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Connectedness in Fuzzy bitopological Spaces

M.K.Gupta
Department of Mathematics. Ch. Charan Singh University
Meerut-250005. (INDIA)

Rupen Pratap Singh [1]
Department of Mathematics. Ch. Charan Singh University
Meerut-250005. (INDIA)
r_p_rp@rediffmail.com

ABSTRACT

In this paper, we extend the four notions of connectedness introduced by Ajmal and Kohli [1] to pairwise connectedness for an arbitrary fuzzy set in fuzzy bitopological spaces (X, τ_1, τ_2) and discuss the implications that exist between them. These conditions are called c_k- pairwise connectedness (k = 1, 2, 3, 4). We establish that the union of an arbitrary family of c_k- pairwise connected (k = 1, 2) fuzzy set which are pairwise intersecting is c_k- pairwise connected (k = 1, 2). Also the union of arbitrary family of c_k- pairwise connected (k = 3, 4) fuzzy set which are overlapping is c_k- pairwise connected (k = 3, 4). It is also shown that (τ_i, τ_j)- closure of a c_1- pairwise connected fuzzy set need not be a c_1- pairwise connected fuzzy set. We also discuss the preservation of c_k- pairwise connectedness (k = 1, 2, 3, 4)under fuzzy pairwise continuous mapping and fuzzy pairwise open mapping.

RESUMEN

En este artículo extendemos las cuatro nociones de conexidad introducidas por Ajmal y Kohli [1] a conexidad por parejas para un conjunto difuso arbitrario en espacios bitopológicos difusos (X, τ_1, τ_2) y discutimos las implicaciones que

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existen entre ellos. Estas condiciones se llaman conexidad por parejas c_k (k = 1, 2, 3, 4). Establecemos que la uniíon de una familia arbitraria de conjuntos conexos por parejas c_k, (k = 1, 2) que se intersectan por parejas es conexo por parejas c_k, (k = 1, 2). Además, la uniíon de una familia arbitraria de conjuntos conexos por parejas c_k, (k = 3,4) que se sobreponen es conexo por parejas c_k, (k = 3,4). También se muestra que la cerradura (τ_i, τ_j) de un conjunto difuso por parejas c_1 no necesita ser un conjunto difuso por parejas c_1. Además discutimos la preservación de la conexidad por pares c_k (k=1, 2, 3, 4) bajo la aplicación continua difusa por parejas y la aplicación abierta difusa por parejas.

Key words and phrases: *Fuzzy bitopological spaces, fuzzy pairwise connectedness, fuzzy pairwise continuity, overlapping.*

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1 Introduction

A fuzzy bitopological space [5] (in short, fbts) is a triple (X, τ_1, τ_2) where X is a set and τ_1, τ_2 are two fuzzy topologies on X.
Let $f : X \longrightarrow Y$ be a mapping from X into Y. If λ is a fuzzy set in X and μ be a fuzzy set in Y then $f(\lambda)$ and $f^{-1}(\mu)$ are defined as follows:

$$f(\lambda)(y) = \begin{cases} sup\{\lambda(x)\} & \text{if} f^{-1}(y) \text{is non-empty} \\ 0 & \text{otherwise} \end{cases}$$

and $f^{-1}(\mu)(x) = \mu(f(x))$ for every $x \in X$.
A fuzzy mapping $f : X \longrightarrow Y$is said to be fuzzy continuous if the inverse image of every fuzzy open (closed) set in Y is fuzzy open (closed) in X.
A fuzzy mapping $f : X \longrightarrow Y$is said to be fuzzy open (resp. fuzzy closed) if the image of every fuzzy open (resp. closed) set in X is fuzzy open (closed) in Y.
A mapping from fuzzy bitopological space (X, τ_1, τ_2) to (Y, σ_1, σ_2) is called fuzzy pairwise continuous (rsep. Fuzzy pairwise open) if the induced maps $f : (X, \tau_i) \longrightarrow (Y, \sigma_i)$ $i = 1, 2$ are fuzzy continuous (rsep. fuzzy open).
For a fuzzy set λ of X, the τ_i - closure and τ_i - interior are defined respectively, as

$$\tau_i - cl(\lambda) = \inf\{\nu : \nu \geq \lambda, 1 - \nu \in \tau_i\}$$

$$\tau_i - int(\lambda) = \sup\{\nu : \nu \leq \lambda, \nu \in \tau_i\}$$

2 Fuzzy Pairwise Connectedness

Definition 2.1 *Let (X, τ_1, τ_2) be a fuzzy bitopological space. A fuzzy set λ of X is said to have a τ_k-pairwise disconnection (k = 1, 2, 3, 4) if there exists τ_i- fuzzy open* *set* μ *and* τ_j*- fuzzy open set* ν *in* X *for* $i \neq j$ $i, j = 1, 2$ *such that, respectively,*

c_1 : $\lambda \leq \mu \vee \nu$ $\quad \mu \wedge \nu \leq 1 - \lambda$ $\quad \lambda \wedge \mu \neq 0$ *and* $\lambda \wedge \nu \neq 0$

c_2 : $\lambda \leq \mu \vee \nu$ $\quad \mu \wedge \nu \wedge \lambda = 0$ $\quad \lambda \wedge \mu \neq 0$ *and* $\lambda \wedge \nu \neq 0$

c_3 : $\lambda \leq \mu \vee \nu$ $\quad \mu \wedge \nu \leq 1 - \lambda$ $\quad \mu \not\leq 1 - \lambda$ *and* $\nu \not\leq 1 - \lambda$

c_4 : $\lambda \leq \mu \vee \nu$ $\quad \mu \wedge \nu \wedge \lambda = 0$ $\quad \mu \not\leq 1 - \lambda$ *and* $\nu \not\leq 1 - \lambda$

Definition 2.2 *A fuzzy set* λ *in a fbts* (X, τ_1, τ_2) *is said to be* c_k*-pairwise connected* $(k = 1, 2, 3, 4)$ *if there exists no* c_k*-pairwise disconnection* $(k = 1, 2, 3, 4)$ *of* λ *in* X.

In a fbts (X, τ_1, τ_2), c_k-pairwise connected ($k = 1, 2, 3, 4$) fuzzy sets can be described by the following diagram:

$$c_1 \Rightarrow c_2, \quad c_1 \Rightarrow c_3$$

$$c_2 \Rightarrow c_4, \quad c_3 \Rightarrow c_4$$

We demonstrate through examples that the inclusions are proper, moreover the intersection of the classes of c_2- pairwise connected and c_3- pairwise connected fuzzy sets may not be empty. In that, there exist fuzzy sets in fbts which are c_2- pairwise connected as well as c_3- pairwise connected but not c_1- pairwise connected.So, c_2-pairwise connectedness and c_3- pairwise connectedness even together do not imply c_1- pairwise connectedness.
Implications of the above diagram are immediate from the definitions. Here, we illustrate all the reverse implications by counter examples.

Example 2.3 *Fuzzy set which is* c_4 *but not* c_3.
Let $X = [\,0,\ 1]$ *and define fuzzy sets* μ *and* ν *as follows:*

$$\mu(x) = \begin{cases} 1 & \text{if} \quad 0 \leq x \leq \frac{1}{3} \\ \frac{1}{3} & \text{if} \quad \frac{1}{3} \leq x \leq 1 \end{cases} \qquad \nu(x) = \begin{cases} \frac{1}{3} & \text{if} \quad 0 \leq x \leq \frac{1}{3} \\ 1 & \text{if} \quad \frac{1}{3} \leq x \leq 1 \end{cases}$$

Then $\tau_1 = \{0, \mu, 1\}$ *and* $\tau_2 = \{0, \nu, 1\}$ *are fuzzy topologies on* X. *Define fuzzy set* λ *by* $\lambda(x) = \frac{2}{3}$ *if* $0 \leq x \leq 1$. *Then* λ *is* c_4*- pairwise connected but not* c_3*- pairwise connected.*

Example 2.4 *Fuzzy set which is* c_4 *but not* c_2.
Let $X = [\,0,\ 1]$ *and define fuzzy sets* μ *and* ν *as follows:*

$$\mu(x) = \begin{cases} \frac{1}{3} & \text{if} \quad 0 \leq x \leq \frac{1}{3} \\ 0 & \text{if} \quad \frac{1}{3} \leq x \leq 1 \end{cases} \qquad \nu(x) = \begin{cases} 0 & \text{if} \quad 0 \leq x \leq \frac{1}{3} \\ \frac{1}{3} & \text{if} \quad \frac{1}{3} \leq x \leq 1 \end{cases}$$

Then $\tau_1 = \{0, \mu, 1\}$ *and* $\tau_2 = \{0, \nu, 1\}$ *are fuzzy topologies on* X. *Define fuzzy set* λ *by* $\lambda(x) = \frac{1}{3}$ *if* $0 \leq x \leq 1$. *Then* λ *is* c_4*- pairwise connected but not* c_2*- pairwise connected.*

Example 2.5 *Fuzzy set which is c_3 and c_2 but not c_1.*
Let $X = [\,0,\ 1]$ and define fuzzy sets μ and ν as follows:

$$\mu(x) = \begin{cases} \frac{2}{3} & if \quad 0 \le x \le \frac{1}{3} \\ \frac{1}{3} & if \quad \frac{1}{3} \le x \le 1 \end{cases} \qquad \nu(x) = \begin{cases} \frac{1}{3} & if \quad 0 \le x \le \frac{1}{3} \\ \frac{2}{3} & if \quad \frac{1}{3} \le x \le 1 \end{cases}$$

Then $\tau_1 = \{0, \mu, 1\}$ and $\tau_2 = \{0, \nu, 1\}$ are fuzzy topologies on X. Define fuzzy set λ by $\lambda(x) = \frac{1}{3}$ for all $x \in X$. Here λ is c_3- pairwise connected and c_2- pairwise connected but not c_1- pairwise connected.

Remark 2.6 *Example 2.5 also establishes the fact that the intersection of the classes of c_2-pairwise connected and c_3-pairwise connected fuzzy sets in a fbts may not be empty.*

Example 2.7 *Fuzzy set which is c_3 but not c_2.*
Let $X = [\,0,\ 1]$ and define fuzzy sets μ and ν as follows:

$$\mu(x) = \begin{cases} \frac{2}{3} & if \quad 0 \le x \le \frac{2}{3} \\ 0 & if \quad \frac{2}{3} \le x \le 1 \end{cases} \qquad \nu(x) = \begin{cases} 0 & if \quad 0 \le x \le \frac{2}{3} \\ \frac{1}{3} & if \quad \frac{2}{3} \le x \le 1 \end{cases}$$

Then $\tau_1 = \{0, \mu, 1\}$ and $\tau_2 = \{0, \nu, 1\}$ are fuzzy topologies on X. Define fuzzy set λ by

$$\lambda(x) = \begin{cases} \frac{2}{3} & if \quad 0 \le x \le \frac{2}{3} \\ \frac{1}{3} & if \quad \frac{2}{3} \le x \le 1 \end{cases}$$

Then λ is c_3- pairwise connected but not c_2- pairwise connected.

Example 2.8 *Fuzzy set which is c_2 but not c_3.*
Let $X = [\,0,\ 1]$ and define fuzzy sets μ and ν as follows:

$$\mu(x) = \begin{cases} \frac{2}{3} & if \quad 0 \le x \le \frac{1}{3} \\ \frac{1}{3} & if \quad \frac{1}{3} \le x \le 1 \end{cases} \qquad \nu(x) = \begin{cases} \frac{1}{3} & if \quad 0 \le x \le \frac{1}{3} \\ \frac{2}{3} & if \quad \frac{1}{3} \le x \le 1 \end{cases}$$

Then $\tau_1 = \{0, \mu, 1\}$ and $\tau_2 = \{0, \nu, 1\}$ are fuzzy topologies on X. Define fuzzy set λ by $\lambda(x) = \frac{2}{3}$ for all $x \in X$. Here λ is c_2- pairwise connected but not c_3- pairwise connected.

Remark 2.9 *Example 2.7 and 2.8 establish the fact that the classes of c_2-pairwise connected and c_3-pairwise connected fuzzy sets in a fbts may not be comparable.*

By choosing fuzzy set λ to be fuzzy whole space (constant function 1_X) in all four forms of pairwise disconnections, it is easily verified that we get the same pairwise disconnection of fuzzy space 1_X which we refer as c-pairwise disconnection of fuzzy spaces 1_X i.e. there exits non-zero τ_i-fuzzy open set μ and τ_j-fuzzy open set ν such that $\mu \vee \nu = 1$ and $\mu \wedge \nu = 0$.

Definition 2.10 *A fuzzy bitopological space* (X, τ_1, τ_2) *is said to be c-pairwise connected if there exits no c-pairwise disconnection of* 1_X.
We observe that a fuzzy point x_α *is* c_2- *as well as* c_3 *pairwise connected and hence* c_4 *pairwise connected but not necessarily* c_1 *pairwise connected.*

Example 2.11 *Let* $X = \{x, y\}$ *define fuzzy sets* μ *and* ν *as* $\mu(x) = \frac{1}{3}$, $\mu(y) = \frac{2}{3}$ *and* $\nu(x) = \frac{2}{3}$, $\nu(y) = \frac{1}{3}$. *Then* $\tau_1 = \{0, \mu, 1\}$ *and* $\tau_2 = \{0, \nu, 1\}$ *are fuzzy topologies on* X. *Here the fuzzy point* $x_{\frac{1}{3}}$ *is not* c_1*-pairwise connected. Moreover, we observe that the fuzzy set 0 is c-pairwise connected and hence* c_k*-pairwise connected (k = 1, 2, 3, 4).*

Definition 2.12 *Two fuzzy sets* λ_1 *and* λ_2 *in a fuzzy bitopological space* (X, τ_1, τ_2) *are said to be intersecting if* $\lambda_1 \wedge \lambda_2 \neq 0$.

Definition 2.13 *Two fuzzy sets* λ_1 *and* λ_2 *in a fuzzy bitopological space* (X, τ_1, τ_2) *are said to be non-overlapping if* $\lambda_1 \leq 1 - \lambda_2 . \lambda_1$ *and* λ_2 *are overlapping if there exits a point* $x \in X$ *such that* $\lambda_1(x) > 1 - \lambda_2(x)$.

In this case λ_1 and λ_2 are said to overlap at x.

Theorem 2.14 *If* λ_1 *and* λ_2 *are intersecting* c_1*-pairwise connected fuzzy sets in* (X, τ_1, τ_2), *then* $\lambda_1 \vee \lambda_2$ *is a* c_1*-pairwise connected fuzzy set in* X.

Proof. Since λ_1 and λ_2 are c_1-pairwise connected fuzzy sets therefore they have no c_1-pairwise disconnection i.e. there exits μ and ν be τ_i-,τ_j- fuzzy open sets in X such that
$\lambda_1 \leq \mu \vee \nu \quad \mu \wedge \nu \leq 1 - \lambda_1 \quad \lambda_1 \wedge \mu = 0$ or $\lambda \wedge \nu = 0$
$\lambda_2 \leq \mu \vee \nu \quad \mu \wedge \nu \leq 1 - \lambda_2 \quad \lambda_2 \wedge \mu = 0$ or $\lambda \wedge \nu = 0$
$\Rightarrow \lambda_1 \vee \lambda_2 \leq \mu \vee \nu \quad \mu \wedge \nu \leq 1 - (\lambda_1 \vee \lambda_2)$ ———(1)
$(\lambda_1 \wedge \mu = 0$ or $\lambda_1 \wedge \nu = 0)$ and $(\lambda_2 \wedge \mu = 0$ or $\lambda_2 \wedge \nu = 0)$———(2)
Suppose $\lambda_1 \wedge \mu = 0$. Since λ_1 and λ_2 are intersecting, there exits $x_1 \in X$ such that $\lambda_1(x_1) \neq 0 \neq \lambda_2(x_1)$. we claim that $\lambda_2 \wedge \nu \neq 0$. Let if possible $\lambda_2 \wedge \nu = 0$ then $(\lambda_2 \wedge \nu)(x_1) = 0$ but $\lambda_2(x_1) \neq 0$ implies that $\mu(x_1) = 0$. Hence $(\mu \vee \nu)(x_1) = 0$ this contradict (1) as $(\lambda_1 \vee \lambda_2)(x_1) = 0$. Therefore we have $\lambda_2 \wedge \nu \neq 0$ this gives from (2) $\lambda_2 \wedge \mu = 0$ and hence $(\lambda_1 \vee \lambda_2) \wedge \mu = 0$———(3)
Again suppose $\lambda_1 \wedge \nu = 0$.we can show as above that $\lambda_2 \wedge \mu = 0$ is not possible, hence $\lambda_2 \wedge \nu = 0$. Therefore $(\lambda_1 \vee \lambda_2) \wedge \nu = 0$ ———(4)
So $\lambda_1 \vee \lambda_2$ is c_1-pairwise connected. ∎

The proof is similar for c_2-pairwise connectedness.
The following example shows that the above theorem is not valid for non-intersecting fuzzy sets.

Example 2.15 *Let* $X = [\,0,\ 1]$ *and define fuzzy sets* μ *and* ν *as follows:*

$$\mu(x) = \begin{cases} \frac{2}{3} & if \quad 0 \leq x \leq \frac{2}{3} \\ 0 & if \quad \frac{2}{3} \leq x \leq 1 \end{cases} \qquad \nu(x) = \begin{cases} 0 & if \quad 0 \leq x \leq \frac{2}{3} \\ \frac{2}{3} & if \quad \frac{2}{3} \leq x \leq 1 \end{cases}$$

Then $\tau_1 = \{0, \mu, 1\}$ *and* $\tau_2 = \{0, \nu, 1\}$ *are fuzzy topologies on* X. *Let* λ_1 *and* λ_2 *be defined as*

$$\lambda_1(x) = \begin{cases} 0 & if \quad 0 \le x \le \frac{2}{3} \\ \frac{1}{3} & if \quad \frac{2}{3} \le x \le 1 \end{cases} \qquad \lambda_2(x) = \begin{cases} \frac{1}{3} & if \quad 0 \le x \le \frac{2}{3} \\ 0 & if \quad \frac{2}{3} \le x \le 1 \end{cases}$$

Here $\lambda_1 \wedge \lambda_2 = 0$, *also it can be easily verified that* λ_1 *and* λ_2 *are* c_2*-pairwise connected fuzzy sets but* $\lambda_1 \vee \lambda_2$ *is not* c_2*-pairwise connected.*

Theorem 2.16 *Let* $\{\lambda_i\}_{i \in I}$ *be a family of* c_1*- pairwise connected (*c_2*-pairwise connected) fuzzy sets in fbts* (X, τ_1, τ_2) *such that for* $i \neq j$ $i, j = 1, 2$ *the fuzzy sets* λ_i *and* λ_j *are intersecting then* $\vee_{i \in I} \lambda_i$ *is a* c_1*- pairwise connected (*c_2*-pairwise connected) fuzzy set in* X.

Proof. Let $\lambda = \vee_{i \in I} \lambda_i$. Let μ and ν be τ_i-,τ_j- fuzzy open sets in X such that $\lambda \le \mu \vee \nu$ and $\mu \wedge \nu \le 1 - \lambda$. Now let λ_{i_0} be any fuzzy set of given family then $\lambda_{i_0} \le \mu \vee \nu$ and $\mu \wedge \nu \le 1 - \lambda \Rightarrow \mu \wedge \nu \le 1 - \vee_{i \in I} \lambda_i \Rightarrow \mu \wedge \nu \le \wedge_{i \in I}(1 - \lambda_i) \Rightarrow \mu \wedge \nu \le \lambda_{i_0}$.
But λ_{i_0} is c_1- pairwise connected therefore $\lambda_{i_0} \wedge \mu = 0$ or $\lambda_{i_0} \wedge \nu = 0$.
Now if $\lambda_{i_0} \wedge \mu = 0$ then as in theorem 2.14, we can prove $\lambda_i \wedge \nu = 0$ for each $i \in I - \{i_0\}$ Therefore $\vee_{i \in I}(\lambda_i \wedge \mu) = 0$ this implies $\lambda \wedge \mu = 0$. Therefore λ is c_1-pairwise connected. ∎

Corollary 2.17 *Let* $\{\lambda_i\}_{i \in I}$ *be a family of* c_1*- pairwise connected (*c_2*-pairwise connected) fuzzy sets in fbts* (X, τ_1, τ_2) *and* $\wedge_{i \in I} \lambda_i \neq 0$ *then* $\vee_{i \in I} \lambda_i$ *is a* c_1*- pairwise connected (*c_2*-pairwise connected) fuzzy set in* X.

Proof. - Follows by using theorem 2.16. ∎

Corollary 2.18 *If* $< \lambda_n >$ *be a sequence of* c_1*- pairwise connected (*c_2*-pairwise connected) fuzzy sets in fbts* (X, τ_1, τ_2) *such that for each* n, λ_n *and* λ_{n+1} *are intersecting. Then* $\vee_{n=1}^{\infty} \lambda_n$ *is a* c_1*- pairwise connected (*c_2*-pairwise connected) fuzzy set in* X.

Proof. Follows by induction and using theorem 2.14. ∎ The following example shows that the theorem 2.14 fails for c_3- pairwise connectedness (c_4 - pairwise connectedness).

Example 2.19 *Let* $X = [0, 1]$ *and define fuzzy sets* μ *and* ν *as follows:*

$$\mu(x) = \begin{cases} \frac{2}{5} & if \quad 0 \le x \le \frac{4}{5} \\ \frac{4}{5} & if \quad \frac{4}{5} \le x \le 1 \end{cases} \qquad \nu(x) = \begin{cases} \frac{4}{5} & if \quad 0 \le x \le \frac{4}{5} \\ \frac{2}{5} & if \quad \frac{4}{5} \le x \le 1 \end{cases}$$

Then $\tau_1 = \{0, \mu, 1\}$ *and* $\tau_2 = \{0, \nu, 1\}$ *are fuzzy topologies on* X. *Let* λ_1 *and* λ_2 *be* *defined as*

$$\lambda_1(x) = \begin{cases} \frac{2}{5} & if \quad 0 \le x \le \frac{4}{5} \\ \frac{1}{5} & if \quad \frac{4}{5} \le x \le 1 \end{cases} \qquad \lambda_2(x) = \begin{cases} \frac{1}{5} & if \quad 0 \le x \le \frac{4}{5} \\ \frac{2}{5} & if \quad \frac{4}{5} \le x \le 1 \end{cases}$$

Here $\lambda_1 \wedge \lambda_2 \neq 0$, and we can be easily verified that λ_1 and λ_2 are c_3-pairwise connected fuzzy sets but $\lambda_1 \vee \lambda_2$ is not c_3-pairwise connected fuzzy set in X.

For c_3- pairwise connectedness (c_4 - pairwise connectedness) we have the following theorem.

Theorem 2.20 *If λ_1 and λ_2 are overlapping c_3-pairwise connected (c_4-pairwise connected)fuzzy sets in (X, τ_1, τ_2), then $\lambda_1 \vee \lambda_2$ is a c_3-pairwise connected (c_4-pairwise connected) fuzzy set in X.*

Proof. Since λ_1 and λ_2 are c_3-pairwise connected fuzzy sets therefore they have no c_3-pairwise disconnection i.e. there exits μ and ν be τ_i-,τ_j- fuzzy open sets in X such that

$\lambda_1 \le \mu \vee \nu \quad \mu \wedge \nu \le 1 - \lambda_1 \quad \mu \le 1 - \lambda_1$ or $\nu \le 1 - \lambda_1$

$\lambda_2 \le \mu \vee \nu \quad \mu \wedge \nu \le 1 - \lambda_2 \quad \mu \le 1 - \lambda_2$ or $\nu \le 1 - \lambda_2$

$\Rightarrow \lambda_1 \vee \lambda_2 \le \mu \vee \nu \qquad \mu \wedge \nu \le 1 - (\lambda_1 \vee \lambda_2)$ ———-(1)

$(\mu \le 1 - \lambda_1$ or $\nu \le 1 - \lambda_1)$ and $(\mu \le 1 - \lambda_2$ or $\nu \le 1 - \lambda_2)$———-(2)

Moreover λ_1 and λ_2 are overlapping fuzzy sets, there exists $x_1 \in X$ such that

$$\lambda_1(x_1) > 1 - \lambda_2(x_1) - - - - - - - - - (3)$$

Suppose $\mu \le 1 - \lambda_1 \Rightarrow \mu(x_1) \le 1 - \lambda_1(x_1)$ this implies in view of (3)

$$\mu(x_1) \le \lambda_2(x_1) - - - - - - - - - (4)$$

We claim $\nu \not\le 1 - \lambda_2$. For if $\nu \le 1 - \lambda_2$

$$\Rightarrow \nu(x_1) \le 1 - \lambda_2(x_1)$$

$$\Rightarrow \nu(x_1) \le \lambda_1(x_1) \qquad [Using(3)] - - - - - - - - - (5)$$

By (4) and (5) $(\mu \vee \nu)(x_1) < (\lambda_1 \vee \lambda_2)(x_1) \Rightarrow \lambda_1 \vee \lambda_2 \not\le \mu \vee \nu$ this contradicts (1). Hence $\nu \not\le 1 - \lambda_2$ Therefore $\mu \le 1 - \lambda_2$.

We have $\mu \le 1 - \lambda_1$ and $\mu \le 1 - \lambda_2$ implies $\mu \le (1 - \lambda_1) \wedge (1 - \lambda_2)$.

Again suppose $\nu \le 1 - \lambda_1$. We can show as above $\mu \le 1 - \lambda_2$ is not possible, hence $\nu \le 1 - \lambda_2$ and therefore $\nu \le 1 - (\lambda_1 \vee \lambda_2)$. Thus $\lambda_1 \vee \lambda_2$ is c_3- pairwise connected. ∎

Theorem 2.21 *Let $\{\lambda_i\}_{i\in I}$ be a family of c_3- pairwise connected (c_4-pairwise connected) fuzzy sets in fbts (X,τ_1,τ_2) such that for $i \neq j$ $i,j = 1, 2$ the fuzzy sets λ_i and λ_j are overlapping then $\vee_{i\in I}\lambda_i$ is a c_3- pairwise connected (c_4-pairwise connected) fuzzy set in X.*

Proof. Let $\lambda = \vee_{i\in I}\lambda_i$. Let μ and ν be τ_i-,τ_j- fuzzy open sets in X such that $\lambda \leq \mu\vee\nu$ and $\mu \wedge \nu \leq 1-\lambda$. Now let λ_{i_0} be any fuzzy set of given family then $\lambda_{i_0} \leq \mu \vee \nu$ and $\mu\wedge\nu \leq 1-\lambda \Rightarrow \mu\wedge\nu \leq 1-\vee_{i\in I}\lambda_i \Rightarrow \mu\wedge\nu \leq \wedge_{i\in I}(1-\lambda_i) \Rightarrow \mu\wedge\nu \leq \lambda_{i_0}$.
But λ_{i_0} is c_3- pairwise connected therefore $\mu \leq \lambda_{i_0}$ or $\nu \leq \lambda_{i_0}$.
Now if $\mu \leq \lambda_{i_0}$ then as in theorem 2.20, we can prove $\mu \leq \lambda_i$ for each $i \in I - \{i_0\}$. Since λ_{i_0} and λ_i are overlapping, c_3- pairwise connected fuzzy sets we have

$$\mu \leq \wedge_{i\in I}(1-\lambda_i) = 1 - \vee_{i\in I}\lambda_i = 1-\lambda$$

Similarly if $\nu \leq \lambda_{i_0}$ we can easily show that $\nu \leq \lambda$. Hence λ is c_3 - pairwise connected. ∎

Corollary 2.22 *If $<\lambda_n>$ be a sequence of c_3- pairwise connected (c_4-pairwise connected) fuzzy sets in fbts (X,τ_1,τ_2) such that for each n, λ_n and λ_{n+1} are overlapping. Then $\vee_{n=1}^{\infty}\lambda_n$ is a c_3- pairwise connected (c_4-pairwise connected) fuzzy set in X.*

Theorem 2.23 *If λ is a c_3-pairwise connected(c_4-pairwise connected)fuzzy set in (X,τ_1,τ_2) and $\lambda \leq \delta \leq (\tau_i,\tau_j)cl-\lambda$. Then δ is also a c_3- pairwise connected (c_4-pairwise connected) fuzzy set in X.*

Proof. Let μ and ν be τ_i-,τ_j-fuzzy open sets in X such that $\delta \leq \mu\vee\nu$ and $\mu\wedge\nu \leq 1-\delta$.
Since λ is a c_3- pairwise connected then $\lambda \leq 1-\mu$ or $\lambda \leq 1-\nu$
If $\lambda \leq 1-\mu \Rightarrow (\tau_i,\tau_j)-cl\lambda \leq (\tau_i,\tau_j)-cl(1-\mu)$

$$= 1-(\tau_i,\tau_j)-int\mu = 1-\mu.$$

Therefore $\delta \leq (\tau_i,\tau_j)-cl\lambda \leq 1-\mu$.
If $\lambda \leq 1-\nu \Rightarrow (\tau_i,\tau_j)-cl\lambda \leq (\tau_i,\tau_j)-cl(1-\nu)$

$$= 1-(\tau_i,\tau_j)-int\nu = 1-\nu.$$

Therefore $\delta \leq (\tau_i,\tau_j)-cl\lambda \leq 1-\nu$.
Hence δ is c_3- pairwise connected. ∎ However the above theorem fails in the case of c_1- pairwise connectedness (c_2 -pairwise connectedness). This example shows that the (τ_i,τ_j)- closure of a c_1 - pairwise connected fuzzy set need not be a c_1 - pairwise connected fuzzy set.

Example 2.24 *Let $X = [0, 1]$ and define fuzzy sets μ_1,μ_2,μ_3 and μ_4 as follows:*

$$\mu_1(x) = \begin{cases} 0 & if \quad 0 \leq x \leq \frac{2}{3} \\ 1 & if \quad \frac{2}{3} \leq x \leq 1 \end{cases} \qquad \mu_2(x) = \begin{cases} \frac{1}{3} & if \quad 0 \leq x \leq \frac{2}{3} \\ 0 & if \quad \frac{2}{3} \leq x \leq 1 \end{cases}$$

$$\mu_3(x) = \begin{cases} \frac{1}{3} & if \quad 0 \leq x \leq \frac{2}{3} \\ 1 & if \quad \frac{2}{3} \leq x \leq 1 \end{cases} \qquad \mu_4(x) = \begin{cases} \frac{2}{3} & if \quad 0 \leq x \leq \frac{2}{3} \\ 0 & if \quad \frac{2}{3} \leq x \leq 1 \end{cases}$$

Then $\tau_1 = \{0, \mu_1, \mu_2, \mu_3, 1\}$ *and* $\tau_2 = \{0, \mu_4, 1\}$ *are fuzzy topologies on* X.
Define fuzzy set λ *by* $\lambda(x) = \begin{cases} 0 & if \ \ 0 \le x \le \frac{2}{3} \\ 1 & if \ \ \frac{2}{3} \le x \le 1 \end{cases}$ *Here* λ *is a* c_1*- pairwise connected fuzzy set but* $(\tau_i, \tau_j) - cl\lambda$ *is not* c_1*- pairwise connected.*

3 Fuzzy Pairwise Connectedness And Fuzzy Pairwise Mappings

In the following theorems, we discuss the preservation of c_k-pairwise connectedness (k = 1, 2, 3, 4) under fuzzy pairwise continuity, pairwise open mappings.

Theorem 3.1 *If* $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ *is a fuzzy pairwise continuous bijection and* λ *is a* c_1*- pairwise connected fuzzy set in* X*, then* $f(\lambda)$ *is a* c_1*- pairwise connected fuzzy set in* Y.

Proof. Suppose $f(\lambda)$ is not c_1- pairwise connected then $f(\lambda)$ has a c_1- pairwise disconnection i.e. there exists σ_i- fuzzy open set μ and σ_j - fuzzy open set ν in Y, for $i \neq j, i, j=$ 1, 2 such that,
$f(\lambda) \le \mu \vee \nu \quad \mu \wedge \nu \le 1 - f(\lambda) \quad f(\lambda) \wedge \mu \neq 0$ or $\quad f(\lambda) \wedge \nu \neq 0$
Therefore $\lambda \le f^{-1}(\mu) \vee f^{-1}(\nu)$ and $f^{-1}(\mu) \wedge f^{-1}(\nu) \le 1 - f^{-1}f(\lambda) \le 1 - \lambda$
where $f^{-1}(\mu)$ and $f^{-1}(\nu)$ are τ_i, τ_j-fuzzy open sets in X. since f is fuzzy pairwise continuous mapping . We claim that $\lambda \wedge f^{-1}(\mu) \neq 0$ and $\lambda \wedge f^{-1}(\nu) \neq 0$. suppose $\lambda \wedge f^{-1}(\mu) = 0 \Rightarrow f(\lambda) \wedge f f^{-1}(\mu) = 0 \Rightarrow f(\lambda) \wedge \mu = 0$. Which contradict $f(\lambda) \wedge \mu \neq 0$. Thus λ is not a c_1- pairwise connected fuzzy set. ∎ The proof is similar for c_2-pairwise connectedness.

Theorem 3.2 *If* $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ *is a fuzzy pairwise continuous surjection and* λ *is a* c_3*- pairwise connected fuzzy set in* X*, then* $f(\lambda)$ *is a* c_3*- pairwise connected fuzzy set in* Y.

Proof. Suppose $f(\lambda)$ is not c_3- pairwise connected then $f(\lambda)$ has a c_3- pairwise disconnection i.e. there exists σ_i- fuzzy open set μ and σ_j - fuzzy open set ν in Y, for $i \neq j, i, j=$ 1, 2 such that,
$f(\lambda) \le \mu \vee \nu \quad \mu \wedge \nu \le 1 - f(\lambda) \quad \mu \not\le 1 - f(\lambda)$ and $\quad \nu \not\le 1 - f(\lambda)$
Therefore $\lambda \le f^{-1}(\mu) \vee f^{-1}(\nu)$ and $f^{-1}(\mu) \wedge f^{-1}(\nu) \le 1 - f^{-1}f(\lambda) \le 1 - \lambda$
where $f^{-1}(\mu)$ and $f^{-1}(\nu)$ are τ_i, τ_j-fuzzy open sets in X. since f is fuzzy pairwise continuous mapping. Now since $\mu \not\le 1 - f(\lambda)$ and $\nu \not\le 1 - f(\lambda)$ therefore there exits $y_1, y_2 \in Y$ such that

$$\mu(y_1) > 1 - f(\lambda)(y_1)$$
$$\nu(y_2) > 1 - f(\lambda)(y_2)$$

Also since f is onto,$f^{-1}(\mu)$ and $f^{-1}(\nu)$ are non-empty subsets of X.

By definition $f^{-1}(\mu)(x_i) = \mu(y_1)$ for every $x_i \in f^{-1}(y_1)$
and $f(\lambda)(y_1) = Sup\{\lambda(x_i)\}$
We claim $f^{-1}(\mu) \not\leq 1-\lambda$ and $f^{-1}(\nu) \not\leq 1-\lambda$. Suppose if possible $f^{-1}(\mu) \leq 1-\lambda$
$\Rightarrow f^{-1}(\mu)(x_i) \leq 1-\lambda(x_i)$ for every $x_i \in f^{-1}(y_1)$
$\Rightarrow \lambda(x_i) \leq 1-f^{-1}(\mu)(x_i)$ for every $x_i \in f^{-1}(y_1)$
$\Rightarrow \lambda(x_i) \leq 1-(\mu)(y_1)$ for every $x_i \in f^{-1}(y_1)$
$\Rightarrow Sup\{\lambda(x_i)\} \leq 1-\mu(y_1) \Rightarrow f(\lambda)(y_1) \leq 1-\mu(y_1)$ this contradicts(1).
Similarly $f^{-1}(\nu) \leq 1-\lambda$ contradicts (2). Hence λ is not a c_3- pairwise connected. ∎
The proof is similar for c_4- pairwise connectedness.

Theorem 3.3 *If $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a fuzzy pairwise open bijection and λ be a fuzzy set in Y then λ is c_1- pairwise connected implies that $f^{-1}(\lambda)$ is c_1- pairwise connected.*

Proof. Suppose $f^{-1}(\lambda)$ is not c_1- pairwise connected then $f^{-1}(\lambda)$ has a c_1-pairwise disconnection i.e. there exists τ_i- fuzzy open set μ and τ_j - fuzzy open set ν in X, for $i \neq j, i, j=$ 1, 2 such that,
$f^{-1}(\lambda) \leq \mu \vee \nu \quad \mu \wedge \nu \leq 1-f^{-1}(\lambda) \quad f^{-1}(\lambda) \wedge \mu \neq 0$ or $f^{-1}(\lambda) \wedge \nu \neq 0$
$\Rightarrow ff^{-1}(\lambda) \leq f(\mu) \vee f(\nu)$, $f(\mu) \wedge f(\nu) \leq 1-ff^{-1}(\lambda)$
$\Rightarrow \lambda \leq f(\mu) \vee f(\nu)$, $f(\mu) \wedge f(\nu) \leq 1-\lambda$
Where $f(\mu)$ is σ_i-fuzzy open set and $f(\nu)$ is σ_j- fuzzy open set in Y, since f is fuzzy pairwise open mapping.
Now we claim that $\lambda \wedge f(\mu) \neq 0$ and $\lambda \wedge f(\nu) \neq 0$.
Suppose $\lambda \wedge f(\mu) = 0 \Rightarrow f^{-1}\lambda \wedge f^{-1}f(\mu) = 0 \Rightarrow f^{-1}(\lambda) \wedge \mu = 0$
Which contradicts the $f^{-1}(\lambda) \wedge \mu \neq 0$. ∎ The proof is similar for c_2-pairwise connectedness.

Theorem 3.4 *If $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a fuzzy pairwise open bijection and λ be a fuzzy set in Y then λ is c_1- pairwise connected implies that $f^{-1}(\lambda)$ is c_1- pairwise connected.*

Proof. Suppose $f^{-1}(\lambda)$ is not c_3- pairwise connected then $f^{-1}(\lambda)$ has a c_3-pairwise disconnection i.e. there exists τ_i- fuzzy open set μ and τ_j - fuzzy open set ν in X, for $i \neq j, i, j=$ 1, 2 such that,
$f^{-1}(\lambda) \leq \mu \vee \nu \quad \mu \wedge \nu \leq 1-f^{-1}(\lambda) \quad \mu \not\leq 1-f^{-1}(\lambda)$ and $\nu \not\leq 1-f^{-1}(\lambda)$
$\Rightarrow ff^{-1}(\lambda) \leq f(\mu) \vee f(\nu)$, $f(\mu) \wedge f(\nu) \leq 1-ff^{-1}(\lambda)$
$\Rightarrow \lambda \leq f(\mu) \vee f(\nu)$, $f(\mu) \wedge f(\nu) \leq 1-\lambda$
Where $f(\mu)$ is σ_i-fuzzy open set and $f(\nu)$ is σ_j- fuzzy open set in Y, since f is fuzzy pairwise open mapping.
Now since $\mu \not\leq 1-f^{-1}(\lambda)$ and $\nu \not\leq 1-f^{-1}(\lambda)$ therefore there exits $x_1, x_2 \in X$ such that

$$\mu(x_1) > 1-f^{-1}(\lambda)(x_1)$$
$$\nu(x_2) > 1-f^{-1}(\lambda)(x_2)$$

Also $f(x_1) = y_1$ and $f(x_2) = y_2$ are non-empty subsets of Y.

By definition $f(\mu)(y_1) = Sup\{\mu(x_i)\}$
and $f^{-1}(\lambda)(x_i) = \lambda(y_1)$ for every $x_i \in f^{-1}(y_1)$.
We claim that $f(\mu) \nleq 1 - \lambda$ and $f(\nu) \nleq 1 - \lambda$
Suppose $f(\mu) \leq 1 - \lambda \Rightarrow f(\mu)(y_1) \leq 1 - \lambda(y_1)$ for $x_i \in f^{-1}(y_1)$
$\Rightarrow f(\mu)(y_1) \leq 1 - \lambda(y_1)$ for $x_i \in f^{-1}(y_1)$
$\Rightarrow Sup\{\mu(x_i)\} \leq 1 - f^{-1}(\lambda)(x_1)$
$\Rightarrow \mu(x_1) \leq 1 - f^{-1}(\lambda)(x_1)$ this contradicts (1).
Similarly we can show $f(\nu) \nleq 1 - \lambda$. Therefore λ is not c_3- pairwise connected. ∎
The proof is similar for c_4- pairwise connectedness.

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