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# Variational approach for identifying a coefficient of the wave equation

Cheok Choi
Department of Mathematics, Seoul National University
56-1 Shillim-dong, Kwanak-ku, Seoul 151-742, Korea
chuck01@snu.ac.kr

Gen Nakamura [1]
Graduate School of Science, Hokkaido University,
Kita 8, Nishi 5, Kita-ku, Sapporo, Hokkaido 060-0808, Japan
gnaka@math.sci.hokudai.ac.jp

Kenji Shirota
Faculty of Science, Ibaraki University,
Bunkyo 2-1-1, Mito, Ibaraki 310-8512, Japan
shirota@mx.ibaraki.ac.jp

## ABSTRACT

An inverse initial-boundary value problem for identifying the coefficient of some second order hyperbolic equation by single set of boundary measurement is considered. The problem is transformed to a minimization problem of a functional. After a concrete expression of the Gateaux derivative of the functional is obtained, an algorithm for identifying the coefficient is given based on the projected gradient method. A numerical result is presented to verify the algorithm.

## RESUMEN

Es considerado un problema inverso de valores inicial y de borde para identificar el coeficiente de cierta ecuación hiperbólica de segundo orden, para un único

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conjunto de medidas en el borde. El problema es transformado a un problema de minimización de un funcional. Además es obtenida una expresión concreta de la derivada de Gateaux del funcional, es dado un algoritmo para identificar los coeficientes, basado en el método del gradiente proyectado. Un resultado numérico es presentado para verificar el algoritmo.

**Key words and phrases:** *Inverse initial-boundary value problem, coefficient identification, wave equation, variational method, projected gradient method*

**Math. Subj. Class.:** *65M32, 35R30*

# 1 Introduction

Let $\Omega \subset \mathbb{R}^n$ ($n = 2$ or $3$) be a bounded domain with $C^\infty$-class boundary $\partial\Omega$. Of course, we only need to assume that $\partial\Omega$ is smooth enough, but it is convenient just to assume it is $C^\infty$. Let $K(x) \in L^\infty(\Omega)$ satisfy the following conditions:

$$\begin{cases} 0 < C_1 \le K(x) \le C_2 & \text{in } \overline{\Omega}, \\ K(x) \text{ is } C^\infty\text{-class} & \text{in } \Omega \setminus F, \\ |\nabla K(x)| \le C_3 & \text{in } \Omega \setminus F, \end{cases} \tag{1.1}$$

where $C_1, C_2, C_3$ are fixed positive constants and $F \subset \Omega$ is a compact set with $\partial\Omega \cap F = \emptyset$. We notice that our coefficient $K$ is smooth function near the boundary from the assumption of the compact set $F$. This will be used later to show the regularity of the solutions of boundary value problems and initial boundary value problems. For a given $\overline{u} \in C^6([0,T]; H_{(\frac{5}{2})}(\partial\Omega))$ with $\partial_t^i \overline{u}(x,0) = 0$ ($x \in \partial\Omega, 0 \le i \le 5$), we consider an initial boundary value problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \nabla \cdot (K(x)\nabla u) = 0 & \text{in } \Omega \times (0,T), \\ u(\cdot,0) = 0, \quad \dfrac{\partial u}{\partial t}(\cdot,0) = 0 & \text{in } \Omega, \\ u = \overline{u} & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{1.2}$$

Here $H_{(\frac{5}{2})}(\partial\Omega)$ is the Sobolev space of order $\dfrac{5}{2}$ defined over $\partial\Omega$, and $C^m([0,T];X)$ with $m \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ denotes the set of all $C^m$ class functions defined in $[0,T]$ taking their values in some Banach space $X$.

The problem of (1.2) admits a unique solution $u \in \bar{H}_{(5)}((0,T); \bar{H}_{(1)}(\Omega))$. We denote this $u$ by $u = u[K](x,t)$ to clarify the dependency on $K$.

Here and hereafter, we define $\bar{H}_{(s)}(\Omega)$ for $s \in \mathbb{R}$ by $g \in \bar{H}_{(s)}(\Omega)$ if and only if there exists an extension $f \in H_{(s)}(\mathbb{R}^n)$ of $g$ to the ambient space $\mathbb{R}^n$ of $\Omega$ and the norm $||g||_{\bar{H}_{(s)}(\Omega)}$ is defined by $||g||_{\bar{H}_{(s)}(\Omega)} := \inf\{||f||_{H_{(s)}(\mathbb{R}^n)}; f|_\Omega = g\}$. We also define

$\dot{H}_{(s)}(\Omega) := \{g \in \bar{H}_{(s)}(\Omega);\ \mathrm{supp}\ g \subset \overline{\Omega}\}$ with norm $||g||_{\dot{H}_{(s)}(\Omega)} := ||g||_{\bar{H}_{(s)}(\Omega)}$. These kind of Sobolev spaces are discussed systematically in [3].

Now, suppose we do not know $K(x)$, but we are given $\overline{q} := K(x)\dfrac{\partial u}{\partial n}$ on $\partial\Omega \times (0,T)$ beside $\overline{u}$ for $T$ large enough. Then, we are interested in the following inverse problem (IP):

**Inverse problem (IP):** Reconstruct $K(x)$ from $\{\overline{u}, \overline{q}\}$.

Let $\mathcal{K}$ be the set of all $K(x) \in L^\infty(\Omega)$ satisfying (1.1). For any $L \in \mathcal{K}$, we define $J(L)$ by

$$J(L) = \int_0^T \int_{\partial\Omega} |L(x)\frac{\partial u[L]}{\partial n} - \overline{q}|^2\, d\sigma dt,$$

where $u[L]$ is the solution of (1.2) for $K = L$. Since the absolute minimum of $J(L)$ is attained when $L(x) = K(x)$, we can expect to recover $K(x)$ by minimizing $J(L)$.

One of our coauthor Shirota started the numerical study of the inverse problem (IP) in his paper [9]. He used the projected gradient method to minimize the functional $J(L)$. There are many methods for minimizing $J(L)$. The projected gradient method is one of them. It needs to compute the Gateaux derivative $J'(L)$ of $J(L)$. Shirota computed $J'(L)$ approximately by a formal argument. Some of the numerical results in [9] were quite good.

The aim of this paper is to justify his formal argument, give the complete form of $J'(L)$ and provide some numerical results using $J'(L)$. The complete form of $J'(L)$ is given by the following theorem.

**Theorem 1.1** *Let $L \in \mathcal{K}$ and $M \in L^\infty(\Omega)$. For any $\varepsilon > 0$ with $L + \varepsilon M \in \mathcal{K}$, it holds for the Gateaux derivative $J'(L)$ that*

$$\begin{cases} J(L+\varepsilon M) - J(L) = \varepsilon J'(L)M + o(\varepsilon)\,, \\ J'(L)M = \displaystyle\int_0^T \int_\Omega M\nabla u[L]\cdot\nabla v dx dt + \int_\Omega \frac{\partial U}{\partial t}(T) w dx, \end{cases}$$

*where $w \in \dot{H}_{(1)}(\Omega)$ is the weak solution of the elliptic equation*

$$\begin{cases} \nabla\cdot(L\nabla w) = 0 & \text{in } \Omega\,, \\ w = 2\left(L\dfrac{\partial u[L]}{\partial n}(T) - \overline{q}(T)\right) & \text{on } \partial\Omega\,, \end{cases} \tag{1.3}$$

*and $v \in L^2((0,T); \bar{H}_{(1)}(\Omega))$ is the weak solution of the equation*

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}v - \nabla\cdot(L\nabla v) = 0 & \text{in } \Omega\times(0,T)\,, \\ v(T) = w, \quad \dfrac{\partial v}{\partial t}(T) = 0 & \text{in } \Omega\,, \\ v = 2\left(L\dfrac{\partial u[L]}{\partial n} - \overline{q}\right) & \text{on } \partial\Omega\times(0,T)\,. \end{cases} \tag{1.4}$$

*Moreover,* $U \in \bar{H}_{(5)}((0,T);L^2(\Omega))$ *is the weak solution of the equation*

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}U - \nabla\cdot(L\nabla U) = \nabla\cdot(M\nabla u[L]) & in \quad \Omega\times(0,T), \\ U(0)=0, \quad \dfrac{\partial U}{\partial t}(0)=0 & in \quad \Omega, \\ U=0 & on \quad \partial\Omega\times(0,T). \end{cases} \tag{1.5}$$

**Remark 1.2** *In the proof of Theorem 1.1 given later, we showed that the mapping* $\Phi : L^\infty(\Omega) \ni M \mapsto \int_\Omega \frac{\partial U}{\partial t}(T) w\, dx \in \mathbb{R}$ *is bounded linear. Hence, by Example 5 in page 118 of [12], there exists a unique* $h \in L^1(\Omega)$ *such that*

$$\int_\Omega \frac{\partial U}{\partial t}(T) w\, dx = \int_\Omega M h\, dx. \tag{1.6}$$

The assumption which $\overline{u} \in C^6([0,T]; H_{(\frac{5}{2})}(\partial\Omega))$ with $\partial_t^i \overline{u}(x,0) = 0$ $(x \in \partial\Omega, 0 \leq i \leq 5)$ is too enough for the existence of the solution to (1.2). However, this assumption is needed to guarantee the existence and regularity of the adjoint problems (1.3) and (1.4). The details for the existence and regularity of $u$, $v$, and $w$ are given in Appendix. The proof of this theorem and the existence of $U$ are given in section 5 and section 4, respectively.

To the best of our knowledge, there is not any paper other than [9] which tried projected gradient method to obtain some good numerical results for the inverse problem (IP). The inverse problem (IP) is only a prototype. The same method can be applied to similar inverse problems with other corresponding equations. When we consider elastic equations, the problem becomes more practical because it really models the nondestructive testing of a material using ultrasound.

The rest of our paper is organized as follows. In section 2, we will show preliminary computation of the variation of $J(L)$ with respect to $L$ as an intermediate step to get the complete form of the Gateaux derivative $J'(L)$ of $J(L)$. In section 3, we will present some theorems, which play a major role to prove Theorem 1.1. In sections 4 and 5, using the theorems given in section 3, we will prove Theorem 1.1. Finally in the last section, we show a numerical algorithm based on the complete form of $J'(L)$ and its example.

## 2 Preliminary computation for $J'(L)$

In order to obtain a concrete expression of the Gateaux derivative of the functional $J$, we have to calculate the difference $J(L+\varepsilon M) - J(L)$. For this purpose, we have the following result.

**Lemma 2.1**

$$J(L+\varepsilon M)-J(L)=\varepsilon\int_0^T\int_\Omega M\nabla u[L]\cdot\nabla v dxdt+\int_\Omega\frac{\partial\delta u}{\partial t}(T)wdx$$
$$+\int_0^T\int_\Omega \varepsilon M\nabla\delta u\cdot\nabla v dxdt+\int_0^T\int_{\partial\Omega}|\delta q|^2 d\sigma dt\,, \qquad (2.1)$$

*where* $\delta u=u[L+\varepsilon M]-u[L]$, $\delta q=(L+\varepsilon M)\dfrac{\partial u[L+\varepsilon M]}{\partial n}-L\dfrac{\partial u[L]}{\partial n}$, *and* $w\in\bar H_{(1)}(\Omega)$ *is the weak solution of the elliptic equation*

$$\begin{cases}\nabla\cdot(L\nabla w)=0 & in\ \ \Omega\,,\\ w=2\left(L\dfrac{\partial u}{\partial n}(T)-\overline{q}(T)\right) & on\ \ \partial\Omega\,.\end{cases} \qquad (2.2)$$

*The function* $v\in L^2((0,T);\bar H_{(1)}(\Omega))$ *is the weak solution of the equation*

$$\begin{cases}\dfrac{\partial^2}{\partial t^2}v-\nabla\cdot(L\nabla v)=0 & in\ \ \Omega\times(0,T)\,,\\ v(T)=w,\quad \dfrac{\partial v}{\partial t}(T)=0 & in\ \ \Omega\,,\\ v=2\left(L\dfrac{\partial u}{\partial n}-\overline{q}\right) & on\ \ \partial\Omega\times(0,T)\,.\end{cases} \qquad (2.3)$$

**Proof** Let $q[L]:=L\dfrac{\partial u[L]}{\partial n}\Big|_{\partial\Omega\times(0,T)}$. For $L,M\in\mathcal{K}$, we have

$$\begin{aligned}J(L+\varepsilon M)-J(L)&=\int_0^T\int_{\partial\Omega}\left[(q[L+\varepsilon M]-\overline{q})^2-(q[L]-\overline{q})^2\right]d\sigma dt\\ &=\int_0^T\int_{\partial\Omega}(q[L+\varepsilon M]+q[L]-2\overline{q})\,\delta q\,d\sigma dt\\ &=\int_0^T\int_{\partial\Omega}2(q[L]-\overline{q})\,\delta q\,d\sigma\,dt+\int_0^T\int_{\partial\Omega}|\delta q|^2 d\sigma dt\,.\end{aligned} \qquad (2.4)$$

Integrating by parts and reminding $\dfrac{\partial u[L]}{\partial t}(0)=0$ and $v(T)=w$, we can see

$$\begin{aligned}\int_0^T\int_\Omega\frac{\partial u[L]}{\partial t}\frac{\partial v}{\partial t}dxdt&=\int_\Omega\left(\left[\frac{\partial u[L]}{\partial t}v\right]_0^T-\int_0^T\frac{\partial^2 u[L]}{\partial t^2}\,v\,dt\right)dx\\ &=\int_\Omega\frac{\partial u[L]}{\partial t}(T)\,w\,dx-\int_0^T\int_\Omega\frac{\partial^2 u[L]}{\partial t^2}\,v\,dxdt\,.\end{aligned}$$

By another integration by parts, we also see

$$\int_0^T\int_\Omega L\nabla u[L]\cdot\nabla v\,dxdt=\int_0^T\left(\int_{\partial\Omega}v\,q[L]\,d\sigma-\int_\Omega\nabla\cdot(L\nabla u[L])\,v\,dx\right)dt\,.$$

Hence, we can get

$$\int_0^T \int_\Omega \Big( \frac{\partial u[L]}{\partial t}\frac{\partial v}{\partial t} - L\nabla u[L]\cdot \nabla v \Big)\, dxdt = \int_\Omega \frac{\partial u[L]}{\partial t}(T)\, w\, dx - \int_0^T \int_{\partial\Omega} v\, q[L]\, d\sigma dt. \tag{2.5}$$

By a similar way, we have

$$\int_0^T \int_\Omega \Big( \frac{\partial u[L+\varepsilon M]}{\partial t}\frac{\partial v}{\partial t} - (L+\varepsilon M)\nabla u[L+\varepsilon M]\cdot \nabla v \Big)\, dxdt$$
$$= \int_\Omega \frac{\partial u[L+\varepsilon M]}{\partial t}(T)\, w\, dx - \int_0^T \int_{\partial\Omega} v\, q[L+\varepsilon M]\, d\sigma dt. \tag{2.6}$$

Using (2.5) and (2.6), we get

$$\int_0^T \int_\Omega \Big[ \frac{\partial \delta u}{\partial t}\frac{\partial v}{\partial t} - ((L+\varepsilon M)\nabla u[L+\varepsilon M] - L\nabla u[L])\cdot \nabla v \Big]\, dxdt$$
$$= \int_\Omega \frac{\partial \delta u}{\partial t}(T)\, w\, dx - \int_0^T \int_{\partial\Omega} v\, \delta q\, d\sigma dt. \tag{2.7}$$

Moreover, applying the integration by parts, we have

$$\int_0^T \int_\Omega \frac{\partial \delta u}{\partial t}\frac{\partial v}{\partial t}\, dxdt = \int_\Omega \left( \Big[ \delta u \frac{\partial v}{\partial t} \Big]_0^T - \int_0^T \delta u \frac{\partial^2 v}{\partial t^2} dt \right) dx$$
$$= -\int_0^T \int_\Omega \delta u\, \frac{\partial^2 v}{\partial t^2}\, dxdt,$$

and

$$\int_0^T \int_\Omega L\nabla \delta u\cdot \nabla v\, dxdt = \int_0^T \left[ \int_{\partial\Omega} \delta u \left( L\frac{\partial v}{\partial n} \right) d\sigma - \int_\Omega \delta u \nabla\cdot (L\nabla v) dx \right] dt$$
$$= -\int_0^T \int_\Omega \delta u\, \nabla\cdot (L\nabla v)\, dxdt.$$

Therefore, we can get

$$\int_0^T \int_\Omega \Big( \frac{\partial \delta u}{\partial t}\frac{\partial v}{\partial t} - L\nabla \delta u\cdot \nabla v \Big)\, dxdt = 0. \tag{2.8}$$

From (2.7) and (2.8), we have

$$-\int_0^T \int_\Omega \varepsilon M\nabla u[L+\varepsilon M]\cdot \nabla v\, dxdt = \int_\Omega \frac{\partial \delta u}{\partial t}(T)\, w\, dx - \int_0^T \int_{\partial\Omega} v\, \delta q\, d\sigma dt.$$

Reminding $v|_{\partial\Omega\times(0,T)} = 2(q[L]-\overline{q})$, we can see

$$\int_0^T \int_{\partial\Omega} 2(q[L]-\overline{q})\, \delta q\, d\sigma dt = \int_0^T \int_\Omega \varepsilon M\nabla u[L]\cdot \nabla v\, dxdt$$
$$+ \int_0^T \int_\Omega \varepsilon M\nabla \delta u\cdot \nabla v\, dxdt + \int_\Omega \frac{\partial \delta u}{\partial t}(T)\, w\, dx. \tag{2.9}$$

Hence, substituting (2.9) into (2.4), we get (2.1). ∎

# 3 Continuous dependence of the solution on the coefficient

Fix $M \in \mathcal{K}$ and take a small $\varepsilon > 0$. We write

$$L_\varepsilon(x) := L(x) + \varepsilon M(x)\,, \quad \mathcal{A}_L := \frac{\partial^2}{\partial t^2} - A_L\,, \quad \mathcal{A}_{L_\varepsilon} := \frac{\partial^2}{\partial t^2} - A_{L_\varepsilon}\,,$$

where the notation $A_L$, $A_{L_\varepsilon}$ are defined as in (A.2) of Appendix.

Let $v \in \bar{H}_{(5)}((0,T); \dot{H}_{(1)}(\bar{\Omega}))$ and $v_\varepsilon \in \bar{H}_{(5)}((0,T); \dot{H}_{(1)}(\bar{\Omega}))$ be the solutions to

$$\begin{cases} \mathcal{A}_L v = f(L) \in C^4([0,T]; \bar{H}_{(1)}(\Omega)) & \text{in } \Omega \times (0,T)\,, \\ v = 0 & \text{on } \partial\Omega \times (0,T)\,, \\ v(0) = \widetilde{u}_0\,, \quad \dfrac{\partial}{\partial t} v(0) = \widetilde{u}_1\,, \end{cases} \tag{3.1}$$

and

$$\begin{cases} \mathcal{A}_{L_\varepsilon} v_\varepsilon = f(L_\varepsilon) \in C^4([0,T]; \bar{H}_{(1)}(\Omega)) & \text{in } \Omega \times (0,T)\,, \\ v_\varepsilon = 0 & \text{on } \partial\Omega \times (0,T)\,, \\ v_\varepsilon(0) = \widetilde{u}_0\,, \quad \dfrac{\partial}{\partial t} v_\varepsilon(0) = \widetilde{u}_1\,, \end{cases} \tag{3.2}$$

respectively. Here, $f(L)$, $f(L_\varepsilon)$, $\widetilde{u}_0$, and $\widetilde{u}_1$ are defined by the formula given in (A.5) of Appendix.

Then, we can obtain the following continuous dependency of the solution on the coefficient in a Gelfand triple $(V, H, V')$, which is defined as in the statements below (A.5) of Appendix.

**Theorem 3.1**

$$v_\varepsilon \to v \quad (\varepsilon \to 0) \quad \text{in} \ \ C([0,T]; V) \cap C^1([0,T]; H)\,.$$

**Proof** If the inhomogeneous terms of the equations (3.1) and (3.2) are same, the continuous dependency of the solution on the coefficient is given as in Theorem 2.8.1 and Theorem 2.8.2 in [10]. The proof for them can be also applied to the present situation without any essential change. So we omit giving further details of the proof. ∎

Now, for given $u_0 \in H$, $u_1 \in V'$ and $f \in L^2((0,T); V')$, we consider the Cauchy problem :

$$\begin{cases} \mathcal{A}_K u = f & \text{in } \Omega \times (0,T)\,, \\ u(0) = u_0\,, \quad \dfrac{\partial}{\partial t} u(0) = u_1 & \text{in } \Omega\,. \end{cases} \tag{3.3}$$

In order to define a weak solution $u$ to (3.3), we first define the test function space $X$.

**Definition 3.2** *(test function space) The test function space $X$ is the set of all $\varphi \in L^2((0,T);V)$ satisfying*

$$\begin{cases} \mathcal{A}_K\varphi \in L^2((0,T);H)\,, \\ \varphi(T) = \dfrac{\partial\varphi}{\partial t}(T) = 0\,. \end{cases}$$

The definition of the weak solution is as follows.

**Definition 3.3** *$u \in L^2((0,T);H)$ with $u' \in L^2((0,T);V')$ is called a weak solution of (3.3) if it satisfies*

$$\int_0^T \int_\Omega u\,\mathcal{A}_K\varphi\,dxdt = \int_0^T \int_\Omega f\varphi\,dxdt + \int_\Omega \left( u_1\varphi(0) - u_0\frac{\partial\varphi}{\partial t}(0)\right) dx \quad \text{for } \forall\varphi \in X\,,$$

*where $'$ means the derivative in distribution sense.*

Then, we have the following existence and uniqueness result from [4].

**Theorem 3.4** *Given $(f,u_0,u_1) \in L^2((0,T);V') \times H \times V'$. Then there is a unique weak solution $u$ with $\left(u, \dfrac{\partial u}{\partial t}\right) \in C([0,T];H) \times C([0,T];V')$.*

For the continuous dependence of the solution to (3.3) on the coefficient, we have the following result.

**Theorem 3.5** *Let $v_\varepsilon$ and $v$ be the weak solution of (3.3) with $K = L + \varepsilon M$ and $K = L$, respectively. Then, $v_\varepsilon \to v$ $(\varepsilon \to 0)$ in $L^2((0,T);H)$ and $v'_\varepsilon \to v'$ $(\varepsilon \to 0)$ in $L^2((0,T);V')$.*

**Proof** Let

$$E(t) := \frac{1}{2}(\langle v,v\rangle_H + \langle A_L^{-1}v', v'\rangle_H)$$

and

$$E_\varepsilon(t) := \frac{1}{2}(\langle v_\varepsilon,v_\varepsilon\rangle_H + \langle A_{L_\varepsilon}^{-1}v'_\varepsilon, v'_\varepsilon\rangle_H)\,.$$

Here, from the well-posedness of (2.2), we can guarantee the existence of the inverse operators $A_L^{-1}$ and $A_{L_\varepsilon}^{-1}$. Then, from the proof of Theorem 9.3 of chapter 3 in [4] and by the same argument with Lemma 2.4.1 in [10], we have

$$E(t),\, E_\varepsilon(t) \le C\left( \|u_0\|_H^2 + \|u_1\|_{V'}^2 + \int_0^T \|f\|_{V'}^2\,dt \right) \tag{3.4}$$

and

$$E(t) - E(0) = \int_0^t \langle A_L^{-1}f, v'\rangle_H\,dt\,, \quad E_\varepsilon(t) - E_\varepsilon(0) = \int_0^t \langle A_{L_\varepsilon}^{-1}f, v'_\varepsilon\rangle_H\,dt\,, \tag{3.5}$$

respectively.

First, we show that $v_\varepsilon$ converges weakly to $v$. From the coercivity in (A.1), we have

$$E_\varepsilon(t) \geq C(||v_\varepsilon||_H^2 + ||v_\varepsilon'||_{V'}^2).$$

So, by (3.4), $v_\varepsilon$ and $v_\varepsilon'$ is uniformly bounded independent of $\varepsilon$ in $L^2((0,T);H)$ and $L^2((0,T);V')$, respectively. Therefore, by the weak compactness of these spaces, there is a sequence $\{\varepsilon(l)\}$, which is the subsequence of $\{\varepsilon\}$, $\alpha \in L^2((0,T);H)$ and $\beta \in L^2((0,T);V')$ satisfying

$$\begin{cases} v_{\varepsilon(l)} \rightharpoonup \alpha & \text{in} \quad L^2((0,T);H), \\ v_{\varepsilon(l)}' \rightharpoonup \beta & \text{in} \quad L^2((0,T);V'). \end{cases} \tag{3.6}$$

Now, we will prove that this $\alpha$ is the same with $v$ by showing

$$\int_0^T \int_\Omega \alpha A_L \varphi \, dx dt = \int_0^T \int_\Omega f\varphi \, dx dt + \int_\Omega \left( u_1 \varphi(0) - u_0 \frac{\partial \varphi}{\partial t}(0) \right) dx \tag{3.7}$$

for $\forall \varphi \in X$. Let $g := A_L\varphi \in L^2((0,T);H)$ and extend $g$ to the whole time interval by putting $g \equiv 0$ in $(-\infty, 0] \cup [T, \infty)$. Also, let $g^l := \chi_{[0,T-\frac{1}{l}]} g \in L^2((0,T);H)$ with the characteristic function $\chi_{[0,T-\frac{1}{l}]}$ of $[0, T-\frac{1}{l}]$ and $g^{l,m} := \rho_m * g^l \in C^\infty([0,T], H)$, where $\rho_m *$ is the mollification with a mollifier $\rho_m(t) := m^{-1}\rho(m^{-1}t)$. Here $\rho \in C_0^\infty(\mathbb{R})$ is a function which satisfies $0 \leq \rho \leq 1$ and $\int_{\mathbb{R}} \rho(t)\,dt = 1$. Then, $g^{l,m}$ is flat at $T$ and we have

$$\begin{cases} g^{l,m} \to g^l \quad (m \to 0) & \text{in} \quad L^2((0,T);H), \\ g^l \to g \quad (l \to 0) & \text{in} \quad L^2((0,T);H). \end{cases} \tag{3.8}$$

Taking $t = T$ as an initial surface and $g^{l,m}$ is flat at $t = T$ into account, we have from Theorem A.1 that there exists a unique $\varphi^{l,m} \in L^2((0,T);V)$ with enough time regularity to

$$\begin{cases} A_L \varphi^{l,m} = g^{l,m}, \\ \varphi^{l,m}(T) = (\varphi^{l,m})'(T) = 0. \end{cases}$$

By defining $\varphi_\varepsilon^{l,m} := A_{L_\varepsilon}^{-1} A_L \varphi^{l,m}$, it has enough time regularity and satisfies

$$\begin{cases} A_{L_\varepsilon} \varphi_\varepsilon^{l,m} = A_{L_\varepsilon}^{-1} A_L (\varphi^{l,m})'' + A_L \varphi^{l,m} \in L^2((0,T);H), \\ \varphi_\varepsilon^{l,m}(T) = (\varphi_\varepsilon^{l,m})'(T) = 0. \end{cases}$$

Hence, $\varphi_\varepsilon^{l,m}$ is a test function. By (3.2) and taking $u = v_\varepsilon$ in (3.3), we can see

$$\int_0^T \int_\Omega v_\varepsilon((\varphi_\varepsilon^{l,m})'' + A_{L_\varepsilon}\varphi_\varepsilon^{l,m}) \, dx dt$$
$$= \int_0^T \int_\Omega f \varphi_\varepsilon^{l,m} \, dx dt + \int_\Omega (u_1 \varphi_\varepsilon^{l,m}(0) - u_0 (\varphi_\varepsilon^{l,m})'(0)) \, dx.$$

From $A_L - A_{L_\varepsilon} = A_{L_\varepsilon}(A_{L_\varepsilon}^{-1} - A_L^{-1})A_L$, we have

$$\begin{aligned} ||\varphi_\varepsilon^{l,m} - \varphi^{l,m}||_V &= ||A_{L_\varepsilon}^{-1} A_L \varphi^{l,m} - \varphi^{l,m}||_V = ||(A_{L_\varepsilon}^{-1} - A_L^{-1}) A_L \varphi^{l,m}||_V \\ &= ||A_{L_\varepsilon}^{-1}(A_L - A_{L_\varepsilon})\varphi^{l,m}||_V \leq C||(A_L - A_{L_\varepsilon})\varphi^{l,m}||_{V'} \\ &\to 0 \quad (\varepsilon \to 0) \quad \text{uniformly with respect to} \quad t \in [0,T]. \end{aligned} \tag{3.9}$$

Here $C$ stands for a generic constant. Similarly, we can show

$$\|(\varphi_\varepsilon^{l,m})' - (\varphi^{l,m})'\|_V,\ \|(\varphi_\varepsilon^{l,m})'' - (\varphi^{l,m})''\|_V \to 0. \tag{3.10}$$

Using (3.6), (3.9), (3.10), and uniform boundedness of $\{v_\varepsilon\}$, we reach

$$\int_0^T \int_\Omega \alpha((\varphi^{l,m})'' + A_{L_\varepsilon}\varphi^{l,m})\,dxdt$$
$$= \int_0^T \int_\Omega f\varphi^{l,m}\,dxdt + \int_\Omega \left(u_1\varphi^{l,m}(0) - u_0(\varphi^{l,m})'(0)\right)\,dx\,.$$

Furthermore, by (3.8) and the continuous dependency of $\varphi^{l,m}$ on the source term in Theorem A.1, we get (3.7).

Also, we can show that $v_\varepsilon$ itself converges to $v$ weakly. In fact, if not, then, we can find $\varepsilon > 0$, one subsequence $\{v_{\varepsilon(m)}\}$ and $\varphi \in X$ satisfying

$$\int_0^T \langle v - v_{\varepsilon(m)}, \varphi\rangle_H > \varepsilon. \tag{3.11}$$

However, $\{v_{\varepsilon(m)}\}$ is uniformly bounded in $L^2((0,T);H)$. As a result, it has convergent subsequence to $v$, which is in contradiction to (3.11).

Second, we show that $v_\varepsilon$ converges strongly to $v$. Reminding $\langle A_{L_\varepsilon}^{-1}f, v_\varepsilon'\rangle_H - \langle A_L^{-1}f, v'\rangle_H = \langle A_{L_\varepsilon}^{-1}f, v_\varepsilon' - v'\rangle_H + \langle (A_{L_\varepsilon}^{-1} - A_L^{-1})f, v_\varepsilon'\rangle_H$ and $A_{L_\varepsilon}^{-1}(A_L - A_{L_\varepsilon})A_L^{-1} = (A_{L_\varepsilon}^{-1} - A_L^{-1})$, we can show in (3.5) as

$$E_\varepsilon(0) \to E(0), \quad \int_0^t \langle A_{L_\varepsilon}^{-1}f, v_\varepsilon'\rangle_H\,dt \to \int_0^t \langle A_L^{-1}f, v'\rangle_H\,dt \quad (\varepsilon \to 0).$$

As a result, $E_\varepsilon(t) \to E(t)$ $(\varepsilon \to 0)$.

Now, let $\xi(t) = \langle v_\varepsilon - v, v_\varepsilon - v\rangle_H + \langle A_L^{-1}(v_\varepsilon' - v'), v_\varepsilon' - v'\rangle_H$. Then, by expanding the right side of $\xi(t)$, we have

$$\xi(t) = 2E(t) + 2E_\varepsilon(t) - \langle (A_{L_\varepsilon}^{-1} - A_L^{-1})v_\varepsilon', v_\varepsilon'\rangle_H - 2(\langle v, v_\varepsilon\rangle_H + \langle A_L^{-1}v', v_\varepsilon'\rangle_H). \tag{3.12}$$

On the other hand, by coercivity (A.1),

$$\xi(t) \geq C(\|v_\varepsilon - v\|_H^2 + \|v_\varepsilon' - v'\|_{V'}^2). \tag{3.13}$$

Therefore, using uniform boundedness of $\{v_\varepsilon\}$ and $\{v\}$, (3.6), (3.12), and (3.13), we get the strong convergence of $v_\varepsilon$ and $v_\varepsilon'$ to $v$ and $v'$ in $L^2((0,T);H)$ and $L^2((0,T);V')$, respectively. ∎

## 4 Asymptotic of $\varepsilon^{-1}\dfrac{\partial \delta u}{\partial t}(T)\ (\varepsilon \to 0)$

This section is devoted to the following theorem which gives a representation formula of the second term on the right side in (2.1).

**Theorem 4.1**
$$\int_\Omega \frac{\partial \delta u}{\partial t}(T) w\, dx = \varepsilon \int_\Omega \frac{\partial U}{\partial t}(T) w\, dx + o(\varepsilon)\,,$$
*where $U$ is the weak solution to (1.5).*

**Proof** Let $u_\varepsilon := u[L+\varepsilon M]$, $u := u[L]$. Then, we define the functions $\widetilde{u}$ and $\widetilde{u}_\varepsilon$ by $\widetilde{u} = u - \phi$ and $\widetilde{u}_\varepsilon = u_\varepsilon - \phi$, respectively. Here $\phi := \Lambda\overline{u}$ and $\Lambda$ is the inverse trace operator given by (A.3). Then we have $\nabla\cdot(M\nabla)u = \nabla\cdot(M\nabla)\widetilde{u} + \nabla\cdot(M\nabla)\phi \in \bar{H}^{(5)}((0,T);V')$, and $u_\varepsilon - u = \widetilde{u}_\varepsilon - \widetilde{u}$.

By defining $U_\varepsilon := \dfrac{\widetilde{u}_\varepsilon - \widetilde{u}}{\varepsilon}$, we have
$$\begin{aligned}\mathcal{A}_{L_\varepsilon} U_\varepsilon &= \frac{1}{\varepsilon}(\mathcal{A}_{L_\varepsilon}\widetilde{u}_\varepsilon - \mathcal{A}_{L_\varepsilon}\widetilde{u}) \\ &= \frac{1}{\varepsilon}(\mathcal{A}_L\widetilde{u} + f(L_\varepsilon) - f(L) - \mathcal{A}_{L_\varepsilon}\widetilde{u}) \\ &= \nabla\cdot(M\nabla\widetilde{u}) + \nabla\cdot(M\nabla\phi)\,,\end{aligned}$$
and
$$U_\varepsilon(0) = U_\varepsilon'(0) = 0\,.$$
Moreover, we have
$$\mathcal{A}_{L_\varepsilon}\frac{\partial^i}{\partial t^i}U_\varepsilon = \frac{\partial^i}{\partial t^i}\left(\nabla\cdot(M\nabla\widetilde{u}) + \nabla\cdot(M\nabla\phi)\right)\,, \qquad \frac{\partial^i}{\partial t^i}U_\varepsilon(0) = \frac{\partial^{i+1}}{\partial t^{i+1}}U_\varepsilon(0) = 0$$
for $0 \le i \le 5$. Therefore, by Theorem 3.5, we have
$$U_\varepsilon \to U(\varepsilon\to 0) \;\text{ in }\; \bar{H}_{(5)}((0,T);H)\,.$$
As a result, we have
$$\frac{\partial \delta u}{\partial t}(T) = \frac{\partial \widetilde{u}_\varepsilon}{\partial t}(T) - \frac{\partial \widetilde{u}}{\partial t}(T) = \varepsilon\frac{\partial U}{\partial t}(T) + o(\varepsilon) \;\text{ in }\; H\,,$$
which completes the proof. ∎

# 5 Completion of the proof of Theorem 1.1

First, we show that the third and fourth terms on the right side in (2.1) are $o(\varepsilon)$.

**Theorem 5.1**
$$\int_0^T\int_\Omega \varepsilon M\nabla\delta u\cdot\nabla v\, dxdt + \int_0^T\int_{\partial\Omega}|\delta q|^2 d\sigma dt = o(\varepsilon) \quad (\varepsilon\to 0)\,.$$

**Proof** Let $u$, $u_\varepsilon$, $\widetilde{u}$ and $\widetilde{u}_\varepsilon$ be those given in the proof of Theorem 4.1. Using Theorem 3.1, we can show easily that the first term is $o(\varepsilon)$.

Let $z_\varepsilon := u_\varepsilon - u = \widetilde{u}_\varepsilon - \widetilde{u} \in \bar{H}_{(5)}((0,T);\dot{H}_{(1)}(\overline{\Omega}))$. If we can show $||z_\varepsilon||_{L^2((0,T);\bar{H}_{(2)}(\Omega\setminus F))} = O(\varepsilon)$ $(\varepsilon \to 0)$, then the proof is done. To begin with, we observe

$$\begin{aligned}\mathcal{A}_L z_\varepsilon &= \mathcal{A}_L \widetilde{u}_\varepsilon - \mathcal{A}_L \widetilde{u} \\ &= \mathcal{A}_{L_\varepsilon}\widetilde{u}_\varepsilon + (\mathcal{A}_L - \mathcal{A}_{L_\varepsilon})\widetilde{u}_\varepsilon - \mathcal{A}_L\widetilde{u} \\ &= (f(L_\varepsilon) - f(L)) + (\mathcal{A}_L - \mathcal{A}_{L_\varepsilon})\widetilde{u}_\varepsilon \\ &= \varepsilon(\nabla\cdot(M\nabla\phi) + \nabla\cdot(M\nabla\widetilde{u}_\varepsilon)) \\ &= \varepsilon h \in \bar{H}_{(5)}((0,T);V') \cap \bar{H}_{(5)}((0,T);\bar{H}_{(1)}(\Omega\setminus F)),\end{aligned} \tag{5.1}$$

where $h := \nabla\cdot(M\nabla u_\varepsilon)$. From (5.1), we have, for $0 \le i \le 5$,

$$\begin{cases} \mathcal{A}_L \dfrac{\partial^i}{\partial t^i} z_\varepsilon = \varepsilon \dfrac{\partial^i}{\partial t^i} h & \text{in} \quad \Omega\times(0,T), \\ \dfrac{\partial^i}{\partial t^i} z_\varepsilon(0) = \dfrac{\partial^{i+1}}{\partial t^{i+1}} z_\varepsilon(0) = 0 & \text{in} \quad \Omega. \end{cases}$$

By (3.4) together with (A.1) given in Appendix and the uniform boundedness of $\{\widetilde{u}_\varepsilon\}$, we can show

$$\left|\left|\frac{\partial^i}{\partial t^i} z_\varepsilon\right|\right|_{L^2((0,T);H)} \le \varepsilon C \left|\left|\frac{\partial^i}{\partial t^i} h\right|\right|_{L^2((0,T);V')} \le \varepsilon C \quad (0 \le i \le 5). \tag{5.2}$$

Now, let $Z_\varepsilon := \alpha z_\varepsilon$ where $\alpha \in C_0^\infty(\mathbb{R}^n)$ satisfying supp $\alpha \subset \mathbb{R}^n \setminus F$ and $\alpha \equiv 1$ near all $x_0 \in \partial\Omega$. Then, we have

$$\begin{aligned}\mathcal{A}_L Z_\varepsilon &= Z_\varepsilon'' - A_L Z_\varepsilon \\ &= \varepsilon\alpha h - z_\varepsilon \nabla L\cdot\nabla\alpha - 2L\nabla\alpha\cdot\nabla z_\varepsilon - Lz_\varepsilon\Delta\alpha.\end{aligned}$$

By defining $g := -\varepsilon\alpha h + z_\varepsilon\nabla L\cdot\nabla\alpha + 2L\nabla\alpha\cdot\nabla z_\varepsilon + Lz_\varepsilon\Delta\alpha$, we get

$$A_L Z_\varepsilon = \alpha z_\varepsilon'' + g \in L^2((0,T);H). \tag{5.3}$$

Moreover, reminding $||z_\varepsilon||_{L^2((0,T);V)} \le C||z_\varepsilon'' - \varepsilon h||_{L^2((0,T);V')}$ from $A_L z_\varepsilon = z_\varepsilon'' - \varepsilon h$, by (5.2) we can show

$$||\alpha z_\varepsilon'' + g||_{L^2((0,T);H)} \le \varepsilon C. \tag{5.4}$$

To get exact inequality of $Z_\varepsilon$ in $L^2((0,T);\bar{H}_{(2)}(\Omega))$, we change the coordinate into a boundary normal coordinate near $x_0$. For example, in the case of dimension 3, by a transform $F:\Omega\to\mathbb{R}^n$ with $F(x) := y(x) = (y_1(x), y_2(x), y_3(x))$, we have near $x_0$

$$\begin{cases} \nabla_y\cdot(\widetilde{L}\nabla_y Z_\varepsilon) = \alpha z_\varepsilon'' + g, \\ \{y_1 > 0\} = F(\Omega), \quad \{y_1 = 0\} = F(\partial\Omega), \end{cases} \tag{5.5}$$

where $\widetilde{L} = (\widetilde{L}_{rs}) = (Lg)(F^{-1}(y))$, $g = (g_{rs}) = \displaystyle\sum_{j=1}^{3} \frac{\partial y_r}{\partial x_j}\frac{\partial y_s}{\partial x_j}$, and we used the same notation $Z_\varepsilon$, $z_\varepsilon$, $g$ to denote their pull back $F^{-1}$. Then, the principal part of $\nabla_y\cdot(\widetilde{L}\nabla_y Z_\varepsilon)$

is $L(\partial_{y_1}^2 + \sum_{i,j=2}^{3} g_{ij}\partial_{y_i}\partial_{y_j})$ due to $g_{11} = 1$, $g_{1s} = 0\,(s \neq 1)$ where we used the same notation $L$ to denote its pull back by $F^{-1}$. By defining $[Z_\varepsilon]_k := \dfrac{Z_\varepsilon(y + ke_j) - Z_\varepsilon(y)}{k}$ and letting $\widetilde{H} := L^2(\mathbb{R}_+^n)$, $\widetilde{V} := \dot{H}_{(1)}(\overline{\mathbb{R}_+^n})$ (Notice that $(\widetilde{V}, \widetilde{H}, \widetilde{V'})$ becomes a Gelfand triple), we have

$$\begin{aligned}\nabla_y \cdot \widetilde{L}\nabla_y[Z_\varepsilon]_k &= [\alpha z_\varepsilon'' + g]_k - \nabla_y \cdot [\widetilde{L}]_k \nabla_y Z_\varepsilon(y + ke_j)\\ &= [\alpha z_\varepsilon'']_k + [g]_k - \nabla_y \cdot [\widetilde{L}]_k \nabla_y Z_\varepsilon(y + ke_j) \in L^2((0,T); \widetilde{V'})\,.\end{aligned}$$

Then, from (5.4),

$$\|[\alpha z_\varepsilon'']_k + [g]_k - \nabla_y \cdot [\widetilde{L}]_k \nabla_y Z_\varepsilon(y + ke_j)\|_{L^2((0,T);\widetilde{V'})} \leq \varepsilon C\,.$$

Hence, $\|[Z_\varepsilon]_k\|_{L^2((0,T);\widetilde{V})} \leq \varepsilon C$. By uniform boundedness of $\{[Z_\varepsilon]_k\}$, it has a subset which converges to a function $W \in L^2((0,T); \widetilde{V})$ weakly. Moreover, $W = \partial_{y_j} Z_\varepsilon\,(2 \leq j \leq 3)$ and $\|W\|_{L^2((0,T);\widetilde{V})} \leq \varepsilon C$. So $\partial_{y_j}^\beta Z_\varepsilon \in L^2((0,T), \widetilde{H})$ for $|\beta| \leq 2$ and $2 \leq j \leq 3$. Also, we can show $\partial_{y_1}^2 Z_\varepsilon \in L^2((0,T), H)$ from (5.3) and observing the principal part of $\nabla_y \cdot (\widetilde{L}\nabla_y Z_\varepsilon)$. Then, using the interpolation theorem of Proposition 3.8 in [5] we have

$$\|Z_\varepsilon(y)\|_{L^2((0,T),\dot{H}_{(2)}(\mathbb{R}_+^n))} \leq \varepsilon C. \tag{5.6}$$

Since we can easily show $\|\alpha z_\varepsilon\|_{L^2((0,T),\dot{H}_{(2)}(\Omega))} \leq C\|Z_\varepsilon(y)\|_{L^2((0,T),\dot{H}_{(2)}(\mathbb{R}_+^n))}$ with some constant $C > 0$ independent of $\varepsilon$, $\|\alpha z_\varepsilon\|_{L^2((0,T),\dot{H}_{(2)}(\Omega))} \leq \varepsilon C$ with another constant $C > 0$. Reminding $\delta q = L_\varepsilon \dfrac{\partial u[L_\varepsilon]}{\partial n} - L\dfrac{\partial u[L]}{\partial n} = L\left(\dfrac{\partial u[L_\varepsilon]}{\partial n} - \dfrac{\partial u[L]}{\partial n}\right) + \varepsilon M \dfrac{\partial u[L_\varepsilon]}{\partial n}$ and (5.6), we arrive at the assertion that the second term is $o(\varepsilon)$. ∎

Now we can finish the proof of Theorem 1.1. From Lemma 2.1, Theorem 4.1 and Theorem 5.1, we have a representation in Theorem 1.1. We clearly have the linearity of the mapping : $L^\infty(\Omega) \ni M \mapsto J'(L)M \in \mathbb{R}$. By using (3.4), we easily have the boundedness of this mapping. Furthermore, $J'(L)$ gives the Gateaux derivative of $J(L)$ at $L$.

## 6 Numerical algorithm and example

To find the minimum of the functional $J$, we make use of the projected gradient method[6]:

$$L_{k+1} = P_C\,(L_k - \alpha_k \nabla J(L_k)) \quad (k = 0,\, 1,\, 2,\, \ldots\ldots), \tag{6.1}$$

where $\alpha_k$ $(0 < \alpha_k \leq 1)$ is a suitable step size and $\nabla J(L)$ is a search direction defined by

$$\langle \nabla J(L),\, M\rangle = J'(L)\,M \quad \text{for } \forall M \in L^\infty(\Omega)\,.$$

Here the map $P_C$ is a clip-off operator such that

$$P_C L(x) = \begin{cases} C_1 & (L(x) < C_1)\,, \\ L(x) & (C_1 \le L(x) \le C_2)\,, \\ C_2 & (L(x) > C_2)\,. \end{cases}$$

From Theorem 1.1 and Remark 1.2, we notice that

$$\nabla J(L) = \int_0^T \nabla u[L] \cdot \nabla v \, dt + h\,.$$

We have to discuss how to obtain numerically the function $h$ in order to use (6.1).

Let $\{B_i\}_{i=1}^N$ be a division of the domain $\Omega$ such that

$$\Omega = \bigcup_{i=1}^N B_i, \quad B_i \cap B_j = \emptyset \quad (i \neq j)\,.$$

We denote by $\chi_i$ a characteristic function, namely,

$$\chi_i(x) := \begin{cases} 1 & (x \in B_i)\,, \\ 0 & (x \notin B_i)\,. \end{cases}$$

Then, we consider finding the approximation of the density function $h$ in a subspace of $L^\infty(\Omega)$ defined by $X_B = \text{span}\,\{\chi_1, \chi_2, \cdots, \chi_N\}$. By using the relation (1.6) and the Galerkin method, we can get

$$\int_\Omega h_B \, \chi_i \, dx = \int_\Omega \frac{\partial U}{\partial t}[\chi_i](T) \, w \, dx$$

for $i = 1, 2, \ldots, N$. Here $h_B \in X_B$ is the approximation of the density function and the function $U[\chi_i]$ is the solution to (1.5) with the source term $\nabla \cdot (\chi_i \nabla u[L])$. We represent $h_B$ by the linear combination of $\chi_i$, namely, $h_B = \sum_{j=1}^N h_j \chi_j$. Then, the linear system can be obtained as follows:

$$\sum_{j=1}^N h_j \int_\Omega \chi_i \, \chi_j \, dx = \int_\Omega \frac{\partial U}{\partial t}[\chi_i](T) \, w \, dx$$

for $i = 1, 2, \ldots, N$. Since $\chi_i$ has the orthogonal relation with respect to $L^2$ inner product, we have

$$h_i = \frac{1}{|B_i|} \int_\Omega \frac{\partial U}{\partial t}[\chi_i](T) \, w \, dx \quad (i = 1, 2, \ldots, N)\,,$$

where $|B_i|$ means the area of $B_i$. Therefore we can get the approximation $h_B$ by solving $N$ initial-boundary value problem (1.5) with $M = \chi_i$. By using this approximation, we define the approximated search direction as follows:

$$\widetilde{\nabla} J(L) := \int_\Omega \nabla u[L] \cdot \nabla v \, dt + h_B\,.$$

Here we notice that our method with $\widetilde{\nabla} J(L)$ is not exactly the projected gradient method but its calculation is very easy.

Hence we summurize an algorithm for our inverse problem as follows:

**Algorithm for coefficient identification**

*Given the division* $\{B_i\}$.

1. *Pick an initial coefficient function* $L_0$ *which belongs to the admissible set* $\mathcal{K}$.
2. *For* $k = 0, 1, 2, \ldots$; *do*
   (a) *Solve (1.2) with* $K = L_k$ *to find* $\nabla u[L_k]$ *and* $L_k \dfrac{\partial u}{\partial n}\Big|_{\partial\Omega\times(0,T)}$.
   (b) *Solve the boundary value problem (1.3) to find* $w$.
   (c) *Solve the initial-boundary value problem (1.4) to find* $\nabla v$.
   (d) *For* $i = 1, 2, \ldots, N$; *do*
      i. *Solve the initial-boundary value problem (1.5) with the source term* $\nabla \cdot (\chi_i \nabla u[L_k])$ *to find* $\dfrac{\partial U}{\partial t}[\chi_i](T)$.
      ii. *Calculate* $h_i$ *by*
      $$h_i = \frac{1}{|B_i|} \int_\Omega \frac{\partial U}{\partial t}[\chi_i](T)\, w \, dx .$$
   (e) *Calculate the approximated search direction* $\widetilde{\nabla} J(L_k)$ *by*
   $$\widetilde{\nabla} J(L_k) = \int_0^T \nabla u[L_k] \cdot \nabla v \, dt + \sum_{i=1}^N h_i \chi_i .$$
   (f) *Choose the step size* $\alpha_k$ *by using some conventional method.*
   (g) *Update the coefficient function:* $L_{k+1} = P_C\left(L_k - \alpha_k \widetilde{\nabla} J(L_k)\right)$.

We show a numerical example for our algorithm. Let $\Omega \subset \boldsymbol{R}^2$ be a unit disk. The coefficient $K$ is given by

$$K(x) = \begin{cases} 1.25 & (|x| < 0.15), \\ 1.0 & (|x| > 0.15) \end{cases}$$

as shown in Fig.1. Here $|\cdot|$ means the Euclidean norm on $\boldsymbol{R}^2$.

The constants in the constraint (1.1) are given by $C_1 = 0.90$ and $C_2 = 1.35$. The Neumann boundary value for this example are supposed to be given by

$$\overline{q}(t) = \begin{cases} -p(t) & \text{on } \partial\Omega_m \times (0, T], \\ 0.0 & \text{on } (\partial\Omega \setminus \overline{\partial\Omega_m}) \times (0, T], \end{cases}$$

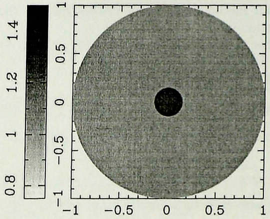


Figure 1: Exact coefficient

where

$$p(t) = \begin{cases} 0.25\sin(12.5\pi\, t) & (0 \leq t \leq 0.16)\ , \\ 0.0 & (t > 0.16)\ . \end{cases}$$

Here $\partial\Omega_m$ $(m = 1, 2, \ldots, 5)$ are set as

$$\partial\Omega_m = \left\{ (\cos\theta,\, \sin\theta) \,\middle|\, -\frac{\pi}{50} < \theta - (m-1)\frac{\pi}{4} < \frac{\pi}{50} \right\} .$$

The Dirichlet boundary value $\overline{u}$ is generated by solving numerically the wave equation with the exact coefficient $K$ and the Neumann boundary value $\overline{q}$. In order to solve this problem numerically, we make use of the Newmark method[2] for time integration with linear triangular finite elements in space. The measured value $\overline{u}$ is given by $\overline{u}(x,\, t) = u_{\rm cal}(x,\, t) + \delta(x,\, t)$, where $u_{\rm cal}$ means the calculated value on the circle and $\delta(x,\, t)$ is a random small valued function satisfied $|\delta(x,\, t)| < 10^{-10}|u_{\rm cal}|$ on the boundary $\partial\Omega$ for any $t > 0$. This treatment for the measured data is to avoid an inverse crime which the numerical errors may be cancelled out inadvertently if we use the data obtained by using the same finite element. The length of time is set as $T = 4.0$. The division $\{B_i\}$ is supposed to be given by

$$B_i = \{x \in \Omega \mid 0.1(i-1) \leq |x| < 0.1i\}$$

for $1 \leq i \leq 10$. We employ the Armijo criterion[1] in order to find the step size $\alpha_k$ in our algorithm.

We assume that $L_0(x) = K|_{\partial\Omega} = 1.0$ in the whole domain. After 100 times of iterations, we have the calculated coefficient as shown in Figure 2. Figure 3 shows the distribution of the relative error for calculated coefficient. The maximum value of the relative error is about 8.94%. These figures show that calculated coefficient is in good agreement with the exact one.

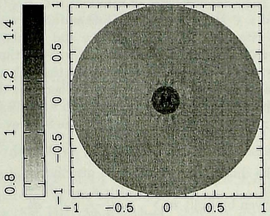


Figure 2: Calculated coefficient

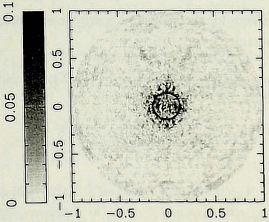


Figure 3: Relative error

# Appendix Proof for existence of the solutions to (1.2), (1.4) and (1.3)

In this Appendix, we show the existence of solutions $u \in \bar{H}_{(5)}((0,T);\bar{H}_{(1)}(\Omega))$, $v \in L^2((0,T);\bar{H}_{(1)}(\Omega))$ and $w \in \bar{H}_{(1)}(\Omega)$ to (1.2), (1.4) and (1.3), respectively.

To begin with we cite from [11] the existence and regularity theorem for the abstract hyperbolic evolution equation of second order in the time variable. Let $V, H$ be real Hilbert spaces and $V$ be separable. Suppose the embedding $i : V \hookrightarrow H$ is continuous, injective and its image is dense in $H$. Then, the dual $i' : H \hookrightarrow V'$ of $i$ is continuous, injective and has a dense range. Such a triple $(V, H, V')$ is called a Gelfand triple. For $T > 0$, $k \in \mathbb{Z}_+$ and $X = H$ or $V$, the Sobolev space $W_2^k((0,T);X)$ is the collection of measurable functions $\varphi : (0,T) \to X$ with $\dfrac{d^l\varphi}{dt^l} \in L^2((0,T);X)$ $(0 \le l \le k)$, where the differentiation is in the distributional sense. The norm of $\varphi \in W_2^k((0,T);X)$ is given by $||\varphi||_k^2 = \displaystyle\sum_{l=0}^{k} \int_0^T \left|\left|\frac{d^l\varphi}{dt^l}(t)\right|\right|_X^2 dt.$

Let $a_K(\varphi,\psi)$ $((\varphi,\psi) \in V)$ be a continuous, symmetric sesquilinear form satisfying the coercivity:

$$\text{there exist } k_0,\ \alpha > 0 \text{ with } a_K(\varphi,\varphi) + k_0||\varphi||_H^2 \ge \alpha||\varphi||_V^2 \quad (\varphi \in V). \tag{A.1}$$

Then, it is well known that there exists a unique $A_K \in L(V, V')$ (i.e. the set of all bounded linear operators from $V$ to $V'$) such that

$$a_K(\varphi,\psi) = (A_K\varphi,\psi)_H\,. \tag{A.2}$$

For the Cauchy problem for the abstract hyperbolic equation $\dfrac{d^2y}{dt^2} + A_K y = f$, we have the following theorem.

**Theorem A.1** *Let $y_0$, $y_1 \in V$ and $f \in W_2^{k-1}((0,T);H)$ with $k \in \mathbb{N}$ satisfy the compatibility condition of degree $k-1$. That is*

$$y_l \in V \quad (0 \le l \le k-1), \quad y_k \in H\,,$$

*where*

$$y_{2l-1} = f^{(2l-3)}(0) - A_K f^{(2l-5)}(0) + \cdots + (-1)^{l-2}A_K^{l-2}f'(0) + (-1)^{l-1}A_K^{l-1}y_1\,,$$
$$y_{2l} = f^{(2l-2)}(0) - A_K f^{(2l-4)}(0) + \cdots + (-1)^{l-1}A_K^{l-1}f(0) + (-1)^{l}A_K^{l}y_0\,.$$

*Then, the Cauchy problem*

$$\begin{cases} \dfrac{d^2y}{dt^2}(t) + A_K y(t) = f(t) \quad in \quad (0,T)\,, \\ y(0) = y_0, \quad y'(0) = y_1 \end{cases}$$

*admits a unique solution $y(t)$ such that*

$$y \in L^2((0,T);V), \quad \frac{dy}{dt} \in L^2((0,T);H)$$

*and they depend linearly and continuously on*

$$(f, y_0, y_1) \in L^2((0,T);H) \times V \times H\,.$$

*Moreover, $y$ has the regularity such that*

$$y \in W_2^{k-1}((0,T);V)\,, \quad y^{(k)} \in L^2((0,T);H)\,, \quad y^{(k+1)} \in L^2((0,T);V')\,,$$

*where* $y^{(k)} := \dfrac{d^k y}{dt^k}$.

For $\delta > 0$ small enough let $B_\delta := \{x \in \mathbb{R}^n;\ \mathrm{dist}\,(x, \partial\Omega) < \delta\}$. Let $(V_j, \Phi_j)$ $(1 \le j \le J)$ be patches of the manifold $B_\delta$ where the collection is an atlas of $B_\delta$. We can assume that $V_j \cap \partial\Omega \neq \emptyset$, $\Phi_j(V_j \cap \partial\Omega) \subset \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$, and $\Phi_j(V_j \cap \Omega) \subset \mathbb{R}^n_+$ for each $j$ $(1 \le j \le J)$. Let $\{\xi_j\}_{1 \le j \le J}$, $\{\eta_j\}_{1 \le j \le J} \subset C_0^\infty(\mathbb{R}^n)$ be partition of unities subordinated to $\{V_j\}_{1 \le j \le J}$. Now, we can construct an inverse trace operator $\Lambda : C^6([0,T]; H_{(\frac{5}{2})}(\partial\Omega)) \to C^6([0,T]; \bar{H}_{(3)}(\Omega))$ i.e. $(\Lambda\ell)|_{\partial\Omega \times [0,T]} = \ell \in C^6([0,T]; H_{(\frac{5}{2})}(\partial\Omega))$ in the following way.

For $\ell \in C^6([0,T]; H_{(\frac{5}{2})}(\partial\Omega))$, let $\ell_j := \xi_j \ell \in C^6([0,T]; H_{(\frac{5}{2})}(\partial\Omega))$, $m_j := \ell_j \circ (I \times (\Phi_j|_{\partial\Omega})^{-1}) \in C^6([0,T]; H_{(\frac{5}{2})}(\mathbb{R}^{n-1}))$, where $I : [0,T] \to [0,T]$ is the identity operator. We define an inverse trace operator $\Lambda_0 : C^6([0,T]; H_{(\frac{5}{2})}(\mathbb{R}^{n-1})) \to C^6([0,T]; H_{(3)}(\mathbb{R}^n_+))$ by

$$(\Lambda_0 m)(t,x) = \frac{1}{(2\pi)^{n-1} d} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{(1+|\xi'|^2)^{\frac{5}{4}}}{(1+|\xi|^2)^3} \hat{m}(t,\xi') d\xi$$

for $m \in C^6([0,T]; H_{(\frac{5}{2})}(\mathbb{R}^{n-1}))$, where $\xi' = (\xi_1, \cdots, \xi_{n-1})$ for $\xi = (\xi_1, \cdots, \xi_n)$, $d := \int_{-\infty}^{\infty} (1+\tau^2)^{-3} d\tau$, and $\hat{m}(t,\xi') := \int_{\mathbb{R}^{n-1}} e^{-ix'\cdot\xi'} m(t,x') dx'$. Then, $\Lambda$ can be given by

$$\Lambda\ell := \sum_{j=1}^{J} (\eta_j((\Lambda_0 m_j) \circ (I \times \Phi_j)))\Bigg|_{\overline{\Omega} \times [0,T]} \tag{A.3}$$

for any $\ell \in C^6([0,T]; H_{(\frac{5}{2})}(\partial\Omega))$. (See [8] for the details.)

We first prove the existence of the solution $u \in \bar{H}_{(5)}((0,T); \bar{H}_{(1)}(\Omega))$ to (1.2) with $u^{(6)} \in L^2((0,T); L^2(\Omega))$ and $u^{(7)} \in L^2((0,T); (\dot{H}_{(1)}(\Omega))')$. Let $\widetilde{u} := u - \phi$ with $\phi := \Lambda\overline{u}$. Then, $\widetilde{u}$ has to satisfy

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}\widetilde{u} - \nabla \cdot (K \nabla \widetilde{u}) = f\,, \\ \widetilde{u}(0) = \widetilde{u}_0\,, \quad \dfrac{\partial}{\partial t}\widetilde{u}(0) = \widetilde{u}_1 \end{cases} \tag{A.4}$$

with

$$\begin{cases} f = f(K) := \nabla \cdot (K\nabla\phi) - \dfrac{\partial^2}{\partial t^2}\phi \in C^4([0,T]; \bar{H}_{(1)}(\Omega)), \\ \widetilde{u}_0 := -\phi|_{t=0}, \quad \widetilde{u}_1 := -\left.\dfrac{\partial \phi}{\partial t}\right|_{t=0} \in V := \dot{H}_{(1)}(\Omega). \end{cases} \tag{A.5}$$

Now $(V, H := L^2(\Omega), V')$ is clearly a Gelfand triple and $a_K(\psi,\omega) := \int_\Omega K\nabla\psi \cdot \nabla\omega\, dx$ $(\psi,\omega \in V)$ is a continuous sesquilinear form satisfying the coercivity condition with $k_0 = 0$. Moreover, it is easy to see that $f$, $\widetilde{u}_0$, and $\widetilde{u}_1$ satisfy the compatibility condition of degree 5. Then, the existence of $u$ to (1.2) with the desired properties immediately follows by applying Theorem A.1 to (A.4).

By observing that $\widetilde{u} \in \bar{H}_{(5)}((0,T); \dot{H}_{(1)}(\Omega))$ satisfies

$$\nabla \cdot (K\nabla\widetilde{u}) = \frac{\partial^2 \widetilde{u}}{\partial t^2} - f \in \bar{H}_{(3)}((0,T); \dot{H}_{(1)}(\Omega)) \subset C^2((0,T); \dot{H}_{(1)}(\Omega)),$$

we have $\widetilde{u} \in C^2([0,T]; \bar{H}_{(3)}(\Omega \setminus F))$ and hence $u \in C^2([0,T]; \bar{H}_{(3)}(\Omega \setminus F))$ by the regularity theorem near the boundary of solutions to the Dirichlet boundary value problem for strongly elliptic equations (See [5], Chapter 3, Proposition 3.7). This implies that $2(L\frac{\partial u}{\partial n}(T) - \overline{q}(T)) \in H_{(\frac{3}{2})}(\partial\Omega)$ and

$$2(L\frac{\partial u}{\partial n} - \overline{q}) \in C^2([0,T]; H_{(\frac{3}{2})}(\partial\Omega)). \tag{A.6}$$

By the well-posedness of (2.2), we immediately have $w \in \bar{H}_{(1)}(\Omega)$.

For the existence of $v \in L^2((0,T); \bar{H}_{(1)}(\Omega))$, we argue likewise we did for the solution $u$ to (1.2) using the inverse trace operator transforming (2.3) to an initial boundary value problem with the corresponding Dirichlet boundary condition. Then, by (A.6), the second term of equation of this initial boundary value problem belongs to $L^2((0,T); H)$ with $H = L^2(\Omega)$. Therefore, by Theorem A.1, we have the existence of $v \in L^2((0,T); \bar{H}_{(1)}(\Omega))$.

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