CUBO A Mathematical Journal
Vol. 9, Nº 3, (39 - 45). December 2007.

Concrete algebraic cohomology for the group $(\mathbb{R}, +)$ or how to solve the functional equation $f(x+y) - f(x) - f(y) = g(x,y)$

Mihai Prunescu
Hornecker Softwareentwicklung, Freiburg, Germany
Institute of Mathematics of the Romanian Academy, Bucharest, Romania.
Mihai.Prunescu@math.uni-freiburg.de

ABSTRACT

The functional equation $f(x+y) - f(x) - f(y) = g(x,y)$ has a solution f that belongs to $C^0(\mathbb{R})$ if and only if the symmetric cocycle g belongs to $C^0(\mathbb{R}^2)$. If the symmetric cocycle g is recursively approximable, there exists a solution f which is recursively approximable also. If g belongs to $C^1(\mathbb{R}^2)$ then there exists an integral expression in g for a solution f that belongs to $C^1(\mathbb{R})$, and the same happens for the classes C^k, C^∞, analytic and polynomial.

RESUMEN

La ecuación funcional $f(x+y) - f(x) - f(y) = g(x,y)$ tiene una solución f que pertenece a $C^0(\mathbb{R})$ si y sólo si el cociclo simétrico g pertenece a $C^0(\mathbb{R}^2)$. Si el cociclo simétrico g es aproximable recursivamente, existe una solución f la cual también es aproximable recursivamente. Si g pertenece a $C^1(\mathbb{R}^2)$, entonces existe una expresión integral en g para una solución f que pertenece a $C^1(\mathbb{R})$ y lo mismo sucede para las clases: C^k, C^∞, analítica, polinomial.

Key words and phrases: *Algebraic Cohomology, Functional Equation, Analytic Properties*
Math. Subj. Class.: *39B05, 39B22*

1 Introduction

The existence of a function $f(x)$ satisfying the functional equation:

$$f(x+y) - f(x) - f(y) = g(x,y)$$

is identical with the 2-coboundary condition for the function $g(x,y)$, as defined in the algebraic cohomology of abelian groups. This theory gives in general non-constructive proofs for the existence of the solution, and studies the obstacles for the functional equation to be soluble (in the so-called cohomologic non-trivial cases). My goal here is to study analytic properties of the solution and its expressibility in the real case. The word *concrete* used in the title can be also understood as the combination of *continuous* and *discrete*.

In general, let K and L be two abelian groups and let $g : L \times L \to K$ be a function. If there is a function $f : L \to K$ verifying the functional equation for all $x, y \in L$ then $g(x,y)$ must verify the following conditions:

– $g(x,y)$ must be symmetric, that is:

$$g(x,y) = g(y,x),$$

– $g(x,y)$ must be a 2-cocycle according to the trivial action of L on K, that is:

$$g(x,y) + g(x+y,z) = g(x,y+z) + g(y,z).$$

We observe that if $f_0 : L \to K$ is a particular solution of the functional equation, then the set of all solutions is $\{f_0 + \delta \,|\, \delta \in \text{Hom}\,(L,K)\} = f_0 + \text{Hom}\,(L,K)$.

The cocycle condition for $y = 0$ gives $g(x,0) = g(0,z) = g(0,0)$. One can always suppose that $g(0,0) = 0$. Indeed, if f is a solution of the functional equation, then $f(0) = -g(0,0)$. If $g(0,0) \neq 0$ then we replace $g(x,y)$ by $g(x,y) - g(0,0)$. The new equation has exactly the solutions $f(x) + g(0,0)$, where $f(x)$ are the solutions for $g(x,y)$.

The following facts are proved in [3], pg. 231 - 239. The results go back to Eilenberg and MacLane, see [2].

If $g : L \times L \to K$ is a symmetric cocycle with $g(0,0) = 0$ than the set $G := K \times L$ with the operation $(u,x) \circ (v,y) := (u + v + g(x,y), x+y)$ is an abelian group such that the abelian groups K, G and L form a short exact sequence:

$$0 \to K \to G \to L \to 0$$

according to the embedding $\iota : u \in K \rightsquigarrow (u,0) \in G$ and to the projection $p : (u,x) \in G \rightsquigarrow x \in L$. In this situation one says that G is an extension of K by L. Two extensions G and G' of K by L are called equivalent if there is an isomorphism of abelian groups $\psi : G \to G'$ such that $\iota' = \psi\iota$ and $p'\psi = p$. Let us denote simply by $K \times L$ the trivial extension of K by L, corresponding to the symmetric cocycle $g(x,y) \equiv 0$. The extension G is equivalent with $K \times L$ if and only if there is an isomorphism $\psi : G \to K \times L$ of the form $\psi(u,x) = (u - f(x), x)$ if and only if

$f : L \to K$ satisfies the identity $f(x+y) - f(x) - f(y) = g(x,y)$. As proven in [3], in the cases:

- L free group, K arbitrary, or
- K divisible group, L arbitrary,

all extensions of K by L are equivalent with the trivial extension. It follows directly:

Corollary 1.1 *For functions $g : \mathbb{Z} \times \mathbb{Z} \to K$ and $g : L \times L \to \mathbb{Q}$ or $\mathbb{R}$ with $g(0,0) = 0$, the functional equation $f(x+y) - f(x) - f(y) = g(x,y)$ has a solution f if and only if g is a symmetric cocycle.*

In particular, there is an $f : \mathbb{R} \to \mathbb{R}$ verifying the functional equation, if and only if $g : \mathbb{R}^2 \to \mathbb{R}$ is a symmetric cocycle. If $f_0 : \mathbb{R} \to \mathbb{R}$ is such a solution, the set of all solutions is given by the sums $f_0 + \delta$, where δ are solutions for the functional equation of Cauchy $\delta(x+y) = \delta(x) + \delta(y)$.

Remark 1.2 *In the case $g : \mathbb{Z} \times \mathbb{Z} \to K$ the solutions have the form:*

$$f(n) = nf(1) + \begin{cases} \sum_{i=1}^{n-1} g(i,1) & n \geq 2 \\ \sum_{i=-1}^{1-n} (g(i,-1) - g(1,-1)) & n < 0 \end{cases}$$

where $f(1) \in K$ is a free parameter, and $f(0) = 0$. This is true for all discrete subgroups $\alpha\mathbb{Z}$ of $\mathbb{R}$ with the only one modification that all integers which are arguments of f or g in this formula must be multiplied with α.

Proof: According to the cited theory, for any symmetric cocycle $g : \mathbb{Z} \times \mathbb{Z} \to K$ there exist solutions. Two solutions differ up to an additive homomorphism of $\mathbb{Z}$, $(n \rightsquigarrow kn)$. Fix a value for $f(1)$. We compute a solution f_0 with $f_0(1) = 0$. By adding the equalities $f_0(i+1) - f_0(i) - f_0(1) = g(i,1)$ for $i = 1$ to $n-1$ one gets the expression for $n > 0$. On the other hand $f(0) = 0$ and $f(-1) = -f(1) - g(-1,1) = -g(-1,1)$. This value is substituted in the similar sum $nf(-1) + \sum_{i=-1}^{1-n} g(i,-1)$. So, if some solution exists, it must be equal with the given expression, and on the other hand we know that there exists a solution. ∎

Example: Let $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be given by $g(x,y) = xy$. This function is a symmetric cocycle. A solution $f : \mathbb{Z} \to \mathbb{Z}$ is given by $f(n) = \frac{n(n+1)}{2}$. These are the triangular numbers, extended over the whole $\mathbb{Z}$. Now let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given again by $g(x,y) = xy$. All functions $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{x^2}{2} + ax$ are solutions.

2 Class C^0

Theorem 2.1 *The functional equation $f(x+y) - f(x) - f(y) = g(x,y)$ has a continuous solution $f : \mathbb{R} \to \mathbb{R}$ if and only if the symmetric cocycle $g : \mathbb{R}^2 \to \mathbb{R}$ is a* *continuous function. In this case for all $x_0 \neq 0$ fixed the following is true: for all $a \in \mathbb{R}$ there exists exactly one continuous solution $f : \mathbb{R} \to \mathbb{R}$ such that $f(x_0) = a$.*

Proof: If f is continuous, then also g. Suppose that g is a continuous symmetric cocycle, without restricting the generality with $g(0,0) = 0$. Construct the short exact sequence of topological groups:

$$0 \to \mathbb{R} \to G \to \mathbb{R} \to 0.$$

Here is $G = \mathbb{R} \times \mathbb{R}$ with the euclidian topology and again $(u,x) \circ (v,y) := (u + v + g(x,y), x + y)$. G is not only an abelian group, but a topological group: the inverse $(u,x)^{-1} := (-u - g(x,-x), -x)$ is also a continuous application. The embedding $\iota : u \in K \rightsquigarrow (u,0) \in G$ and the projection $p : (u,x) \in G \rightsquigarrow x \in L$ are homomorphisms of topological groups. According to a fundamental theorem of Markoff (see [4]) a topological group is isomorphic with some euclidian group $(\mathbb{R}^n, +, 0)$ if and only if it is abelian, Hausdorff, locally compact, connected and the only one compact subgroup is $\{0\}$. Let $(u,x) \neq (0,0)$ be an element of G. If $x \neq 0$ then $\pi_2(< (u,x) >) = x\mathbb{Z}$, which is unbounded. If $x = 0$ then $\pi_1(< (u,0) >) = u\mathbb{Z}$ which is also unbounded. So G hasn't any nontrivial compact subgroup and hence there exists an isomorphism of topological groups $\varphi : G \to \mathbb{R}^2$.

Claim: *$\varphi\iota(\mathbb{R})$ is a vector-line.*

Indeed, $\varphi\iota(\mathbb{Q}x) = \mathbb{Q}\varphi\iota(x)$. If the closed subgroup $\varphi\iota(\mathbb{R})$ contains $\mathbb{R}$-linearly independent elements y_1 and y_2, then it would contain the set of all rational combinations $\mathbb{Q}y_1 + \mathbb{Q}y_2$ and its closure, so it would be the whole $\mathbb{R}^2$, which is a contradiction. So $\varphi\iota(\mathbb{R})$ is the topological closure of $\mathbb{Q}\varphi\iota(1)$, which is a real vector-line. ∎

One can suppose that $\varphi\iota(\mathbb{R}) \neq \{0\} \times \mathbb{R}$; if not, we substitute φ with $\tau\varphi$, where τ is a small rotation. Consider the application $\delta : \mathbb{R} \to \mathbb{R}$ given by $\delta(x) := p\varphi^{-1}(0,x)$. δ is a homomorphism of topological groups, so is additive and continuous. This means that δ is a continuous solution for the functional equation of Cauchy $\delta(x + y) = \delta(x) + \delta(y)$ over $\mathbb{R}$. Hence there is an $a \in \mathbb{R}$ such that $\delta(x) = ax$, and $a \neq 0$ because $\varphi^{-1}(\{0\} \times \mathbb{R}) \not\subset \text{kern } p$.

We construct an application $\theta : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying the following conditions:

$$\theta(\varphi\iota(\mathbb{R})) = \mathbb{R} \times \{0\} \;\; ; \;\; \theta\varphi\iota(1) = (1,0) \;\; ; \;\; \theta^{-1}|_{\{0\}\times\mathbb{R}} = (x \rightsquigarrow \frac{1}{a}x).$$

This is done by the linear application θ such that $\theta(\varphi\iota(1)) = (1,0)$ and $\theta(0,1) = (0,a)$. θ is an isomorphism of topological groups.

Call $\psi := \theta\varphi$, $\iota' := \psi\iota$ and $p' := p\psi^{-1}$. Then $\iota'(u) = (u,0)$ and $p'(u,x) = x$; in particular $p'\iota' = 0$. It follows that ψ is an isomorphism between the exact short sequences of topological groups $(\mathbb{R}, \iota, G, p, \mathbb{R})$ and $(\mathbb{R}, \iota', \mathbb{R}^2, p', \mathbb{R})$; so there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that for all $(u,x) \in \mathbb{R}^2$ it holds $\psi(u,x) = (u + f(x), x)$. According to the results quoted in the Introduction, the continuous function f verifies the functional equation.

Now let us take an $x_0 \neq 0 \in \mathbb{R}$. Any solution has the form $f(x) + \delta(x)$ and is continuous if and only if the additive homomorphism $\delta(x)$ is continuous if and only if $\delta(x) \equiv bx$ for some $b \in \mathbb{R}$. But $b = \frac{a-f(x_0)}{x_0}$ is the only one able to satisfy the given condition.

∎

For a formal definition of recursively approximable functions, see [5]. All recursively approximable functions are continuous, but they build a strict subset of the continuous functions.

Theorem 2.2 *If the continuous symmetric cocycle $g : \mathbb{R}^2 \to \mathbb{R}$ is recursively approximable, then there are continuous solutions $f : \mathbb{R} \to \mathbb{R}$ which are also recursively approximable.*

Proof: Let $f : \mathbb{R} \to \mathbb{R}$ be any solution of the functional equation. As we know, $f = f_0 + \delta$, where f_0 is a continuous solution and δ an additive homomorphism of $\mathbb{R}$. It follows that $f\,|_{\mathbb{Q}} = f_0\,|_{\mathbb{Q}} + ax\,|_{\mathbb{Q}}$ for some $a \in \mathbb{R}$. This means that $f\,|_{\mathbb{Q}} : \mathbb{Q} \to \mathbb{R}$ is always continuous. On the other hand, continuous solutions defined over $\mathbb{Q}$ or over $\mathbb{R}$ are uniquely determined by a value in some $x_0 \neq 0$, for example by $f(1)$. Let $\alpha\mathbb{Z}$ be a cyclic subgroup of $\mathbb{R}$. Considering the similar functional equation corresponding to $g\,|_{\alpha\mathbb{Z}\times\alpha\mathbb{Z}}$ and the form for the solution given in the Remark 1.2 written in integer multiples of α, we see that these discrete solutions are also uniquely determined by $f(1)$.

Lemma 2.3 *Let $g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a symmetric cocycle and $f_\alpha : \alpha\mathbb{Z} \to \mathbb{R}$ a solution of the functional equation written for the symmetric cocycle $g\,|_{\alpha\mathbb{Z}\times\alpha\mathbb{Z}}$. Then there is a unique function $f_{\frac{\alpha}{2}} : \frac{\alpha}{2}\mathbb{Z} \to \mathbb{R}$ satisfying the functional equation for the symmetric cocycle $g\,|_{\frac{\alpha}{2}\mathbb{Z}\times\frac{\alpha}{2}\mathbb{Z}}$ such that $f_{\frac{\alpha}{2}}\,|_{\alpha\mathbb{Z}} = f_\alpha$.*

Proof of the Lemma: According to the Remark 1.2, the function f_α is uniquely determined by the value $f_\alpha(\alpha)$ and $f_{\frac{\alpha}{2}}$ by the value $f_{\frac{\alpha}{2}}(\frac{\alpha}{2})$. But $f_{\frac{\alpha}{2}}\,|_{\alpha\mathbb{Z}}$ is a solution for the same problem as f_α. Hence, the only thing to do is to choose $f_{\frac{\alpha}{2}}(\frac{\alpha}{2})$ such that $f_{\frac{\alpha}{2}}(\alpha) = f_\alpha(\alpha)$. By solving the equation $f_\alpha(\alpha) - 2f_{\frac{\alpha}{2}}(\frac{\alpha}{2}) = g(\alpha, \frac{\alpha}{2})$ one gets the value:

$$f_{\frac{\alpha}{2}}(\frac{\alpha}{2}) = \frac{f_\alpha(\alpha) - g(\alpha, \frac{\alpha}{2})}{2}.$$

∎

So, what we have to do, is to construct the sequence of discrete functions f_1, $f_{\frac{1}{2}}$, $f_{\frac{1}{4}}$, ..., $f_{\frac{1}{2^n}}$, ..., with the property that all $f_{\frac{1}{2^{n+1}}}\,|_{2^{-n}\mathbb{Z}} = f_{\frac{1}{2^n}}$. They are all restrictions of the continuous solution $f : \mathbb{R} \to \mathbb{R}$ determined by $f(1) = f_1(1)$, which has to be taken a recursive real. The union of all these domains are the dyadic numbers, which are dense in $\mathbb{R}$, and the union of all graphs is dense in the graph of f. So f can be recursively approximated. ∎

3 Class C^1 and more

Lemma 3.1 *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a symmetric cocycle of class C^1. Then the following identities hold:*

1. $g(x,0) = g(0,z) = g(0,0)$.
2. $(\partial_1 g)(u,v) = (\partial_2 g)(v,u)$.
3. $(\partial_2 g)(x,y) = (\partial_2 g)(x+y,0) - (\partial_2 g)(y,0)$.
4. $(\partial_1 g)(x,y) = (\partial_1 g)(0,x+y) - (\partial_1 g)(0,x)$.

Proof: Point 1 has been proved in the introduction. Point 2 follows by symmetry. Point 4 follows from 2 and 3. To prove 3, consider the following reformulations for the cocycle-axiom, for $z \neq 0$:

$$g(x,y+z) - g(x,y) = g(x+y,z) - g(y,z)$$

$$\frac{g(x,y+z) - g(x,y)}{z} = \frac{g(x+y,z) - g(x+y,0)}{z} - \frac{g(y,z) - g(y,0)}{z}$$

Make now $z \to 0$ and recall that $g \in C^1$. It follows:

$$(\partial_2 g)(x,y) = (\partial_2 g)(x+y,0) - (\partial_2 g)(y,0).$$

∎

Theorem 3.2 *The functional equation $f(x+y) - f(x) - f(y) = g(x,y)$ has a solution $f : \mathbb{R} \to \mathbb{R}$ of class C^1 if and only if the symmetric cocycle $g : \mathbb{R}^2 \to \mathbb{R}$ is also of class C^1. In this case the function given by:*

$$f(x) = \int_0^x (\partial_2 g)(u,0)\ du,$$

is a solution. Consequently, if g is a symmetric cocycle of class C^k, C^∞, real-analytic or polynomial, then the functional equation has solutions f of the same kind.

Proof: Let again g be a symmetric cocycle of class C^1 with $g(0,0) = 0$. Take f to be the function given in the statement and consider the function: $h(x,y) := f(x+y) - f(x) - f(y)$. Of course, h is a symmetric cocycle, and a function of class C^1. By applying Lemma 3.1 several times, one computes:

$$(\partial_1 h)(x,y) = (\partial_2 g)(x+y,0) - (\partial_2 g)(x,0) = (\partial_1 g)(0,x+y) - (\partial_1 g)(0,x) = (\partial_1 g)(x,y)$$

$$(\partial_2 h)(x,y) = (\partial_2 g)(x+y,0) - (\partial_2 g)(y,0) = (\partial_2 g)(x,y)$$

Let now $l(x,y) := (h-g)(x,y) \in C^1$. Because $(\partial_1 l)(x,y) \equiv 0$ and $(\partial_2 l)(x,y) \equiv 0$, the function $l(x,y)$ must be constant. But $l(0,0) = 0$, so $h(x,y) \equiv g(x,y)$. ∎

Again if the symmetric cocycle g of class C^k (or C^∞, and so on...) is recursively approximable, the the solutions f of the corresponding class are recursively approximable too. The proof of the Theorem 2.2 works in all these cases.

Received: April 2006. Revised: May 2006.

References

[1] J. Aczel, *Vorlesung über Funktionalgleichungen und ihre Anwendungen*. Birkhäuser Verlag, Basel und Stuttgart, 1961.

[2] S. Eilenberg, S. MacLane, *Group extensions and homology*. The Annals of Mathematics, 43, 757 - 831, 1942.

[3] L. Fuchs, *Abelian groups*. Pergamon Press, Oxford 1967.

[4] A. Markoff, *Über endlich-dimensionale Vektorräume*. The Annals of Mathematics, Ser. 2, 36, 2, 464 - 506, 1935.

[5] K. Weihrauch, *Computable analysis*. Springer Verlag, 2000.