_l)}. \tag{3.9}$$

Let's now turn to the terms of the first equation in (3.7). Since (3.3),

$$\langle \Sigma_1^n \dot{\varphi}_i^k e_i, e_l \rangle_{L^2} = \dot{\varphi}_l^k, \tag{3.10}$$

$$\langle (\Sigma_1^n \varphi_i^k e_i + \hat{g}) \times \Sigma_1^n \dot{\varphi}_i^k e_i, e_l \rangle_{L^2} = \Sigma_{s,t=1}^n \langle e_s e_t, e_l \rangle_{L^2} \varphi_s^k \times \dot{\varphi}_t^k + \hat{g} \times \dot{\varphi}_l^k, \tag{3.11}$$

$$\langle \Sigma_1^n \varphi_i^k \Delta e_i, e_l \rangle_{L^2} = -\langle \Sigma_1^n \varphi_i^k \lambda_i e_i, e_l \rangle_{L^2} = -\lambda_l \varphi_l^k, \tag{3.12}$$

$$\langle (\Sigma_1^n \varphi_{1,i}^k e_i + g_1) \hat{k}_1, e_l \rangle_{L^2} = \varphi_{1,l}^k \hat{k}_1 + g_1 \langle 1, e_l \rangle_{L^2} \hat{k}_1, \tag{3.13}$$

$$\langle (\Sigma_1^n \varphi_{3,i}^k e_i + g_3) \hat{k}_3, e_l \rangle_{L^2} = \varphi_{3,l}^k \hat{k}_3 + g_3 \langle 1, e_l \rangle_{L^2} \hat{k}_3, \tag{3.14}$$

$$\langle \hat{h}, e_l \rangle_{L^2} = \langle 1, e_l \rangle_{L^2} \hat{h}, \tag{3.15}$$

and

$$\begin{aligned}
&\langle (|\Sigma_1^n \varphi_i^k e_i + \hat{g}|^2 - 1)(\Sigma_1^n \varphi_i^k e_i + \hat{g}), e_l \rangle_{L^2} \\
= \ &\langle (\Sigma_{s,t=1}^n \varphi_s^k \cdot \varphi_t^k e_s e_t + 2\Sigma_{s=1}^n \varphi_s^k \cdot \hat{g} e_s)(\Sigma_1^n \varphi_i^k e_i + \hat{g}), e_l \rangle_{L^2} \\
= \ &\Sigma_{s,t,i=1}^n \langle e_s e_t e_i, e_l \rangle_{L^2} (\varphi_s^k \cdot \varphi_t^k) \varphi_i^k + 2\Sigma_{s,i=1}^n \langle e_s e_i, e_l \rangle_{L^2} (\varphi_s^k \cdot \hat{g}) \varphi_i^k \\
&+ \Sigma_{s,t=1}^n \langle e_s e_t, e_l \rangle_{L^2} (\varphi_s^k \cdot \varphi_t^k) \hat{g} + 2(\varphi_l^k \cdot \hat{g}) \hat{g}.
\end{aligned} \tag{3.16}$$

By (3.9—3.16), (3.7) can be written as an ODE system of

$$\Phi_n^k(t) \equiv \left(\varphi_{1,1}^k, \varphi_{2,1}^k, \varphi_{3,1}^k, \cdots, \varphi_{1,n}^k, \varphi_{2,n}^k, \varphi_{3,n}^k\right)^T, \tag{3.17}$$

which has $3n$ components of $\varphi_{i,j}^k(t)$, $1 \le i \le 3, 1 \le j \le n$. Namely

$$\begin{cases}
(\tilde{\lambda} I + T)\dot{\Phi}_n^k = F(\Phi_n^k) \\
\Phi_n^k(0) = \begin{pmatrix} \langle v_0(x), e_1(x) \rangle_{H^1(\Omega)} / (1+\lambda_1) \\ \cdots \\ \langle v_0(x), e_n(x) \rangle_{H^1(\Omega)} / (1+\lambda_n) \end{pmatrix},
\end{cases} \tag{3.18}$$

where I is the identity matrix, $T = T(\Phi_n^k)$ is a $3n \times 3n$ matrix of $\varphi_{i,j}^k(t)$ and $F(\Phi_n^k)$ has $3n$ components which are polynomial functions of $\varphi_{i,j}^k(t)$ as linear combinations of (3.12—3.16). In detail

$$T = \tilde{\rho} \begin{pmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & \cdots & \\ T_{n1} & \cdots & T_{nn} \end{pmatrix} + \tilde{\rho} \begin{pmatrix} R_{11} & \cdots & R_{1n} \\ \vdots & \cdots & \\ R_{n1} & \cdots & R_{nn} \end{pmatrix},$$

where

$$T_{pq} = \begin{pmatrix} 0 & -\Sigma_{s=1}^{n}\langle e_s e_q, e_p\rangle_{L^2}\varphi_{3,s}^k & \Sigma_{s=1}^{n}\langle e_s e_q, e_p\rangle_{L^2}\varphi_{2,s}^k \\ \Sigma_{s=1}^{n}\langle e_s e_q, e_p\rangle_{L^2}\varphi_{3,s}^k & 0 & -\Sigma_{s=1}^{n}\langle e_s e_q, e_p\rangle_{L^2}\varphi_{1,s}^k \\ -\Sigma_{s=1}^{n}\langle e_s e_q, e_p\rangle_{L^2}\varphi_{2,s}^k & \Sigma_{s=1}^{n}\langle e_s e_q, e_p\rangle_{L^2}\varphi_{1,s}^k & 0 \end{pmatrix},$$

$$R_{pq} = \begin{pmatrix} 0 & -g_3 & g_2 \\ g_3 & 0 & -g_1 \\ -g_2 & g_1 & 0 \end{pmatrix} \text{ for } p = q, \text{ and } R_{pq} = 0 \text{ for } p \neq q.$$

Because of the antisymmetry of $T(\Phi_n^k)$, the eigenvalues of $T(\Phi_n^k)$ are either pure imaginary or zero. Hence $\tilde{\lambda}I + T(\Phi_n^k)$ is always reversible, moreover

$$|\det(\tilde{\lambda}I + T)| \geq \tilde{\lambda}^{3n}.$$

Since $F(\Phi_n^k)$ is Lipshitz continuous provided that $|\Phi_n^k|$ is bounded, there is a unique local solution $\Phi_n^k(t) \in C^1[0, \delta_n^k)$ to (3.18), for all n and k by ODE Theory. We can obtain $\Phi_n^k(t) \in C^\infty[0, \delta_n^k)$ because both $\tilde{\lambda}I + T$ and F are smooth for Φ_n^k. In other words, $u_n^k = v_n^k + \hat{g}$, where $v_n^k = \Sigma_{i=1}^n \varphi_i^k(t)e_i(x) \in C^\infty([0, \delta_n^k); (C_0^\infty(\Omega))^3)$, is the unique local solution to (3.6). ∎

If we multiply the first identity in (3.6) by $\dot{\varphi}_l^k$ and summate it for $1 \leq l \leq n$, then

$$\begin{aligned} \tilde{\lambda}\int_\Omega |\partial_t u_n^k|^2 \mathrm{d}x &= -\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |\nabla u_n^k|^2 \mathrm{d}x + \frac{\beta_1}{2\alpha}\frac{\partial}{\partial t}\int_\Omega |\hat{k}_1 \cdot u_n^k|^2 \mathrm{d}x + \frac{\beta_3}{2\alpha}\frac{\partial}{\partial t}\int_\Omega |\hat{k}_3 \cdot u_n^k|^2 \mathrm{d}x \\ &\quad + \frac{H_0}{\alpha}\frac{\partial}{\partial t}\int_\Omega \hat{h} \cdot u_n^k \mathrm{d}x - \frac{k}{4}\frac{\partial}{\partial t}\int_\Omega ||u_n^k|^2 - 1|^2 \mathrm{d}x \end{aligned} \tag{3.19}$$

is implied by the facts $u_n^k|_{\partial\Omega}(t, x) = \hat{g}$, $\partial_t u_n^k|_{\partial\Omega}(t, x) = 0$, and

$$\int_\Omega \Delta u_n^k \cdot \partial_t u_n^k \mathrm{d}x = \int_{\partial\Omega} \vec{n} \cdot \nabla u_n^k \cdot \partial_t u_n^k \mathrm{d}\sigma - \int_\Omega \nabla u_n^k \cdot \frac{\partial}{\partial t}\nabla u_n^k \mathrm{d}x = -\frac{\partial}{\partial t}\int_\Omega \frac{|\nabla u_n^k|^2}{2}\mathrm{d}x. \tag{3.20}$$

Define

$$\begin{aligned} E_n^k(t) &\equiv \frac{1}{2}\|\nabla u_n^k\|_{L^2(\Omega)}^2(t) - \frac{\beta_1}{2\alpha}\|\hat{k}_1 \cdot u_n^k\|_{L^2(\Omega)}^2(t) - \frac{\beta_3}{2\alpha}\|\hat{k}_3 \cdot u_n^k\|_{L^2(\Omega)}^2(t) \\ &\quad - \frac{H_0}{\alpha}\int_\Omega \hat{h} \cdot u_n^k \mathrm{d}x(t) + \frac{k}{4}\||u_n^k|^2 - 1\|_{L^2(\Omega)}^2(t). \end{aligned} \tag{3.21}$$

Integrating (3.19) over $[0, t]$, it becomes the energy equality

$$E_n^k(t) + \tilde{\lambda}\int_{V_t} |\partial_t u_n^k|^2 \mathrm{d}x\mathrm{d}s = E_n^k(0), \qquad V_t = [0, t] \times \Omega \tag{3.22}$$

where

$$\begin{aligned} E_n^k(0) &= \frac{1}{2}\|\nabla u_n^k(0)\|_{L^2(\Omega)}^2 - \frac{\beta_1}{2\alpha}\|\hat{k}_1 \cdot u_n^k(0)\|_{L^2(\Omega)}^2 - \frac{\beta_3}{2\alpha}\|\hat{k}_3 \cdot u_n^k(0)\|_{L^2(\Omega)}^2 \\ &\quad - \frac{H_0}{\alpha}\int_\Omega \hat{h} \cdot u_n^k(0) \mathrm{d}x + \frac{k}{4}\||u_n^k(0)|^2 - 1\|_{L^2(\Omega)}^2. \end{aligned} \tag{3.23}$$

and

$$u_n^k(0) = v_n^k(0) + \hat{g} = \Sigma_{i=1}^n \frac{\langle v_0(x), e_i(x)\rangle_{H^1(\Omega)}}{1+\lambda_i} e_i(x) + \hat{g}. \tag{3.24}$$

Since $v_0 \in \mathbb{H}_0^1(\Omega)$, we have $\|v_n^k(0)\|_{H^1(\Omega)}^2 \le \|v_0\|_{H^1(\Omega)}^2$ and then, $\|\nabla u_n^k(0)\|_{L^2(\Omega)}^2 \le C_0$, $\|\hat{k}_1 \cdot u_n^k(0)\|_{L^2(\Omega)}^2 \le C_0$, $\|\hat{k}_3 \cdot u_n^k(0)\|_{L^2(\Omega)}^2 \le C_0$, $|\int_\Omega \hat{h} \cdot u_n^k(0) \mathrm{d}x| \le C_0$ and $\||u_n^k(0)|^2 - 1\|_{L^2(\Omega)}^2 \le C_0$ by the Sobolev embedding theorem, where C_0 is used to denote the constant depending only on $\|v_0\|_{H^1(\Omega)}$ and $|\Omega|$ (maybe also on $|\lambda|, |\rho|, |\alpha|, |\beta_1|, |\beta_3|, |H_0|$ and K_0 in some cases below), but not on n and k. Hence $E_n^k(0) \le kC_0$.

The second, third and fourth term in the definition of E_n^k can be dominated by the fifth term (for k large enough), since

$$\begin{aligned} \|\hat{k}_i \cdot u_n^k\|_{L^2(\Omega)}^2(t) &\le \int_\Omega |u_n^k|^2 \mathrm{d}x = \int_\Omega |u_n^k|^2 - 1 \ \mathrm{d}x + |\Omega| \\ &\le \frac{1}{2}\||u_n^k|^2 - 1\|_{L^2(\Omega)}^2(t) + \frac{3}{2}|\Omega|, \end{aligned} \tag{3.25}$$

and

$$\left|\int_\Omega \hat{h} \cdot u_n^k \mathrm{d}x\right|(t) \le \|u_n^k\|_{L^2(\Omega)}(t)|\Omega|^{1/2} \le \frac{1}{4}\||u_n^k|^2 - 1\|_{L^2(\Omega)}^2(t) + \frac{5}{4}|\Omega|. \tag{3.26}$$

Thus, there exists a fixed $K_0 > 0$, only depending on α, β_1, β_3 and H_0, such that

$$\begin{aligned} -\frac{\beta_1}{2\alpha}\|\hat{k}_1 \cdot u_n^k\|_{L^2(\Omega)}^2(t) - \frac{\beta_3}{2\alpha}\|\hat{k}_3 \cdot u_n^k\|_{L^2(\Omega)}^2(t) - \frac{H_0}{\alpha}\int_\Omega \hat{h} \cdot u_n^k \mathrm{d}x(t) \\ + \frac{K_0}{4}\||u_n^k|^2 - 1\|_{L^2(\Omega)}^2(t) + K_0|\Omega| \ge 0. \end{aligned} \tag{3.27}$$

When, $k > K_0$, the domination mentioned above being achieved, there hold

$$0 \le \frac{1}{2}\|\nabla u_n^k\|_{L^2(\Omega)}^2 + \frac{k - K_0}{4}\||u_n^k|^2 - 1\|_{L^2(\Omega)}^2 \le E_n^k(t) + K_0|\Omega|, \tag{3.28}$$

and

$$0 \le E_n^k(0) + K_0|\Omega| \le kC_0. \tag{3.29}$$

As a consequence of the energy equality (3.22), (3.28) and (3.29), for all $k > K_0, n = 1, 2, \ldots$ and $t \in [0, \delta_n)$, there hold

$$\|\nabla u_n^k\|_{L^2(\Omega)}^2(t) \le kC_0, \quad \||u_n^k|^2 - 1\|_{L^2(\Omega)}^2(t) \le kC_0, \tag{3.30}$$

$$\|u_n^k\|_{L^2(\Omega)}^2(t) = \int_\Omega |u_n^k|^2 - 1 \ \mathrm{d}x + |\Omega| \le \||u_n^k|^2 - 1\|_{L^2(\Omega)}|\Omega|^{1/2} + |\Omega| \le kC_0, \tag{3.31}$$

and

$$\int_{V_t} |\partial_t u_n^k|^2 \mathrm{d}x\mathrm{d}s = \int_0^t \|\partial_t u_n^k\|_{L^2(\Omega)}^2(s)\mathrm{d}s \le kC_0. \tag{3.32}$$

Proposition 3.2. *There exists a $K_0 > 0$ such that for all $k > K_0$ and for all n, the unique local solution of (3.6) obtained in Proposition 3.1 can be extended globally.*

Proof. Let $k > K_0$ and n be fixed. Since (3.3—3.5), we get from (3.31)

$$\Sigma_{i=1}^{n}(\varphi_i^k(t))^2 = \|v_n^k\|_{L^2(\Omega)}^2(t) \le kC_0, \qquad \text{for all } t \in [0, \delta_n^k).$$

This implies that $\Phi_n^k(t)$ is bounded on $[0, \delta_n^k)$, and that $|F(\Phi_n^k(t))| \le C_1$ uniformly on $[0, \delta_n^k)$ for some constant $C_1 > 0$, due to the continuity of F. It follows standardly from the known inequality (3.19) that $\Phi_n^k(t)$ can be extended to $[0, \infty)$, that is, the solution $u_n^k(t, x)$ of (3.6) is global. ∎

From now on, we consider only the case of k large enough when the global solution $u_n^k = v_n^k + \hat{g}$, where $v_n^k \in C^\infty(\mathbb{R}^+; (C_0^\infty(\Omega))^3)$, exists from Proposition 3.2. It is easy to see that the energy equality (3.22) and the inequalities (3.30—3.32) preserve for all $t \in [0, \infty)$. Hence from (3.30) and (3.31), one gets that

$$\|v_n^k\|_{L^\infty(\mathbb{R}^+;\mathbb{H}^1(\Omega))}^2 \text{ and } \|u_n^k\|_{L^\infty(\mathbb{R}^+;\mathbb{H}^1(\Omega))}^2 \le kC_0. \tag{3.33}$$

And from (3.32),

$$\|\partial_t v_n^k\|_{L^2(\mathbb{R}^+;\mathbb{L}^2(\Omega))}^2 = \|\partial_t u_n^k\|_{L^2(\mathbb{R}^+;\mathbb{L}^2(\Omega))}^2 \le kC_0. \tag{3.34}$$

Proposition 3.3. *For the sequence $\{u_n^k(t, x)\}_{n=1}^\infty$ obtained in Proposition 3.2, there exists a weak limit $u^k(t, x)$ in W, which is a global weak solution to (3.2).*

Proof. Since (3.33—3.34), there exists a $v^k(t, x) \in W_0$ written as $v^k \equiv (v_1^k, v_2^k, v_3^k)^T$, such that

$$\begin{cases} v_n^k(t, x) \overset{*}{\rightharpoonup} v^k(t, x) \text{ in } L^\infty(\mathbb{R}^+; \mathbb{H}_0^1(\Omega)) \\ \partial_t v_n^k(t, x) \rightharpoonup \partial_t v^k(t, x) \text{ in } L^2(\mathbb{R}^+; \mathbb{L}^2(\Omega)), \end{cases} \tag{3.35}$$

up to subsequences, as $n \to \infty$ (in the following proof we may take limit up to subsequences, as $n \to \infty$). Let $u^k \equiv (u_1^k, u_2^k, u_3^k)^T = v^k + \hat{g}$. We deduce easily from (3.35), by the Sobolev embedding theorem and the Lions compact theorem, that for arbitrary $T > 0$,

$$u_n^k \overset{*}{\rightharpoonup} u^k \qquad \text{in } L^\infty([0, T]; \mathbb{L}^6(\Omega)), \tag{3.36}$$

$$u_n^k \rightharpoonup u^k \qquad \text{in } \mathbb{H}^1([0, T] \times \Omega), \tag{3.37}$$

and

$$u_n^k \to u^k \text{ in } C([0, T]; \mathbb{L}^2(\Omega)) \cap L^q([0, T]; \mathbb{L}^p(\Omega)) \text{ for } q < +\infty, p < 6. \tag{3.38}$$

(See for instance J.L. Lions [15] for very general compactness results of that sort.) Hence (3.35—3.38) together with the well-known fact,

$$e_i(x) \in H_0^1(\Omega) \hookrightarrow L^p(\Omega) \quad \text{for } p < 6, \tag{3.39}$$

give the following (3.40—3.44). First of all,

$$\int_0^T \langle \partial_t u_n^k, e_i\rangle_{L^2(\Omega)}\mathrm{d}t = \int_0^T\int_\Omega \partial_t u_n^k(t,x)e_i(x)\mathrm{d}x\mathrm{d}t \to \int_0^T \langle \partial_t u^k, e_i\rangle_{L^2(\Omega)}\mathrm{d}t, \quad (3.40)$$

and

$$\int_0^T \langle u_n^k \times \partial_t u_n^k, e_i\rangle_{L^2(\Omega)}\mathrm{d}t \to \int_0^T \langle u^k \times \partial_t u^k, e_i\rangle_{L^2(\Omega)}\mathrm{d}t. \quad (3.41)$$

Using the divergence theorem,

$$\int_0^T \langle \Delta u_n^k, e_i\rangle_{L^2(\Omega)}(t)\mathrm{d}t = -\int_0^T\int_\Omega \nabla u_n^k(t,x)\cdot\nabla e_i \to -\int_0^T \langle \nabla u^k, \nabla e_i\rangle_{L^2(\Omega)}(t)\mathrm{d}t. \quad (3.42)$$

For $j = 1, 2, 3$,

$$\int_0^T \langle u_{j,n}^k, e_i\rangle_{L^2(\Omega)}(t)\mathrm{d}t = \int_0^T \langle u_n^k\cdot\hat{k}_j, e_i\rangle_{L^2(\Omega)}(t)\mathrm{d}t \to \int_0^T \langle u^k\cdot\hat{k}_j, e_i\rangle_{L^2(\Omega)}(t)\mathrm{d}t. \quad (3.43)$$

Finally,

$$\int_0^T \langle(|u_n^k|^2-1)u_n^k, e_i\rangle_{L^2(\Omega)}(t)\mathrm{d}t \to \int_0^T \langle(|u^k|^2-1)u^k, e_i\rangle_{L^2(\Omega)}(t)\mathrm{d}t. \quad (3.44)$$

From (3.40—3.44) and (3.6), we obtain the global weak solution to (3.2) in the sense that

$$\int_0^T \langle \tilde{\lambda}\partial_t u^k + \tilde{\rho}u^k \times \partial_t u^k, e_i\rangle_{L^2(\Omega)}(t)\mathrm{d}t + \int_0^T \langle \nabla u^k, \nabla e_i\rangle_{L^2(\Omega)}(t)\mathrm{d}t$$
$$= \int_0^T \langle \frac{\beta_1 u_1^k}{\alpha}\hat{k}_1 + \frac{\beta_3 u_3^k}{\alpha}\hat{k}_3 + \frac{H_0}{\alpha}\hat{h} - k(|u^k|^2-1)u^k, e_i\rangle_{L^2(\Omega)}(t)\mathrm{d}t. \quad (3.45)$$

Since T is taken arbitrary, we can say

$$\langle \tilde{\lambda}\partial_t u^k + \tilde{\rho}u^k \times \partial_t u^k, e_i\rangle_{L^2(\Omega)}(t) + \langle \nabla u^k, \nabla e_i\rangle_{L^2(\Omega)}(t)$$
$$-\langle \frac{\beta_1 u_1^k}{\alpha}\hat{k}_1 + \frac{\beta_3 u_3^k}{\alpha}\hat{k}_3 + \frac{H_0}{\alpha}\hat{h} - k(|u^k|^2-1)u^k, e_i\rangle_{L^2(\Omega)}(t) = 0 \quad \text{a.e. on } \mathbb{R}^+. \quad (3.46)$$

Since $\{e_i(x)\}$ is the orthogonal basis of $H_0^1(\Omega)$, $v_n^k(0,x) \to v_0(x)$ in $\mathbb{H}_0^1(\Omega)$ from (3.6); on the other hand, $v_n^k(0,x) \to v^k(0,x)$ in $\mathbb{L}^p(\Omega)$ since (3.38); thus

$$u^k(0,x) = u_0(x) \qquad \text{in the sense of traces.} \quad (3.47)$$

Since $v^k(t,\cdot) \in \mathbb{H}_0^1(\Omega)$, the boundary value condition also holds,

$$u^k(t,x) = \hat{g} \qquad \text{on } \partial\Omega. \quad (3.48)$$

∎

3.2 End of the Proof

Consider $E_n^k(0)$. Since

$$u_n^k(0) = v_n^k(0) + \hat{g} \to v_0 + \hat{g} = u_0 \text{ as } n \to \infty \text{ in } \mathbb{H}^1(\Omega) \hookrightarrow \mathbb{L}^4(\Omega), \tag{3.49}$$

one has

$$\begin{aligned} \int_\Omega ||u_n^k(0)|^2 - 1|^2 \mathrm{d}x &\le \int_\Omega |u_n^k(0) - u_0|^2 |u_n^k(0) + u_0|^2 \mathrm{d}x \\ &\le \Big(\int_\Omega |u_n^k(0) - u_0|^4 \mathrm{d}x\Big)^{\frac{1}{2}} \Big(\int_\Omega |u_n^k(0) + u_0|^4 \mathrm{d}x\Big)^{\frac{1}{2}} \\ &\to 0 \end{aligned} \tag{3.50}$$

and

$$\begin{aligned} &\tfrac{1}{2}\|\nabla u_n^k(0)\|^2_{L^2(\Omega)} - \tfrac{\beta_1}{2\alpha}\|\hat{k}_1 \cdot u_n^k(0)\|^2_{L^2(\Omega)} - \tfrac{\beta_3}{2\alpha}\|\hat{k}_3 \cdot u_n^k(0)\|^2_{L^2(\Omega)} - \tfrac{H_0}{\alpha}\textstyle\int_\Omega \hat{h} \cdot u_n^k(0)\mathrm{d}x \\ &\to \tfrac{1}{2}\|\nabla u_0\|^2_{L^2(\Omega)} - \tfrac{\beta_1}{2\alpha}\|\hat{k}_1 \cdot u_0\|^2_{L^2(\Omega)} - \tfrac{\beta_3}{2\alpha}\|\hat{k}_3 \cdot u_0\|^2_{L^2(\Omega)} - \tfrac{H_0}{\alpha}\textstyle\int_\Omega \hat{h} \cdot u_0 \mathrm{d}x \\ &\equiv E(0) \qquad \text{as } n \to \infty. \end{aligned} \tag{3.51}$$

Since (3.35—3.38), we have

$$\frac{1}{2}\|\nabla u^k\|^2_{L^2(\Omega)}(t) + \bar{\lambda}\int_{V_t} |\partial_t u^k|^2 \le \liminf_{n\to\infty}\Big(\frac{1}{2}\|\nabla u_n^k\|^2_{L^2(\Omega)}(t) + \bar{\lambda}\int_{V_t} |\partial_t u_n^k|^2\Big) \tag{3.52}$$

by the lower semi-continuity, and

$$\|\hat{k}_i \cdot u_n^k\|^2_{L^2(\Omega)}(t) \to \|\hat{k}_i \cdot u^k\|^2_{L^2(\Omega)}(t), \tag{3.53}$$

$$\int_\Omega \hat{h} \cdot u_n^k \mathrm{d}x(t) \to \int_\Omega \hat{h} \cdot u^k \mathrm{d}x(t), \tag{3.54}$$

$$\||u_n^k|^2 - 1\|^2_{L^2(\Omega)}(t) \to \||u^k|^2 - 1\|^2_{L^2(\Omega)}(t), \tag{3.55}$$

as $n \to \infty$, by the continuity. Thus, in view of (3.21—3.23), there holds

$$E^k(t) + \bar{\lambda}\int_{V_t} |\partial_t u^k|^2 \mathrm{d}x\mathrm{d}s \le E(0) \le C_0 \qquad \text{for a.e. } t \ge 0, \tag{3.56}$$

where

$$\begin{aligned} E^k(t) \equiv\ & \frac{1}{2}\|\nabla u^k\|^2_{L^2(\Omega)}(t) - \frac{\beta_1}{2\alpha}\|\hat{k}_1 \cdot u^k\|^2_{L^2(\Omega)}(t) - \frac{\beta_3}{2\alpha}\|\hat{k}_3 \cdot u^k\|^2_{L^2(\Omega)}(t) \\ & - \frac{H_0}{\alpha}\int_\Omega \hat{h} \cdot u^k \mathrm{d}x(t) + \frac{k}{4}\||u^k|^2 - 1\|^2_{L^2(\Omega)}(t). \end{aligned} \tag{3.57}$$

Like the arguments in the subsection 3.1, one can deduce from the energy inequality (3.56) that for $k > K_0$,

$$\|\nabla u^k\|^2_{L^2(\Omega)}(t) \le C_0, \quad \||u^k|^2 - 1\|^2_{L^2(\Omega)}(t) \le C_0, \quad \|u^k\|^2_{L^2(\Omega)}(t) \le C_0, \tag{3.58}$$

and

$$\int_{V_t} |\partial_t u^k|^2 \mathrm{d}x\mathrm{d}s = \int_0^t \|\partial_t u^k\|^2_{L^2(\Omega)}(s)\mathrm{d}s \le C_0. \tag{3.59}$$

Thus

$$\|v^k\|^2_{L^\infty(\mathbb{R}^+;\mathbb{H}^1(\Omega))} \text{ and } \|u^k\|^2_{L^\infty(\mathbb{R}^+;\mathbb{H}^1(\Omega))} \le C_0, \tag{3.60}$$

and

$$\|\partial_t v^k\|^2_{L^2(\mathbb{R}^+;\mathbb{L}^2(\Omega))} = \|\partial_t u^k\|^2_{L^2(\mathbb{R}^+;\mathbb{L}^2(\Omega))} \le C_0. \tag{3.61}$$

There exists some $v(t,x) \equiv (v_1, v_2, v_3)^T \in W_0$ such that

$$\begin{cases} v^k(t,x) \overset{*}{\rightharpoonup} v(t,x) \text{ in } L^\infty(\mathbb{R}^+;\mathbb{H}^1_0(\Omega)) \\ \partial_t v^k(t,x) \rightharpoonup \partial_t v(t,x) \text{ in } L^2(\mathbb{R}^+;\mathbb{L}^2(\Omega)), \end{cases} \tag{3.62}$$

up to subsequences, as $k \to \infty$. Let $u \equiv (u_1, u_2, u_3)^T = v + \hat{g}$. Then for all $T \ge 0$,

$$u^k \overset{*}{\rightharpoonup} u \qquad \text{in } L^\infty([0,T];\mathbb{L}^6(\Omega)), \tag{3.63}$$

$$u^k \rightharpoonup u \qquad \text{in } \mathbb{H}^1([0,T]\times\Omega), \tag{3.64}$$

and

$$u^k \to u \text{ in } C([0,T];\mathbb{L}^2(\Omega)) \cap L^q([0,T];\mathbb{L}^p(\Omega)) \text{ for } q < +\infty, p < 6. \tag{3.65}$$

And thus up to subsequences,

$$u^k(t,\cdot) \to u(t,\cdot) \qquad \text{a.e. on } \Omega \text{ for all } t \ge 0. \tag{3.66}$$

Let $k \to \infty$ in (3.56), we have the energy inequality

$$E(t) + \tilde{\lambda}\int_{V_t} |\partial_t u|^2 \mathrm{d}x\mathrm{d}s \le E(0) \le C_0 \qquad \text{for all } t \ge 0, \tag{3.67}$$

by the lower semi-continuity and the continuity as in (3.52—3.56), and have

$$|u^k|^2 - 1 \to 0 \qquad \text{in } L^2(\Omega) \text{ and a.e. on } \Omega \text{ for all } t, \tag{3.68}$$

where

$$E(t) \equiv \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} - \frac{\beta_1}{2\alpha}\|\hat{k}_1 \cdot u\|^2_{L^2(\Omega)} - \frac{\beta_3}{2\alpha}\|\hat{k}_3 \cdot u\|^2_{L^2(\Omega)} - \frac{H_0}{\alpha}\int_\Omega \hat{h}\cdot u\mathrm{d}x. \tag{3.69}$$

As a consequence of (3.66) and (3.68), there holds

$$|u| = 1 \qquad \text{a.e.} \tag{3.70}$$

End of the Proof. Take $\psi^k = u^k \times \varphi$ where $\varphi \in C^\infty(\mathbb{R}^+;(C_0^\infty(\Omega))^3)$. Since $u^k \in W$, one gets $\psi^k(t,\cdot) \in \mathbb{H}^1_0(\Omega)$. Noticing (3.46) we obtain by density that

$$\langle \tilde{\lambda}\partial_t u^k + \tilde{\rho} u^k \times \partial_t u^k, u^k \times \varphi\rangle_{L^2(\Omega)}(t) + \langle \nabla u^k, u^k \times \nabla\varphi\rangle_{L^2(\Omega)}(t)$$

$$-\langle\frac{\beta_1 u_1^k}{\alpha}\hat{k}_1+\frac{\beta_3 u_3^k}{\alpha}\hat{k}_3+\frac{H_0}{\alpha}\hat{h},u^k\times\varphi\rangle_{L^2(\Omega)}(t)=0\quad\text{a.e. on }\mathbb{R}^+. \tag{3.71}$$

Integrating the above identity over $[0,T]$, and letting $k\to\infty$, we have

$$\int_0^T\langle\tilde{\lambda}\partial_t u\times u,\varphi\rangle_{L^2(\Omega)}+\langle\tilde{\rho}(u\cdot u)\partial_t u,\varphi\rangle_{L^2(\Omega)}-\langle\tilde{\rho}(u\cdot\partial_t u)u,\varphi\rangle_{L^2(\Omega)}$$

$$+\langle\nabla u\times u,\nabla\varphi\rangle_{L^2(\Omega)}-\langle\frac{\beta_1 u_1}{\alpha}\hat{k}_1\times u+\frac{\beta_3 u_3}{\alpha}\hat{k}_3\times u+\frac{H_0}{\alpha}\hat{h}\times u,\varphi\rangle_{L^2(\Omega)}\mathrm{d}t=0, \tag{3.72}$$

by (3.62—3.65). Now we have gotten from (3.70) that

$$\int_{V_T}\partial_t u\cdot\varphi-\frac{\lambda}{\rho}(u\times\partial_t u)\cdot\varphi-\frac{\alpha(\rho^2+\lambda^2)}{\rho}\sum_{i=1}^{3}(u\times\partial_i u)\cdot\partial_i\varphi$$

$$+\frac{(\rho^2+\lambda^2)}{\rho}\Big(\beta_1 u_1(u\times\hat{k}_1)\cdot\varphi+\beta_3 u_3(u\times\hat{k}_3)\cdot\varphi+H_0(u\times\hat{h})\cdot\varphi\Big)\,\mathrm{d}x\mathrm{d}t=0, \tag{3.73}$$

for all $\varphi\in C^\infty(\mathbb{R}^+;(C_0^\infty(\Omega))^3)$, that is, the weak solution to system (2.2) in the divergence-form. By an argument of density, we deduce that (3.73) holds for all $\varphi\in L^2(0,T;\mathbb{H}_0^1(\Omega))$. It is trivial to check the initial condition and the boundary condition, like that in the proof of Proposition 3.3. Together with (3.67) and (3.70),we end the proof of Theorem 2.2. ∎

4 Proof of Theorem 2.4

We can use the same methods as in the proof of Theorem 2.2 to prove Theorem 2.4. In this section, the sketch of the proof is given as follows.

Step 1. We introduce the standard orthogonal basis of $L^2(\Omega)$, $\{w_i\}_{i=1}^{\infty}$, whose elements are the eigenvectors of the Laplace operator with the homogenous Neumann boundary condition. The sequence $\{w_i\}$ is also an orthogonal basis of $H^1(\Omega)$. It satisfies for all $i,j=1,2,\ldots$

$$\begin{cases} w_i(x)\in H^1(\Omega), & (w_i(x)\in C^\infty(\Omega),\ \text{elliptic property})\\ \langle-\Delta w_i,\omega\rangle_{L^2(\Omega)}=\lambda_i\langle w_i,\omega\rangle_{L^2(\Omega)} & \text{for all }\omega\in H^1(\Omega)\\ \frac{\partial w_i}{\partial\vec{n}}=0 & \text{on }\partial\Omega\\ \langle w_i,w_j\rangle_{L^2(\Omega)}=\delta_{ij}\\ \langle w_i,w_j\rangle_{H^1(\Omega)}=(1+\lambda_i)^{1/2}(1+\lambda_j)^{1/2}\delta_{ij}, \end{cases} \tag{4.1}$$

where λ_i's denote the eigenvalues.

Step 2. Consider the inner problem

$$\begin{cases} \langle\tilde{\lambda}\partial_t u_n^k+\tilde{\rho}u_n^k\times\partial_t u_n^k,w_l(x)\rangle_{L^2(\Omega)}\\ =\langle\Delta u_n^k+\frac{\beta_1 u_{1,n}^k}{\alpha}\hat{k}_1+\frac{\beta_3 u_{3,n}^k}{\alpha}\hat{k}_3+\frac{H_0}{\alpha}\hat{h}-k(|u_n^k|^2-1)u_n^k,w_l(x)\rangle_{L^2(\Omega)}\\ \langle u_n^k(0,x)-u_0(x),w_l(x)\rangle_{H^1(\Omega)}=0 \qquad \text{on }\Omega, \end{cases} \tag{4.2}$$

for all $l \leq n$, where

$$\begin{aligned} u_n^k &\equiv (u_{1,n}^k(t,x), u_{2,n}^k(t,x), u_{3,n}^k(t,x))^T \\ &= \Sigma_{i=1}^n \varphi_i^k(t) w_i(x) \qquad n = 1, 2, \ldots, \end{aligned} \tag{4.3}$$

where $\varphi_i^k(t) = (\varphi_{1,i}^k(t), \varphi_{2,i}^k(t), \varphi_{3,i}^k(t))^T \in \mathbb{R}^3$. Problem (4.2) is also globally solved:

Proposition 4.1. *There exists a $K_0 > 0$ such that for all $k > K_0$ and for all n, there exists a unique global solution $u_n^k(t,x) \in C^\infty(\mathbb{R}^+; (C^\infty(\Omega))^3)$ of (4.2). Moreover, u_n^k satisfies the same energy equality (3.22) and the estimates (3.33—3.34).*

Proof. Noticing that

$$\begin{aligned} \int_{\partial\Omega} \vec{n} \cdot \nabla u_n^k(x) f(x) \mathrm{d}\sigma &= \int_{\partial\Omega} \Sigma_{i=1}^n \varphi_i^k(t) \vec{n} \cdot \nabla w_i(x) f(x) \mathrm{d}\sigma \\ &= \int_{\partial\Omega} \Sigma_{i=1}^n \varphi_i^k(t) \frac{\partial w_i}{\partial \vec{n}}(x) f(x) \mathrm{d}\sigma = 0, \end{aligned}$$

and $R_{pq} \equiv 0$, what is showed for the basis $\{e_i\}$ in the proofs of Proposition 3.1 and Proposition 3.2 is also hold for $\{w_i\}$. Then the similar arguments end this proof. ∎

Step 3. Let k be large enough. Passing to the limit ($n \to \infty$), we find a weak solution $u^k(t,x) \in W$ of equation

$$\tilde{\lambda}\partial_t u^k - \Delta u^k = -\tilde{\rho} u^k \times \partial_t u^k + \frac{\beta_1 u_1^k}{\alpha}\hat{k}_1 + \frac{\beta_3 u_3^k}{\alpha}\hat{k}_3 + \frac{H_0}{\alpha}\hat{h} - k(|u^k|^2 - 1)u^k \equiv S(t,x), \tag{4.4}$$

in the sense of (3.45) or (3.46) where e_i is replaced by w_i, as shown in Proposition 3.3. Moreover, (3.47) and (3.56) also hold.

As shown in subsection 3.2, passing to the limit ($k \to \infty$), we find a weak solution $u(t,x) \in W$ to the system (2.3) such that (i) and (ii) hold in Theorem 2.4.

In the following proposition, it is showed that there is an improvement for the regularity of u_k ([1]).

Proposition 4.2. *Assume that the parameters $\beta_1 \leq 0$, $\beta_3 \leq 0$ and $H_0 = 0$ in (4.4), then $\|u^k(t,x)\|_{L^\infty} \leq 1$.*

Proof. By a density argument, we deduce that for all $\phi \in \mathbb{H}^1(\Omega)$ there holds

$$\begin{aligned} &\int_0^T \langle \tilde{\lambda}\partial_t u^k + \tilde{\rho} u^k \times \partial_t u^k, \phi \rangle_{L^2(\Omega)}(t)\mathrm{d}t + \int_0^T \langle \nabla u^k, \nabla\phi \rangle_{L^2(\Omega)}(t)\mathrm{d}t \\ &= \int_0^T \langle \frac{\beta_1 u_1^k}{\alpha}\hat{k}_1 + \frac{\beta_3 u_3^k}{\alpha}\hat{k}_3 - k(|u^k|^2 - 1)u^k, \phi \rangle_{L^2(\Omega)}(t)\mathrm{d}t. \end{aligned} \tag{4.5}$$

Let $G(x)$ be the following

$$\begin{aligned} G(x) &= 0 && \text{if } x \leq 0; \\ G(x) &= x^2/2 && \text{if } 0 < x \leq 1; \\ G(x) &= x - 1/2 && \text{otherwise.} \end{aligned}$$

Then

$$\frac{\partial}{\partial t}G(|u^k|^2-1)(t,x)=2g(|u^k|^2-1)u^k\cdot\frac{\partial}{\partial t}u^k,$$

where $g(x)=G'(x)$. Take ϕ under the form

$$\phi(t,x)=2g(|u^k|^2-1)u^k. \tag{4.6}$$

Obviously, $\phi\in\mathbb{L}^2(V_T)$ and $\nabla\phi\in L^\infty(\mathbb{R}^+;\mathbb{L}^2(\Omega))$. Carrying (4.6) in (4.5) leads to

$$\int_{V_T}\bar\lambda 2g(|u^k|^2-1)u^k\cdot\partial_t u^k+\sum_{i=1}^{3}(4g'(|u^k|^2-1)(u^k\cdot\partial_i u^k)^2+2g(|u^k|^2-1)|\partial_i u^k|^2)\mathrm{d}x\mathrm{d}t$$

$$=\int_{V_T}2g(|u^k|^2-1)\left(\frac{\beta_1u_1^k}{\alpha}\hat k_1+\frac{\beta_3u_3^k}{\alpha}\hat k_3-k(|u^k|^2-1)u^k\right)\cdot u^k\mathrm{d}x\mathrm{d}t. \tag{4.7}$$

But,

$$g'(|u^k|^2-1)(u^k\cdot\partial_i u^k)^2\geq 0;\qquad g(|u^k|^2-1)|\partial_i u^k|^2\geq 0;$$
$$g(|u^k|^2-1)(|u^k|^2-1)|u^k|^2\geq 0;$$
$$g(|u^k|^2-1)|u_1^k|^2 \text{ and } g(|u^k|^2-1)|u_3^k|^2\geq 0;$$

and they all belong to $L^1(V_T)$. Since the assumptions that $\beta_1\leq 0$, $\beta_3\leq 0$ and $k\geq 0$, we get that

$$\int_{V_T}\frac{\partial}{\partial t}G(|u^k|^2-1)(t,x)\mathrm{d}x\mathrm{d}t=\int_{V_T}2g(|u^k|^2-1)u^k\cdot\partial_t u^k\mathrm{d}x\mathrm{d}t\leq 0;$$

furthermore, by (3.47),

$$\int_\Omega G(|u^k|^2-1)(T,x)\mathrm{d}x\leq\int_\Omega G(|u^k|^2-1)(0,x)\mathrm{d}x=\int_\Omega G(|u_0|^2-1)(x)\mathrm{d}x=0.$$

We conclude that $|u^k(t,x)|\leq 1$ a.e. ■

By the properties of parabolic equations, the formula (4.4) then gives u^k belongs to $W\cap L^2(0,T;\mathbb{H}^2(\Omega))$ since $S(t,x)\in L^2(0,T;\mathbb{L}^2(\Omega))$.

5 Constant Solutions

According to the sense of weak solutions in Theorems 2.2 and 2.4, any classical solution to the Landau-Lifshitz equation is also a weak solution. For the classical solutions to (2.2) or (2.3), one may deduce the energy equality.

Proposition 5.1. *For any classical solution u to (2.2) or (2.3), the energy equality holds,*

$$E(t)+\frac{\lambda}{\alpha(\rho^2+\lambda^2)}\int_{V_t}|\partial_t u|^2dxds=E(0), \tag{5.1}$$

where $E(t)$ is the same as in (3.69).

Proof. Since $u \cdot \partial_t u = 0$, and u satisfies (2.2) ($\hat{g}$ constant) or (2.3), one gets

$$\int_{\partial\Omega} (\vec{n} \cdot \nabla u) \cdot \partial_t u d\sigma = \int_{\partial\Omega} \frac{\partial u}{\partial \vec{n}} \cdot \partial_t u d\sigma \int_{\partial\Omega} 0 \, d\sigma = 0. \quad (5.2)$$

Hence taking inner product of (2.5) with $\partial_t u$ and integrating over V_t, we get (5.1). ∎

We consider the constant solutions in the following. If $u(t,x) \equiv \hat{g}$ is a constant solution to the Landau-Lifshitz equation, it can be treated as the global classical solution both to system (2.2) and to system (2.3) with the initial data $u_0 = \hat{g}$. We could ask whether the uniqueness holds for the solution with this initial data.

Problem: If $v(x) \equiv \hat{g}$ is a constant solution to the static Landau-Lifshitz equation:

$$H_{eff}(v) = (H_{eff}(v) \cdot v)v. \quad (5.3)$$

Let $u_0(x) = \hat{g}$ in system (2.2) or (2.3). Is the solution to (2.2) or (2.3), $u(t,x) \equiv \hat{g}$, unique? ∎

We give a positive answer to the above problem in some special cases. Suppose $u^*(t,x) = (u_1^*, u_2^*, u_3^*)^T$ is a classical solution to (2.2) or (2.3) with $u^*(0,x) = \hat{g}$. Employing the energy identity (5.1), one can deduce that:

$$\begin{aligned} 0 &\le \alpha \int_\Omega |\nabla u^*|^2 dx + 2\alpha\tilde{\lambda} \int_{V_t} |\partial_t u^*|^2 dx dt \\ &= \beta_1 \int_\Omega u_1^{*2}(t) - g_1^2 dx + \beta_3 \int_\Omega u_3^{*2}(t) - g_3^2 dx + 2H_0 \int_\Omega \hat{h} \cdot u^*(t) - \hat{h} \cdot \hat{g} dx \\ &\equiv \delta E(t). \end{aligned} \quad (5.4)$$

Obviously, if there holds $\delta E(t) \le 0$, then $\partial_t u^* = 0$ and $\nabla u^* = 0$ on V_t; Uniqueness is obtained.

(a) If $H_0 = 0, \beta_1 = 0$ and $\beta_3 = 0$, then for an arbitrary $\hat{g} \in \mathbb{S}^2$, $v(x) = \hat{g}$ is a solution satisfying (5.3) and $u(t,x) = \hat{g}$ is a constant solution to the dynamic equation. Moreover one can deduce that $\delta E(t) = 0$ and then uniqueness is obtained. This tells us that the isotropic ferromagnet could be oriented randomly.

(b) If $H_0 = 0, \beta_1 \ne 0$ and $\beta_3 = 0$, either $v(x) = \pm\hat{k}_1$ or $v(x) = (0, g_2, g_3)^T \in \mathbb{S}^2$ satisfies (5.3). By checking $\delta E(t) \le 0$, we can obtain that with $u_0 = (0, g_2, g_3)^T$, the solution to (2.2) or (2.3) is unique if $\beta_1 < 0$; and with $u_0 = \pm\hat{k}_1$, the solution for $\beta_1 > 0$ is unique. These assert that in the ground state of an easy-plane ferromagnet the vector u lies in the easy plane in the absence of an external magnetic field and can be directed arbitrarily in this plane; and in the ground state of an easy-axis ferromagnet the magnetization vector is directed only along this axis. ([14])

(c) If $H_0 = 0, \beta_1 \ne 0$ and $\beta_3 \ne 0$, the constant solutions to (5.3) are either $v(x) = \pm\hat{k}_1, \pm\hat{k}_2, \pm\hat{k}_3$ for arbitrary β_1 and β_3, or $v(x) = (g_1, 0, g_3)^T \in \mathbb{S}^2$ for $\beta_1 = \beta_3$. With $u_0 = \pm\hat{k}_1$, the solution to (2.2) or (2.3) is unique for $\beta_1 > 0$ and $\beta_3 \le \beta_1$, since

$$\begin{aligned} \delta E(t) &= \beta_1 \int_\Omega u_1^{*2}(t) - 1 dx + \beta_3 \int_\Omega u_3^{*2}(t) dx \\ &= \beta_1 \int_\Omega u_1^{*2}(t) + u_3^{*2}(t) - 1 dx + (\beta_3 - \beta_1) \int_\Omega u_3^{*2}(t) dx \le 0; \end{aligned} \quad (5.5)$$

with $u_0 = \pm\hat{k}_2$, the solution is unique for $\beta_1 < 0$ and $\beta_3 < 0$; with $u_0 = \pm\hat{k}_3$, the solution is unique for $\beta_3 > 0$ and $\beta_1 \le \beta_3$; and when $\beta_1 = \beta_3 > 0$, the solution with $u_0 = (g_1, 0, g_3)^T$ is uniquely obtained since

$$\begin{aligned}\delta E(t) &= \beta_1 \int_\Omega u_1^{*2}(t) - g_1^2 \mathrm{d}x + \beta_3 \int_\Omega u_3^{*2}(t) - g_3^2 \mathrm{d}x \\ &= \beta_1 \int_\Omega u_1^{*2}(t) + u_3^{*2}(t) - 1 \mathrm{d}x \le 0.\end{aligned} \tag{5.6}$$

From these uniqueness conditions we see that the size of β_i will decide the leading axis for a biaxial ferromagnet.

(d) If $H_0 \neq 0, \beta_1 = 0$ and $\beta_3 = 0$, then $v(x) = \pm\hat{h}$ are solutions to (5.3). When u_0 equals to $\hat{h}$ and $H_0 > 0$, the constant solution $u = \hat{h}$ whose direction agrees with that of the external field $\hat{H}$ is uniquely solved to (2.2) or (2.3). With $u_0 = -\hat{h}$ and $H_0 < 0$, then $u = -\hat{h}$ whose direction also agrees with that of $\hat{H} = H_0\hat{h}$ is uniquely solved. In other words, the external field will be forcible for an isotropy ferromagnet. Indeed, the properties of the static Landau-Lifshitz equation with constant boundary value are studied in [12, 13].

(e) Let $H_0 \neq 0, \hat{h} = \hat{k}_1, \beta_1 \neq 0$ and $\beta_3 = 0$. In (5.3), $v(x)$ can take the constant either $\hat{g} = \pm\hat{k}_1$ or $\hat{g} = (-H_0/\beta_1, g_2, g_3)^T \in \mathbb{S}^2$ provided $|H_0/\beta_1| \le 1$. If $u_0 = \hat{k}_1$,

$$\begin{aligned}\delta E(t) &= \beta_1 \int_\Omega u_1^{*2}(t) - 1 \mathrm{d}x + 2H_0 \int_\Omega u_1^*(t) - 1 \mathrm{d}x \\ &= \int_\Omega (u_1^*(t) - 1)(\beta_1(u_1^*(t) + 1) + 2H_0)\mathrm{d}x.\end{aligned} \tag{5.7}$$

Thus when $\beta_1 > -H_0$, $\delta E(t_1) \le 0$ since $u_1^*(t)$ can be chosen near 1. Hence the dynamical solution $u = \hat{k}_1$ to (2.2) or (2.3) has uniqueness for $\beta_1 > -H_0$. Similarly, with $u_0 = -\hat{k}_1$ one gets that the dynamical solution $u(t, x) = -\hat{k}_1$ is unique for $\beta_1 > H_0$.

(f) If $H_0 \neq 0, \hat{h} = \hat{k}_2, \beta_1 \neq 0$ and $\beta_3 = 0$, $v(x)$ may take either $\hat{g} = \pm\hat{k}_2$ or $\hat{g} = (g_1, H_0/\beta_1, 0)^T \in \mathbb{S}^2$ provided $|H_0/\beta_1| \le 1$ in (5.3). For $u_0 = \hat{k}_2$, and $H_0 > \beta_1 > 0$, there is uniqueness result for $u(t, x) = \hat{k}_2$ to (2.2) or (2.3), since

$$\begin{aligned}\delta E(t) &= \beta_1 \int_\Omega u_1^{*2}(t)\mathrm{d}x + 2H_0 \int_\Omega u_2^*(t) - 1 \mathrm{d}x \\ &= \int_\Omega (\beta_1(1 + u_2^*(t)) - 2H_0)(1 - u_2^*(t))\mathrm{d}x - \beta_1 \int_\Omega u_3^{*2}(t)\mathrm{d}x \le 0.\end{aligned} \tag{5.8}$$

Similarly, for $u_0 = -\hat{k}_2$, the condition $-H_0 > \beta_1 > 0$ gives the uniqueness for dynamical solution $u(t, x) = -\hat{k}_2$.

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