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Notes on the Isomorphism and Splitting Problems for Commutative Modular Group Algebras

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ABSTRACT

It is proved that the p-Splitting Group Basis Problem for arbitrary abelian groups and the Splitting Group Basis Problem for p-mixed groups are, in fact, equivalent in the commutative group algebra of characteristic p. In addition, a new scheme of obtaining the complete set of invariants for such a group algebra of a p-splitting group is given. In particular, as application, the full system of invariants for a group algebra when the group is p-splitting Warfield is established as well. This supplies own recent results in (Bol. Soc. Mat. Mexicana, 2004) and (Acta Math. Sinica, 2005).

RESUMEN

Se prueba que el Problema de Bases agrupadas p-dividida para grupos abelianos arbitrarios y el Problema de Bases agrupadas dividida para grupos mezclados p es, en efecto, equivalente en el álgebra de grupo conmutativa de característica p. Además, se entrega un esquema nuevo para obtener el conjunto completo de invariantes para esa álgebra de grupo de un grupo p-dividido. En particular, como aplicación, se establece además el sistema completo de invariantes para un álgebra de grupo cuando el grupo es Warfield p-dividido. Esto entrega resultados propios y novedosos en (Bol. Soc. Mat. Mexicana, 2004) y (Acta Math. Sinica, 2005).

Key words and phrases: *isomorphism problem, p-splitting groups, p-mixed groups, Warfield groups, simply presented groups.*
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1 Introduction

Throughout this brief paper we shall use, in a traditional manner, the letter FG as the group algebra of an abelian group G written multiplicatively over a field F. For such a group G, the symbols G_t and G_p are reserved for its maximal torsion subgroup (= torsion part in other terms) and p-primary component, respectively.

The *Splitting Problem* for arbitrary mixed groups, going from May [10] as a special part of the *Isomorphism Problem*, asks of whether the separation of G_t as a direct factor of G can invariantly be retrieved from the group algebra FG. Unfortunately, this question has a negative settling in general firstly proven again by May (see, for example, [12]). Major advantage in various aspects on the presented theme was done in [1] (see [5] too for commutative group algebras over all fields).

As was firstly observed and conjectured in [13], the weak version of the Splitting Problem for p-mixed groups can, eventually, to have a positive solution over the commutative group algebra with prime characteristic p. For this purpose, May, however, assumed that the *Direct Factor Problem* for p-mixed groups is true. In what follows, we shall show that the same can also be expected under the validity of the Isomorphism Problem for p-mixed groups.

The difficulties encountered in the theory of mixed splitting abelian groups can become decidedly less complex, if it is possible to reduce the question to splitting mixed abelian groups whose maximal torsion subgroup is p-primary, that is $G_t = G_p$. Call such an abelian group a *p-mixed* group. In the next paragraph we show that the p-splitting problem for a mixed group is reducible to the same problem for p-mixed groups. After this, we look at certain classes of p-splitting groups for which the commutative modular group algebra with characteristic p possesses a complete set of invariants.

2 Main Theorems

As usual, the mixed abelian group is said to be *p-splitting*, respectively *splitting*, if G_p, respectively G_t, is a direct factor of G.

Before presenting the central results, we need the following technical assertion which is the crucial point.

Lemma. *Let G be an abelian group. Then G is p-splitting $\iff$ $G/\coprod_{q\neq p} G_q$ is splitting.*

Proof. First of all, note that $(G/\coprod_{q\neq p} G_q)_t = G_t/\coprod_{q\neq p} G_q = (G_p \times \coprod_{q\neq p} G_q)/\coprod_{q\neq p} G_q \cong$

G_p.

1. "$\Rightarrow$": Write $G = G_p \times M$ for some $M \leq G$. Therefore, it is self-evident that $G/\coprod_{q\neq p} G_q = (G_t/\coprod_{q\neq p} G_q)(M(\coprod_{q\neq p} G_q)/\coprod_{q\neq p} G_q) = (G/\coprod_{q\neq p} G_q)_t(M(\coprod_{q\neq p} G_q)/\coprod_{q\neq p} G_q)$. What remains to demonstrate is that the intersection between the two factors is equal to one. This, certainly, is accomplished by showing that $G_t \cap (M\coprod_{q\neq p} G_q) = \coprod_{q\neq p} G_q$. Indeed, with the aid of the modular law, we calculate that

$$\begin{aligned} G_t \cap (M\coprod_{q\neq p} G_q) &= (\coprod_{q\neq p} G_q)(G_t \cap M) = (\coprod_{q\neq p} G_q)((G_p \times M_t) \cap M) \\ &= (\coprod_{q\neq p} G_q)((G_p \times \coprod_{q\neq p} M_q) \cap M) = (\coprod_{q\neq p} G_q)(G_p \cap M) \\ &= \coprod_{q\neq p} G_q, \text{ so completing this part-half.} \end{aligned}$$

2. "$\Leftarrow$": Write $G/\coprod_{q\neq p} G_q = (G/\coprod_{q\neq p} G_q)_t \times (N/\coprod_{q\neq p} G_q)$ or equivalently $G/\coprod_{q\neq p} G_q = (G_t/\coprod_{q\neq p} G_q) \times (N/\coprod_{q\neq p} G_q)$ for some $N \leq G$. Consequently, since $N \supseteq \coprod_{q\neq p} G_q$, it is not hard to detect that $G = G_t N = G_p N$ with $G_t \cap N = \coprod_{q\neq p} G_q$. Hence $G_p \cap N \subseteq (\coprod_{q\neq p} G_q)_p = 1$. Thereby, $G = G_p \times N$ and we are finished. ∎

J. Oppelt has considered in ([14], p. 1260, Corollary) another interesting reduction of the splitting mixed groups to the p-splitting ones.

The following equivalence plays a key role in our further exploration.

Proposition. *Let G be an abelian group and F a field of char$(F) = p$. Then the following two implications are equivalent:*

(1) If G is p-mixed splitting and $FG \cong FH$ as F-algebras for some group $H \Rightarrow H$ is p-mixed splitting.

(2) If G is p-splitting and $FG \cong FH$ as F-algebras for some group $H \Rightarrow H$ is p-splitting.

Proof. The left implication (1) $\Rightarrow$ (2) follows like this. Choose G to be p-splitting. Since, in view of [16], the F-isomorphism between FG and FH forces an F-isomorphism between $F(G/\coprod_{q\neq p} G_q)$ and $F(H/\coprod_{q\neq p} H_q)$, we appeal to the Lemma to conclude that H is also p-splitting, as stated.

The right implication (1) $\Leftarrow$ (2) is straightforward because the splitting groups are known to be p-splitting and also the p-mixed p-splitting groups are splitting. ∎

As already indicated above, W. May showed in [13] that if the Direct Factor Problem for p-mixed groups holds in the affirmative, then the same is true for the Splitting Problem for p-mixed groups; as will be demonstrated in the sequel (for example our Main Theorems Invariants stated below) the positive solution of the Isomorphism Problem for p-mixed groups assures much more, namely that both the

p-Splitting Problem and the Isomorphism Problem for arbitrary p-splitting groups hold positively. Besides, we claim that the well-known Direct Factor Problem for p-groups, which is weaker than the Direct Factor Problem for p-mixed groups, is absolutely enough for the full resolution of the general p-Splitting Problem (a sketch of proof was first given in [2] but the complete confirmation is a problem of another research article where a new approach might work).

Foremost, before beginning with the main theorem, for the convenience of the reader, we recollect some more questions in details.

Isomorphism Problem for p-Mixed Groups. Let G be a p-mixed abelian group and let F be a field of characteristic p. Does it follow that the F-isomorphism between FH and FG for any group H implies that H and G are isomorphic?

Isomorphism Problem for p-Splitting Groups. Let G be a p-splitting abelian group and let F be a field of characteristic p. Find the complete system of invariants for FG.

In the next lines, we shall illustrate that, the positive answering of the Isomorphism Problem for p-mixed groups yields the positive solution of the Isomorphism Problem for p-splitting groups. And so, we have accumulated all the information necessary to proceed by proving the following.

Main Theorem (Invariants). *Suppose G is an abelian p-splitting group and F is a field of $char(F) = p \neq 0$. Under the assumption of truthfulness of the Isomorphism Problem for p-mixed groups, $FH \cong FG$ are F-isomorphic for some group H if and only if the following conditions hold:*

(1) H is p-splitting abelian;

(2) $H_p \cong G_p$;

(3) $H/H_t \cong G/G_t$;

(4) $F(H/H_p) \cong F(G/G_p)$.

Proof. "necessity". According to [16] we obtain that $F(G/\coprod_{q\neq p} G_q) \cong F(H/\coprod_{q\neq p} H_q)$, whence via our assumption $G/\coprod_{q\neq p} G_q \cong H/\coprod_{q\neq p} H_q$. Thus $G_p \cong (G/\coprod_{q\neq p} G_q)_t \cong (H/\coprod_{q\neq p} H_q)_t \cong H_p$, i.e. (2) sustained. Moreover, referring to the Lemma, we infer that (1) is true. Furthermore, (3) follows from [10] (see [4] too) as well as (4).

"sufficiency". Write $G \cong G_p \times G/G_p$ and $H \cong H_p \times H/H_p$, hence $FG \cong FG_p \otimes_F F(G/G_p)$ and $FH \cong FH_p \otimes_F F(H/H_p)$. Consequently, (2) and (4) ensure that $FG \cong FH$, as asserted. ∎

When F is an algebraically closed field, in accordance with [11], the isomorphism (4) can be replaced by (4') $|H_t/H_p| = |G_t/G_p|$. When F is an arbitrary field but the quotient G_t/G_p is finite, owing for instance to [2], the isomorphism (4) may be replaced by (4") $|H_t^{q^i}/H_p| = |G_t^{q^i}/G_p|$ $\forall\ q \neq p$ and $i \in s_q(F) \cup \{0\}$ where $s_q(F) = \{i \in N : F(\eta_i) \neq F(\eta_{i+1})\}$ and η_i is the primitive q^i-th root of unity taken in the algebraic closure of F.

3 Applications

Here we shall now consider concrete classes of groups for which the Main Theorem Invariants holds. However, we bound our attention only on Warfield groups although the same results are valid even for the more large class of μ-elementary W-groups, for an ordinal number μ, defined and studied in [8]. We emphasize that general methods of obtaining the invariants of splitting commutative group algebras were first given in [2].

Theorem 1 (Invariants). *Suppose that G is a p-splitting Warfield abelian group and that K is an algebraically closed field of $char(K) = p > 0$. Then $KH \cong KG$ as K-algebras for another group H if and only if the following conditions are fulfilled:*

(1) H is p-splitting abelian;

(2) $H_p \cong G_p$;

(3) $H/H_t \cong G/G_t$;

(4) $|H_t/H_p| = |G_t/G_p|$.

Proof. In [6] (cf. [7] as well) we have proved that $G/\coprod_{q\neq p} G_q \cong H/\coprod_{q\neq p} H_q$. That is why, the Main Theorem applies to derive the desired claim. ∎

Theorem 2 (Invariants). *Suppose that G is a splitting Warfield abelian group with a finite factor G_t/G_p and that F is an arbitrary field of $char(F) = p > 0$. Then $FH \cong FG$ as F-algebras for another group H if and only if the following conditions are realized:*

(1) H is splitting abelian;

(2) $H_p \cong G_p$;

(3) $H/H_t \cong G/G_t$;

(4) $|H_t^{q^i}/H_p| = |G_t^{q^i}/G_p|$, for all primes $q \neq p$ and $i \in s_q(F) \cup \{0\}$.

Proof. Because in virtue of [6] (cf. [7] too) we have deduced that $G/\coprod_{q\neq p} G_q \cong H/\coprod_{q\neq p} H_q$, we employ at once the Main Theorem to get the wanted claim. ∎

We terminate the paper by the helpful observations that although G and G/G_p are both Warfield, H and H/H_p are not necessarily from this group class. Also, since G_p being an S-group (see e.g. [9] for the p-local case or [15] for the global one) need not be simply presented, the last two theorems are at first look independent from [3]. However, because G_p as being a direct factor of the Warfield group G is also Warfield, hence simply presented, Theorems 1 and 2 are deducible from [3].

Remark. In [5, p. 160, line 8(-)] there is a misprint, namely the formula $T(VP[C]) = VP[T(C)]$ should be written as $T(VP[C]) = T(VP[T(C)])$; see also [4]. Moreover, in [2, p. 14, line 11(+)] the "iff" must be read as "if".

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