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Accuracy on eigenvalues for a Schrödinger operator with a degenerate potential in the semi-classical limit

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ABSTRACT

We consider a semi-classical Schrödinger operator $-h^2\Delta + V$ with a degenerate potential $V(x,y) = f(x)g(y)$.
g is assumed to be a homogeneous positive function of m variables and f is a strictly positive function of n variables, with a strict minimum. We give sharp asymptotic behaviour of low eigenvalues bounded by some power of the parameter h, by improving Born-Oppenheimer approximation.

RESUMEN

Consideramos un operador de Schrödinger semi-clásico $-h^2\Delta + V$ con potencial degenerado $V(x,y) = f(x)g(y)$.
Suponemos que g es una función positiva homogénea de m variables y f es una función estrictamente positiva de n variables, con un mínimo estricto. Damos un comportamiento asintótico óptimo de autovalores acotados por abajo para alguna potencia del parámetro h, mediante perfeccionamiento de la aproximación de Born - Oppenheimer.

Key words and phrases: *eigenvalues, semi-classical asymptotics, Born-Oppenheimer approximation.*
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1 Introduction

In our paper [11] we have considered the Schrödinger operator on $L^2(\mathbb{R}^n_x \times \mathbb{R}^m_y)$

$$\widehat{H}_h = h^2D_x^2 + h^2D_y^2 + f(x)g(y) \tag{1.1}$$

with $g \in C^\infty(\mathbb{R}^m \setminus \{0\})$ homogeneous of degree $a > 0$,

$$g(\mu y) = \mu^a g(y) > 0 , \quad \forall \mu > 0 \text{ and } \forall y \in \mathbb{R}^m \setminus \{0\} . \tag{1.2}$$

$h > 0$ is a semiclassical parameter we assume to be small.

We have investigated the asymptotic behavior of the number of eigenvalues less than λ of $\widehat{H}_h$,

$$N(\lambda, \widehat{H}_h)tr(\chi_{]-\infty,\lambda[}(\widehat{H}_h) = \sum_{\lambda_k(\widehat{H}_h)<\lambda} 1 . \tag{1.3}$$

($tr(P)$ denotes the trace of the operator P).
If P is a self-adjoint operator on a Hilbert space $\mathcal{H}$, we denote respectively by $sp(P)$, $sp_{ess}(P)$ and $sp_d(P)$ the spectrum, the essential spectrum and the discret spectrum of P .

When $-\infty < \inf sp(P) < \inf sp_{ess}(P)$, we denote by $(\lambda_k(P))_{k>0}$ the increasing sequence of eigenvalues of P , repeated according to their multiplicity:

$$sp_d(P) \bigcap]-\infty, \inf sp_{ess}(P)[= \{\lambda_k(P)\} . \tag{1.4}$$

In this paper we are interested in a sharp estimate for some eigenvalues of $\widehat{H}_h$. We make the following assumptions on the other multiplicative part of the potential:

$$\begin{array}{l} f \in C^\infty(\mathbb{R}^n),\ \forall \alpha \in \mathbb{N}^n,\ (|f(x)|+1)^{-1}\partial_x^\alpha f(x) \in L^\infty(\mathbb{R}^n) \\ 0 < f(0) = \inf_{x\in\mathbb{R}^n} f(x) \\ f(0) < \liminf_{|x|\to\infty} f(x) = f(\infty) \\ \partial^2 f(0) > 0 \end{array} \tag{1.5}$$

$\partial^2 f(a)$ denotes the hessian matrix:

$$\partial^2 f(a) = \left(\frac{\partial^2}{\partial x_i \partial x_j} f(a)\right)_{1\le i,j\le n} .$$

By dividing $\widehat{H}_h$ by $f(0)$, we can change the parameter h and assume that

$$f(0) = 1 . \tag{1.6}$$

Let us define : $\hbar = h^{2/(2+a)}$ and change y in $y\hbar$; we can use the homogeneity of g (1.2) to get :

$$sp\,(\widehat{H}_h) \;=\; \hbar^a\; sp\,(\widehat{H}^\hbar)\,, \tag{1.7}$$

with $\widehat{H}^\hbar \;=\; \hbar^2 D_x^2 \;+\; D_y^2 \;+\; f(x)g(y) \;=\; \hbar^2 D_x^2 \;+\; Q(x,y,D_y)$:

$$Q(x,y,D_y) \;=\; D_y^2 \;+\; f(x)g(y)\,.$$

Let us denote the increasing sequence of eigenvalues of $D_y^2 \,+\, g(y)$, (on $L^2(\mathbb{R}^m)$) , by $(\mu_j)_{j>0}$.
The associated eigenfunctions will be denoted by $(\varphi_j)_j$:

$$\begin{aligned} D_y^2\varphi_j(y) \;+\; g(y)\varphi_j(y) \;&=\; \mu_j\varphi_j(y) \\ \langle \varphi_j \mid \varphi_k\rangle \;&=\; \delta_{jk} \end{aligned} \tag{1.8}$$

and $(\varphi_j)_j$ is a Hilbert base of $L^2(\mathbb{R}^m)$.

By homogeneity (1.2) the eigenvalues of $Q_x(y,D_y) \;=\; D_y^2 \;+\; f(x)g(y)$, on $L^2(\mathbb{R}^m)$) , for a fixed x, are given by the sequence $(\lambda_j(x))_{j>0}$, where : $\lambda_j(x) = \mu_j\; f^{2/(2+a)}(x)$.
So as in [11] we get :

$$\widehat{H}^\hbar \;\geq\; \Big[\; \hbar^2 D_x^2 \;+\; \mu_1 f^{2/(2+a)}(x) \Big]\,. \tag{1.9}$$

This estimate is sharp as we will see below.

Then using the same kind of estimate as (1.9), one can see that

$$\inf\; sp_{ess}(\widehat{H}^\hbar) \;\geq\; \mu_1 f^{2/(2+a)}(\infty)\,. \tag{1.10}$$

We are in the Born-Oppenheimer approximation situation described by A. Martinez in [10] : the "effective " potential is given by $\lambda_1(x) = \mu_1\; f^{2/(2+a)}(x)$, the first eigenvalue of Q_x, and the assumptions on f ensure that this potential admits one unique and nondegenerate well $U = \{0\}$, with minimal value equal to μ_1. Hence we can apply theorem 4.1 of [10] and get :

Theorem 1.1 *Under the above assumptions, for any arbitrary $C > 0$, there exists $h_0 > 0$ such that, if $0 < \hbar < h_0$, the operator $(\widehat{H}^\hbar)$ admits a finite number of eigenvalues $E_k(\hbar)$ in $[\mu_1, \mu_1 + C\hbar]$, equal to the number of the eigenvalues e_k of $D_x^2 + \frac{\mu_1}{2+a} < \partial^2 f(0)\, x,\, x >$ in $[0, +C]$ such that :*

$$E_k(\hbar) = \lambda_k(\widehat{H}^\hbar) \;=\; \lambda_k\left(\hbar^2 D_x^2 + \mu_1 f^{2/(2+a)}(x)\right) \;+\; \mathbf{O}(\hbar^2)\,. \tag{1.11}$$

More precisely $E_k(\hbar) = \;\lambda_k(\widehat{H}^\hbar)$ has an asymptotic expansion

$$E_k(\hbar) \;\sim\; \mu_1 \;+\; \hbar\;(\; e_k \;+\; \sum_{j\geq 1} \alpha_{kj}\hbar^{j/2}\;). \tag{1.12}$$

If $E_k(\hbar)$ is asymptotically non degenerated, then there exists a quasimode

$$\phi_k^\hbar(x,y) \sim \hbar^{-m_k} e^{-\psi(x)/\hbar} \sum_{j\geq 0} \hbar^{j/2} a_{kj}(x,y) , \tag{1.13}$$

satisfying

$$\begin{array}{l} C_0^{-1} \leq \|\hbar^{-m_k} e^{-\psi(x)/\hbar} a_{k0}(x,y)\| \leq C_0 \\ \|\hbar^{-m_k} e^{-\psi(x)/\hbar} a_{kj}(x,y)\| \leq C_j \\ \| \left(\widehat{H}^\hbar - \mu_1 - \hbar e_k - \sum_{1\leq j\leq J} \alpha_{kj} \hbar^{j/2}\right) \\ \hbar^{-m_k} e^{-\psi(x)/\hbar} \sum_{0\leq j\leq J} \hbar^{j/2} a_{kj}(x,y)\| \leq C_J \hbar^{(J+1)/2} \end{array} \tag{1.14}$$

The formula (1.12) implies

$$\lambda_k(\widehat{H}^\hbar) = \mu_1 + \hbar\lambda_k \left(D_x^2 + \frac{\mu_1}{2+a} < \partial^2 f(0)\, x \,,\, x > \right) + \mathbf{O}(\hbar^{3/2}) , \tag{1.15}$$

and when $k = 1$, one can improve $\mathbf{O}(\hbar^{3/2})$ into $\mathbf{O}(\hbar^2)$. The function ψ is defined by : $\psi(x) = d(x,0)$, where d denotes the Agmon distance related to the degenerate metric $\mu_1 \ f^{2/(2+a)}(x)dx^2$.

2 Lower energies

We are interested now with the lower energies of $\widehat{H}^\hbar$. Let us make the change of variables

$$(x,\ y) \rightarrow (x,\ f^{1/(2+a)}(x)y) . \tag{2.1}$$

The Jacobian of this diffeomorphism is $f^{m/(2+a)}(x)$, so we perform the change of test functions : $u \rightarrow f^{-m/(4+2a)}(x)u$, to get a unitary transformation.

Thus we get that

$$sp\,(\widehat{H}^\hbar) = sp\,(\widetilde{H}^\hbar) \tag{2.2}$$

where $\widetilde{H}^\hbar$ is the self-adjoint operator on $L^2(\mathbb{R}^n \times \mathbb{R}^m)$ given by

$$\widetilde{H}^\hbar = \hbar^2 L^\star(x,y,D_x,D_y) L(x,y,D_x,D_y) + f^{2/(2+a)}(x) \left(D_y^2 + g(y)\right) , \tag{2.3}$$

with

$$L(x,y,D_x,D_y) = D_x + \frac{1}{(2+a)f(x)}[(yD_y) - i\frac{m}{2}]\nabla f(x) .$$

We decompose $\widetilde{H}^\hbar$ in four parts :

$$\begin{array}{l} \widetilde{H}^\hbar = \hbar^2 D_x^2 + f^{2/(2+a)}(x) \left(D_y^2 + g(y)\right) \\ +\hbar^2 \frac{2}{(2+a)f(x)} (\nabla f(x) D_x)(yD_y) \\ +i\hbar^2 \frac{1}{(2+a)f^2(x)} \left(|\nabla f(x)|^2 - f(x)\Delta f(x)\right) [(yD_y) - i\frac{m}{2}] \\ + \hbar^2 \frac{1}{(2+a)^2 f^2(x)} |\nabla f(x)|^2 [(yD_y)^2 + \frac{m^2}{4}] \end{array} \tag{2.4}$$

Our goal is to prove that the only significant role up to order 2 in $\hbar$ will be played by the first operator, namely : $\widetilde{H}_1^\hbar = \hbar^2 D_x^2 + f^{2/(2+a)}(x)\left(D_y^2 + g(y)\right)$.

Let us denote by $\nu_{j,k}^\hbar$ the eigenvalues of the operator $\hbar^2 D_x^2 + \mu_j f^{2/(2+a)}(x)$ and by $\psi_{j,k}^\hbar$ the associated normalized eigenfunctions .

Let us consider the following test functions :

$$u_{j,k}^\hbar(x,y) = \psi_{j,k}^\hbar(x)\varphi_j(y) ,$$

where the φ_j's are the eigenfunctions defined in (1.8); we have immediately :

$$\widetilde{H}_1^\hbar(u_{j,k}^\hbar(x,y)) = \nu_{j,k}^\hbar u_{j,k}^\hbar(x,y) .$$

We will need the following lemma :

Lemma 2.1 . *For any integer* N , *there exists a positive constant* C *depending only on* N *such that for any* $k \leq N$, *the eigenfunction* $\psi_{j,k}^\hbar$ *satisfies the following inequalities : for any* $\alpha \in \mathbb{N}^n$, $|\alpha| \leq 2$,

$$\begin{array}{rcl} \| \hbar_j^{|\alpha|/2} |D_x^\alpha \psi_{j,k}^\hbar| \| & < & C \\ \| \left(\frac{\nabla f(x)}{f(x)}\right)^\alpha \psi_{j,k}^\hbar \| & < & \hbar_j^{|\alpha|/2} C \end{array} \tag{2.5}$$

with $\hbar_j = \hbar \mu_j^{-1/2}$.

Proof.

Let us recall that it is well known, (see [5]), that

$$\forall\, k \leq N , \quad \mu_j^{-1}\nu_{j,k}^\hbar = 1 + \mathrm{O}(\hbar_j) .$$

It is clear also that

$$\left[\hbar_j^2 D_x^2 + f^{2/(2+a)}(x) - \mu_j^{-1}\nu_{j,k}^\hbar\right]\psi_{j,k}^\hbar(x) = 0 . \tag{2.6}$$

We shall need the following inequality, that we can derive easily from (2.6) and the Agmon estimate (see [5]) : $\forall\, \varepsilon \in]0,1[$,

$$\begin{array}{l} \varepsilon \int \left[f^{2/(2+a)}(x) - \mu_j^{-1}\nu_{j,k}^\hbar\right]_+ e^{2(1-\varepsilon)^{1/2} d_{j,k}(x)/\hbar_j} |\psi_{j,k}^\hbar(x)|^2\, dx \leq \\ \int \left[f^{2/(2+a)}(x) - \mu_j^{-1}\nu_{j,k}^\hbar\right]_- |\psi_{j,k}^\hbar(x)|^2\, dx , \end{array} \tag{2.7}$$

where $d_{j,k}$ is the Agmon distance associated to the metric $\left[f^{2/(2+a)}(x) - \mu_j^{-1}\nu_{j,k}^\hbar\right]_+ dx^2$.

Let us prove the lemma for $|\alpha| = 1$.

As $\int \left[\hbar_j^2 |D_x \psi_{j,k}^\hbar(x)|^2 + (f^{2/(2+a)}(x) - \mu_j^{-1}\nu_{j,k}^\hbar)|\psi_{j,k}^\hbar(x)|^2\right] dx = 0$,

$\mu_j^{-1}\nu_{j,k}^\hbar - 1 = \mathrm{O}(\hbar_j)$, and $f^{2/(2+a)}(x) - 1 > 0$,

we get that $\hbar_j \| \, |D_x \psi_{j,k}^\hbar(x)| \, \|^2 \leq C$.

Furthermore, we use that $C^{-1}|\nabla f(x)|^2 \le f^{2/(2+a)}(x) - 1 \le C|\nabla f(x)|^2$, for $|x| \le C^{-1}$, the exponential decreasing (in $\hbar_j$) of $\psi^{\hbar}_{j,k}$ given by (2.7) and the boundness of $|\nabla f(x)|/f(x)$ to get

$$\|\frac{|\nabla f(x)|}{f(x)}\psi^{\hbar}_{j,k}(x)\|^2 \le C\int [f^{2/(2+a)}(x)-1]\,|\psi^{\hbar}_{j,k}(x)|^2\,dx \le \hbar_j C\,.$$

Now we study the case $|\alpha| = 2$.
If $c_0 \in]0,1]$ is large enough and $|x| \in [\hbar_j^{1/2}c_0,\ 2c_0]$, then we have

$$|x|^2/C \le f^{2/(2+a)}(x) - \mu_j^{-1}\nu^{\hbar}_{j,k} \le C|x|^2 \tag{2.8}$$

Therefore there exists $C_1 > 1$ such that $C_1^{-1}|x|^2 \le d_{j,k}(x) \le C_1|x|^2$, and then

$$|x|^2 \le \hbar_j C e^{d_{j,k}(x)/\hbar_j}\,. \tag{2.9}$$

Then the inequality: $C^{-1}|x| \le |\nabla f(x)| \le C|x|$, with (2.8), (2.9) and (2.7) entail that

$$\begin{aligned}\int_{|x|\ge C_0\hbar_j^{1/2}} \frac{|\nabla f(x)|^4}{f^4(x)}|\psi^{\hbar}_{j,k}(x)|^2dx \\ &\le \hbar_j C\int \left[f^{2/(2+a)}(x)-\mu_j^{-1}\nu^{\hbar}_{j,k}\right]_+ e^{d_{j,k}(x)/\hbar_j}|\psi^{\hbar}_{j,k}(x)|^2dx\\ &\le \hbar_j C\int \left[f^{2/(2+a)}(x)-\mu_j^{-1}\nu^{\hbar}_{j,k}\right]_- |\psi^{\hbar}_{j,k}(x)|^2\,dx\\ &\le \hbar_j^2 C\,.\end{aligned}$$

It remains to estimate $\hbar_j^2\|D_x^\alpha\psi^{\hbar}_{j,k}(x)\|$ with $|\alpha| = 2$.
We use that $-\hbar_j^2\Delta\psi^{\hbar}_{j,k}(x) = [-f^{2/(2+a)}(x)+\mu_j^{-1}\nu^{\hbar}_{j,k}]\psi^{\hbar}_{j,k}(x)$,
and that we have proved that $\|[-f^{2/(2+a)}(x)+\mu_j^{-1}\nu^{\hbar}_{j,k}]\psi^{\hbar}_{j,k}(x)\| \le \hbar_j C$;
so $\|D_x^\alpha\psi^{\hbar}_{j,k}(x)\| \le C/\hbar_j$ if $|\alpha| = 2$.
We will need the following result.

Proposition 2.2 *Let $V(y) \in C^\infty(\mathbb{R}^m)$ such that*

$$\begin{aligned}&\exists\, s>0,\ C_0>0 \text{ s.t. } \ -C_0+|y|^s/C_0 \le V(y) \le C_0(|y|^s+1)\\ &\forall\,\alpha\in\mathbb{N}^m,\ (1+|y|^2)^{(s-|\alpha|)/2}\partial_y^\alpha V(y) \in L^\infty(\mathbb{R}^m)\,.\end{aligned} \tag{2.10}$$

If $u(y) \in L^2(\mathbb{R}^m)$ and $D_y^2u(y) + V(y)u(y) \in S(\mathbb{R}^m)$, then $u \in S(\mathbb{R}^m)$. ($S(\mathbb{R}^m)$ is the Schwartz space).

The proof comes from the fact that there exists a parametrix of $D_y^2 + V(y)$ in some class of pseudodifferential operator: see for the more general case in [7], or for this special case in Shubin book [17].

Theorem 2.3 .
Under the assumptions (1.2) and (1.5), for any fixed integer $N > 0$, *there exists a positive constant* $h_0(N)$ *verifying : for any* $\hbar \in]0, h_0(N)[$, *for any* $k \leq N$ *and any* $j \leq N$ *such that*

$$\mu_j < \mu_1 f^{2/(2+a)}(\infty) ,$$

there exists an eigenvalue $\lambda_{jk} \in sp_d(\widehat{H}^\hbar)$ *such that*

$$| \lambda_{jk} - \lambda_k \left(\hbar^2 D_x^2 + \mu_j f^{2/(2+a)}(x) \right) | \leq \hbar^2 C . \tag{2.11}$$

Consequently, when $k = 1$, *we have*

$$| \lambda_{j1} - \left[\mu_j + \hbar(\mu_j)^{1/2} \frac{tr((\partial^2 f(0))^{1/2})}{(2+a)^{1/2}} \right] | \leq \hbar^2 C . \tag{2.12}$$

Proof.
The first part of the theorem will follow if we prove that :

$$\|(\widehat{H}^\hbar - \widetilde{H}_1^\hbar)(u_{j,k}^\hbar(x,y)) \| = \|(\widehat{H}^\hbar - \nu_{j,k}^\hbar) u_{j,k}^\hbar(x,y)\| = \mathbf{O}(\hbar^2) .$$

Let us consider a function $\chi \in C^\infty(\mathbb{R})$ such that
$\chi(t) = 1$ if $|t| \leq 1/2$ and
$\chi(t) = 0$ if $|t| > 1$.
Then $(D_y^2 + g(y))(1 - \chi(|y|))\varphi_j(y) \in S(\mathbb{R}^m)$,
and Proposition 2.2 shows that $(1 - \chi(|y|))\varphi_j(y) \in S(\mathbb{R}^m)$.
As $D_y^2 \varphi_j(y) = (\mu_j - g(y))\varphi_j(y)$, we get that

$$\forall\, k \in \mathbb{N}, \quad (1 + |y|)^k [|\varphi_j(y)|^2 + |D_y \varphi_j(y)|^2 + |D_y^2 \varphi_j(y)|^2] \in L^1(\mathbb{R}^m) . \tag{2.13}$$

The quantity $(\widehat{H}^\hbar - \widetilde{H}_1^\hbar)(u_{j,k}^\hbar(x,y))$ is, by (2.4) , composed of 3 parts. According to Lemma 2.1 and the estimate (2.13), the two last parts are bounded in L^2-norm by $\hbar^2 C$, $(\mu_j \leq C)$.
To obtain a bound for the first part, we integrate by parts to get that

$$\|\frac{\nabla f(x)}{f(x)} D_x \psi_{j,k}^\hbar\|^2 \leq C \left[\|D_x^2 \psi_{j,k}^\hbar\| \times \|\frac{|\nabla f(x)|^2}{f^2(x)} \psi_{j,k}^\hbar\| + \|D_x \psi_{j,k}^\hbar\| \times \|\frac{|\nabla f(x)|}{f(x)} \psi_{j,k}^\hbar\| \right] ,$$

and then we use again Lemma 2.1. Thus : $\|\frac{\nabla f(x)}{f(x)} D_x \psi_{j,k}^\hbar\| \leq C$.

According to estimate (2.13) we have finally $\|\frac{\nabla f(x)}{f(x)} D_x (y D_y) u_{j,k}^\hbar\| \leq C.$

3 Middle energies

We are going to refine the preceding results when $a \geq 2$ and $f(\infty) = \infty$. It is possible then to get sharp localization near the μ_j's for much higher values of j's. More precisely we prove :

Theorem 3.1 . *We assume (1.5) with* $f(\infty) = \infty$, *(1.2) with* $a \geq 2$ *and with* $g \in C^\infty(\mathbb{R}^m)$.

Let us consider j *such that* $\mu_j \leq \hbar^{-2}$;
then for any integer N , *there exists a constant* C *depending only on* N *such that, for any* $k \leq N$, *there exists an eigenvalue* $\lambda_{jk} \in sp_d\,(\widehat{H}^\hbar)$ *verifying*

$$| \lambda_{jk} - \lambda_k \left(\hbar^2 D_x^2 + \mu_j f^{2/(2+a)}(x) \right) | \leq C\mu_j \hbar^2 . \tag{3.1}$$

Consequently, when $k = 1$, *we have*

$$| \lambda_{j1} - \left[\mu_j + \hbar(\mu_j)^{1/2} \frac{tr((\partial^2 f(0))^{1/2})}{(2+a)^{1/2}} \right] | \leq C\mu_j \hbar^2 . \tag{3.2}$$

Proof :
Let us define the class of symbols $S(p^s(y,\eta))$, $s \in \mathbb{R}$, with $p(y,\eta) = |\eta|^2 + g(y) + 1$.

$$q(y,\eta) \in S(p^s(y,\eta)) \quad \text{iff} \quad q(y,\eta) \in C^\infty(\mathbb{R}^m \times \mathbb{R}^m)$$

and for any α and $\beta \in \mathbb{N}^m$,

$$p^{-s}(y,\eta)(|\eta|+1)^{-|\alpha|}(|y|+1)^{-|\beta|} D_\eta^\alpha D_y^\beta q(y,\eta) \in L^\infty(\mathbb{R}^{2m}) .$$

For such a symbol $q(y,\eta) \in S(p^s(y,\eta))$, we define the operator Q on $S(\mathbb{R}^m)$:

$$Qf(y) = (2\pi)^{-m} \int_{\mathbb{R}^{2m}} q(\frac{y+z}{2}, \eta) e^{i(y-z)\eta} f(z) dz d\eta .$$

We will say that $Q \in OPS(p^s(y,\eta))$.
It is well known, (see [7]) that $(D_y^2 + g(y))^s \in OPS(p^s(y,\eta))$.
As $a \geq 2$, we get that $yD_y \in OPS(p(y,\eta))$, and then that $yD_y(D_y^2 + g(y))^{-1} \in OPS(1)$.
Therefore $yD_y(D_y^2 + g(y))^{-1}$ and $(yD_y)^2(D_y^2 + g(y))^{-2}$ are bounded operator on $L^2(\mathbb{R}^m)$, and we get as a consequence the following bound :

$$\mu_j^{-1}\|yD_y\varphi_j\| + \mu_j^{-2}\|(yD_y)^2\varphi_j\| \leq C . \tag{3.3}$$

As in the proof of Theorem 2.3, using (3.3) instead of (2.13), we get easily that

$$\|(\widehat{H}^\hbar - \widetilde{H}^\hbar)u_{j,k}^\hbar\| \leq C[\,\hbar^2\mu_j + \hbar^3\mu_j^{3/2}] \leq C\hbar^2\mu_j ,$$

and then Theorem 3.1 follows.

4 An application

We consider a Schrödinger operator on $L^2(\mathbb{R}_z^d)$ with $d \geq 2$,

$$P^h = -h^2\Delta + V(z) \tag{4.1}$$

with a real and regular potential $V(z)$ satisfying

$$\begin{array}{c} V \in C^\infty(\mathbb{R}^d\,;\,[0,+\infty[) \\ \liminf_{|z|\to\infty} V(z) > 0 \\ \Gamma = V^{-1}(\{0\}) \quad \text{is a regular hypersurface.} \end{array} \tag{4.2}$$

By hypersurface, we mean a submanifold of codimension 1. Moreover we assume that Γ is connected and that there exist $m \in \mathbb{N}^\star$ and $C_0 > 0$ such that for any z verifying $d(z,\,\Gamma) < C_0^{-1}$

$$C_0^{-1} d^{2m}(z,\,\Gamma) \leq V(z) \leq C_0\, d^{2m}(z,\,\Gamma) \tag{4.3}$$

($d(E,F)$ denotes the euclidian distance between E and F).

We choose an orientation on Γ and a unit normal vector $N(s)$ on each $s \in \Gamma$, and then, we can define the function on Γ,

$$f(s) = \frac{1}{(2m)!}\left(N(s)\frac{\partial}{\partial s}\right)^{2m} V(s)\,, \quad \forall\, s \in \Gamma\,. \tag{4.4}$$

Then by (4.2) and (4.6), $f(s) > 0\,, \quad \forall\, s \in \Gamma$.

Finally we assume that the function f achieves its minimum on Γ on a finite number of discrete points:

$$\Sigma_0 = f^{-1}(\{\eta_0\}) = \{s_1,\ldots,\,s_{\ell_0}\}\,, \quad if \quad \eta_0 = \min_{s\in\Gamma} f(s)\,, \tag{4.5}$$

and the hessian of f at each point $s_j \in \Sigma_0$ is non degenerated:
$\exists\, \eta_1 > 0 \;\; s.t.$

$$\frac{1}{2}\langle d\left(\langle df\,;\,w\rangle\right)\,;\,w\rangle(s_j) \geq \eta_1 |w(s_j)|^2\,, \quad \forall\, w \in T\Gamma\,, \forall\, s_j \in \Sigma_0\,. \tag{4.6}$$

If $g = (g_{ij})$ is the riemannian metric on Γ, then $|w(s)| = (g(w(s),w(s)))^{1/2}$. The hessian of f at each $s_j \in \Sigma_0$, is the symmetric operator on $T_{s_j}\Gamma$, $Hess(f)_{s_j}$, associated to the two-bilinear form defined on $T_{s_j}\Gamma$ by :

$$(v,w) \in (T_{s_j}\Gamma)^2 \to \frac{1}{2}\langle d\left(\langle df\,;\,\widetilde{v}\rangle\right)\,;\,\widetilde{w}\rangle(s_j)\,, \tag{4.7}$$

$\forall\,(\widetilde{v},\widetilde{w}) \in (T\Gamma)^2 \;\; s.t. \;\; (\widetilde{v}(s_j),\widetilde{w}(s_j)) = (v,w)$.

$Hess(f)_{s_j}$ has $d-1$ non negative eigenvalues

$$\rho_1^2(s_j) \leq \ldots \leq \rho_{d-1}^2(s_j)\,, \qquad (\rho_j(s_j) > 0)\,.$$

In local coordinates, those eigenvalues are the ones of the symmetric matrix

$$\frac{1}{2}G^{1/2}(s_j)\left(\frac{\partial^2}{\partial x_k \partial x_\ell} f(s_j)\right)_{1\leq k,\ell\leq d-1} G^{1/2}(s_j)\,, \quad (\,G(x) = (g_{k,\ell}(x))_{1\leq k,\ell\leq d-1}\,)\,.$$

The eigenvalues $\rho_k^2(s_j)$ do not depend on the choice of coordinates. We denote

$$Tr^+(Hess(f(s_j))) = \sum_{\ell=1}^{d-1} \rho_\ell(s_j)\,. \tag{4.8}$$

We denote by $(\mu_j)_{j\geq 1}$ the increasing sequence of the eigenvalues of the operator $-\frac{d^2}{dt^2} + t^{2m}$ on $L^2(\mathbb{R})$,
and by $(\varphi_j(t))_{j\geq 1}$ the associated orthonormal Hilbert base of eigenfunctions.

Theorem 4.1 *Under the above assumptions, for any* $N \in \mathbb{N}^\star$ *, there exist* $h_0 \in]0,1]$ *and* $C_0 > 0$ *such that, if* $\mu_j << h^{-4m/(m+1)(2m+3)}$ *,*
and if $\alpha \in \mathbb{N}^{d-1}$ *and* $|\alpha| \leq N$ *,*
then $\forall\, s_\ell \in \Sigma_0$ *,* $\exists\, \lambda_{j\ell\alpha}^h \in sp_d(P^h)$ *s.t.*

$$\left| \lambda_{j\ell\alpha}^h - h^{2m/(m+1)} \left[\eta_0^{1/(m+1)} \mu_j + h^{1/(m+1)} \mu_j^{1/2}\, \mathcal{A}_\ell(\alpha)\right] \right|$$

$$\leq h^2 \mu_j^{2+3/2m} C_0\,;$$

with $\mathcal{A}_\ell(\alpha) = \dfrac{1}{\eta_0^{m/(2m+2)}(m+1)^{1/2}} \left[2\alpha\rho(s_\ell) + Tr^+(Hess(f(s_\ell)))\right]$.
$(\alpha\rho(s_\ell) = \alpha_1\rho_1(s_\ell) + \ldots \alpha_{d-1}\rho_{d-1}(s_\ell)\,)$.

Proof:

Let $\mathcal{O}_0 \subset \mathbb{R}^d$ be an open neighbourhood of $s_l \in \Sigma_0$, such that there exists $\phi \in C^\infty(\mathcal{O}_0\,;\,\mathbb{R})$ satisfying

$$\begin{aligned} \Gamma_0 &= \Gamma \cap \mathcal{O}_0 = \{z \in \mathcal{O}_0\,;\,\phi(z) = 0\}\,; \\ |\nabla\phi(z)| &= 1\,,\quad \forall\, z \in \mathcal{O}_0\,. \end{aligned} \tag{4.9}$$

After changing $\mathcal{O}_0$ into a smaller neighbourhood if necessary, we can find $\tau \in C^\infty(\mathcal{O}_0\,;\,\mathbb{R}^{d-1})$ such that $\tau(s_l) = 0$ and $\forall\, z \in \mathcal{O}_0$,

$$\begin{aligned} &\nabla\tau_j(z).\nabla\phi(z) = 0\,,\quad \forall j = 1,\ldots,d-1 \\ &\mathrm{rank}\{\nabla\tau_1(z),\ldots,\nabla\tau_{d-1}(z)\} = d-1\,. \end{aligned} \tag{4.10}$$

Then $(x,y) = (x_1,\ldots,x_{d-1},y) = (\tau_1,\ldots,\tau_{d-1},\phi)$ are local coordinates in $\mathcal{O}_0$ such that

$$\begin{aligned} \Delta &= |\widetilde{g}|^{-1/2} \textstyle\sum_{1\leq i,j\leq d-1} \partial_{x_i}\left(|\widetilde{g}|^{1/2}\, \widetilde{g}^{ij}\partial_{x_j}\right) + |\widetilde{g}|^{-1/2}\partial_y\left(|\widetilde{g}|^{1/2}\partial_y\right) \\ V &= y^{2m}\widetilde{f}(x,y) \quad \text{with} \quad \widetilde{f} \in C^\infty(\mathcal{V}_0)\,; \end{aligned} \tag{4.11}$$

$\mathcal{V}_0$ is an open neighbourhood of zero in $\mathbb{R}^d$,
$\widetilde{g}^{ij}(x,y) = \widetilde{g}^{ji}(x,y) \in C^\infty(\mathcal{V}_0;\,\mathbb{R})$, $|\widetilde{g}|^{-1} = det\left(\widetilde{g}^{ij}(x,y)\right) > 0$.
$x = (x_1,\ldots,x_{d-1})$ are local coordinates on Γ_0
and the metric $g = (g_{ij})$ on Γ_0 is given by

$$(g_{ij}(x))_{1\leq i,j\leq d-1} = G(x)\,,\quad \text{with} \quad (G(x))^{-1} = \left(\widetilde{g}^{ij}(x,0)\right)_{1\leq i,j\leq d-1}\,..$$

If $w \in C_0^2(\mathcal{O}_0)$ then

$$\begin{aligned} P^h w &= \widehat{P}^h u \quad \text{with} \\ u &= |\widetilde{g}|^{1/4} w \quad \text{and} \\ \widehat{P}^h &= -h^2 \textstyle\sum_{1\le i,j\le d-1} \partial_{x_i}\left(\widetilde{g}^{ij}\partial_{x_j}\right) - h^2\partial_y^2 + V + h^2 V_0 \,, \end{aligned} \tag{4.12}$$

for some $V_0 \in C^\infty(\mathcal{V}_0\,;\,\mathbb{R})$.

Let us write

$$V(x,y) = y^{2m} f(x) + y^{2m+1} f_1(x) + y^{2m+2}\widetilde{f}_2(x,y) \;: \tag{4.13}$$

$f(x) = \widetilde{f}(x,0)$ and $\widetilde{f}_2 \in C^\infty(\mathcal{V}_0)$.

We perform the change of variable (2.1) and the related unitary transformation,

$$(x,y) \to (x,t) = (x, f^{1/(2(m+1))}(x)y)\,, \quad u \to v = f^{-1/(4(m+1))}u\,,$$

to get that

$$\begin{aligned} \widehat{P}^h u &= \widehat{Q}^h v \qquad \text{with} \\ \widehat{Q}^h &= Q_0^h + t^{2m+1} f_1^0(x) + h^2 R_0 + +h^2 t R_1 + t^{2m+2}\widetilde{f}_2^0 \;: \\ Q_0^h &= -h^2 \textstyle\sum_{1\le i,j\le d-1} \partial_{x_i}\left(g^{ij}\partial_{x_j}\right) + f^{1/(m+1)}(x)\left(-h^2\partial_t^2 + t^{2m}\right) \end{aligned} \tag{4.14}$$

and $R_0 = ta(x,t)(\partial_x f(x)\partial_x)\partial_t + b(x,t)t\partial_t +$

$$\sum_{ij} b_{ij}(x,t)\partial_{x_i} f(x)\partial_{x_j} f(x)(t\partial_t)^2 + c(x,t)\,,$$

$R_1 = \sum_{1\le i,j\le d-1} \partial_{x_i}\left(\alpha_{ij}(x,t)\partial_{x_j}\right)$, all coefficients are regular in a neighbourhood of the zero in $\mathbb{R}^d$.

Let μ_j be as in the theorem 4.1. We define $h_j = h^{1/(m+1)}/\mu_j^{1/2}$.

Let $\mathcal{O}_0'$ be a bounded open neighbourhood of zero in $\mathbb{R}^{d-1}$ such that $\overline{\mathcal{O}}_0' \subset \mathcal{O}_0 \cap \{(x,0)\,;\, x\in\mathbb{R}^{d-1}\}$.

We consider the Dirichlet operator on $L^2(\mathcal{O}_0')$, $H_0^{h_j}$:

$$H_0^{h_j} = -\,h_j^2 \sum_{1\le k,\ell\le d-1} \partial_{x_k}\left(g^{k\ell}(x)\partial_{x_\ell}\right) + f^{1/(m+1)}(x)\,. \tag{4.15}$$

It is well known, (see for example [2] or [5], that for any $\alpha \in \mathbb{N}^{d-1}$ satisfying the assumptions of the theorem 4.1, one has:

$$\exists\, \lambda_{j,\alpha}^h \in sp\,(H_0^{h_j}) \quad \text{s.t.} \quad |\lambda_{j,\alpha}^h - [\eta_0^{1/(m+1)} + h_j\mathcal{A}_l(\alpha)| \le h_j^2 C\,;$$

$\mathcal{A}_l(\alpha)$ is defined in theorem 4.1 in relation with our $s_l \in \Sigma_0$.

C is a constant depending only on N . We will denote by $\psi_{j,\alpha}^{h_j}(x)$ any associated eigenfunction with a L^2-norm equal to 1 . Let $\chi_0 \in C^\infty(\mathbb{R})$ such that

$$\chi_0(t) = 1 \quad \text{if} \quad |t| \le 1/2 \quad \text{and} \quad \chi(t) = 0 \quad \text{if} \quad |t| \ge 1\,.$$

We define the following function :

$$u_{j,\alpha}^h(x,t) = h^{-1/(2m+2)}\chi_0(t/\epsilon_0)\psi_{j,\alpha}^{h_j}(x)\left[\varphi_j(h^{-1/(m+1)}t) - h^{1/(m+1)}F_j^h,(x,t)\right],$$

with

$$F_j^h(x,t) = f_1^0(x)f^{-1/(m+1)}(x)\phi_j(h^{-1/(m+1)}t),$$

where $\phi_j \in S(\mathbb{R})$ is solution of :
$-\frac{d^2}{dt^2}\phi_j(t) + (t^{2m} - \mu_j)\phi_j(t) = t^{2m+1}\varphi_j(t)$,
and $\epsilon_0 \in]0,1]$ is a small enough constant, but independent of h and j .
ϕ_j exists because μ_j is a non-degenerated eigenvalue and the related eigenfunction φ_j (see 1.8) verifies $\int_{\mathbb{R}} t^{2m+1}\varphi_j^2(t)\,dt = 0$, since it is a real even or odd function.
Using the similar estimates as in chapter 3 , one can get easily that

$$\mu_j^{-1}\|t\partial_t\varphi_j\|_{L^2(\mathbb{R})} + \mu_j^{-2}\|(t\partial_t)^2\varphi_j\|_{L^2(\mathbb{R})} \leq C$$

and $\forall k \in \mathbb{N}$, $\exists C_k > 0$ s.t. $\mu_j^{-k/2m}\|t^k\varphi_j\|_{L^2(\mathbb{R})} \leq C_k$.
It is well known that there exists $\epsilon_1 > 0$ s.t.
$|\mu_j - \mu_\ell| \geq \epsilon_1$, $\forall \ell \neq j$, then the inverse of $-\dfrac{d^2}{dt^2} + t^{2m} - \mu_j$ is $L^2(\mathbb{R})$-bounded by $1/\epsilon_1$, (on the orthogonal of φ_j) . So in the same way as in chapter 3 , we get also that

$$\mu_j^{-2-1/2m}\|t\partial_t\phi_j\|_{L^2(\mathbb{R})} + \mu_j^{-3-1/2m}\|(t\partial_t)^2\phi_j\|_{L^2(\mathbb{R})} \leq C$$

and $\forall k \in \mathbb{N}$, $\exists C_k > 0$ s.t. $\mu_j^{-1-(k+1)/2m}\|t^k\phi_j\|_{L^2(\mathbb{R})} \leq C_k$.
As in the proof of Theorem 3.1, we get easily that

$$\|[\widehat{Q}^h - \mu_j\lambda_{j,\alpha}^h]\chi_0(|x|/\epsilon_0)u_{j,\alpha}^{h_j}(x,t)\|_{L^2(\mathcal{O}_0)} \leq h^2\mu_j^{(4m+3)/2m}C$$

and
$|\,\|\chi_0(|x|/\epsilon_0)u_{j,\alpha}^{h_j}(x,t)\|_{L^2(\mathcal{O}_0)} - 1\,| = O(h^{1/(m+1)}\mu_j^{(2m+1)/2m}) = o(1)$.
So the theorem 4.1 follows easily.

Remark 4.2 *If in Theorem 4.1 we assume that j is also bounded by N , then, as in [6], we can get a full asymptotic expansion*

$$\lambda_{j\ell\alpha}^h \sim h^{2m/(m+1)}\sum_{k=0}^{+\infty} c_{j\ell k\alpha}h^{k/(m+1)},$$

and for the related eigenfunction, a quasimode of the form

$$u_{j\ell\alpha}^h(x,t) \sim c(h)e^{-\psi(x)/h^{1/(m+1)}}\chi_0(t/\epsilon_0)\sum_{k=0}^{+\infty} h^{k/(2m+2)}a_{j\ell k\alpha}(x)\phi_{jk}(t/h^{1/(m+1)}).$$

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