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Stochastic model of money flow in economics

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ABSTRACT

In this report a stochastic model of money flow is considered. This model can be applied for both small and big regions. Connection with stochastic differential equations is established. The definitions of absolute stability and relative instability are introduced. A stochastic model with a delay (lag) is considered. The suggestion about a critical total amount of money of organizations working stably is made to get over crisis situation.

RESUMEN

En este artículo se considera un modelo estocástico del flujo de dinero. Este modelo se puede aplicar tanto a regiones pequeñas como grandes. Además, se establece la conexión con las ecuaciones diferenciales estocásticas. Se introducen las definiciones de estabilidad absoluta e inestabilidad relativa y se considera un modelo estocástico con un retardo. Se hace una sugerencia acerca de una cantidad total crítica de dinero de las organizaciones trabajando establemente para superar una situación de crisis.

Key words and phrases: *stochastic equations, differential equations, absolute stable and relatively unstable systems*
Math. Subj. Class.: *93E03, 60H30, 91B02*

1 Introduction

The report is devoted to the study of economical systems with the help of stochastic and differential equations that is the novelty in the study of this kind of systems. Absolute stable and relatively unstable economical systems are defined in terms of Markov's matrices. It is not difficult to transform a Markov's chain, corresponding to our discrete time system, to stochastic differential equations. The definitions of stability of discrete time systems are transformed to the definition of stability of continuous time systems. For completeness Markov's chains with lags are considered.

Each economical system with its own degree of development defines some value $s_{critical}$. If the amount of total money of organizations working stably is less than $s_{critical}$, then the infrastructure will be destroyed in the course of time and any economical system falls into a deep economical crisis.

2 Markov's matrices in economics

Let us have n organizations. Each of them has commercial trades with the others. We can consider a state like one of them. Let us numerate the organizations and the moments of time. Let us consider an organization with the number $i, i \in 1 : n$. Let the i-th organization have at the t-th moment an amount of money s_{t_i} in the bank account. We will denote by p^t_{ij} a fraction of amount of money paid by the i-th organization to the j-th organization at the moment t. It is obvious that

$$0 \leq p^t_{ij} \leq 1, \qquad 0 \leq \sum_{j=1, j\neq i}^{n} p^t_{ij} \leq 1 \quad \forall i, j \in \overline{1:n}.$$

We will also denote by p^t_{ii} the means that the i-th company has at the i-th moment of time. It is clear that $p^t_{ii} = 1$.

Let us introduce the following matrix

$$P^t = \begin{pmatrix} p^t_{11} & p^t_{12} & \cdots & p^t_{1n} \\ p^t_{21} & p^t_{22} & \cdots & p^t_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ p^t_{n1} & p^t_{n2} & \cdots & p^t_{nn} \end{pmatrix}$$

that we will name the industrial matrix of expenses or simply the matrix of expenses at the moment t.

Let us calculate the amount of money that the k-th client (organization) has at the $t+1$-th moment of time if it had s^t_k amount of money at the t-th moment of time. We will write down an equation describing the flow of money from the t-th moment to the $t+1$-th moment.

The sum of money paid by all companies to the k-th company at the $t+1$-th moment is

$$\sum_{j=1, j\neq k}^{n} p^t_{jk}\, s^t_j.$$

The k-th company has the following amount

$$\sum_{j=1,\, j\neq k}^{n} p_{jk}^{t}\, s_j^t + s_k^t. \tag{1}$$

together with its own money without payment to the others companies at the $t+1$-th moment of time.

Let us introduce a vector-column s^t

$$s^t = \begin{pmatrix} s_1^t \\ s_2^t \\ \vdots \\ s_n^t \end{pmatrix},$$

where the element s_i^t, $i \in 1:n$, is the amount of money that the i -th company $, i \in 1:n$, has.

It is obvious that (1) in matrix form is

$$(P^t)' s^t.$$

where the sign $'$ means the transposition.

Now let us calculate the sum of money that the k-th company pays to the others companies with which the first one does business. This sum of all payments is

$$s_k^t \sum_{j, j\neq k} p_{kj}^t. \tag{2}$$

at the $t+1$-th moment of time.

Considering together (1) and (2), we can confirm that the sum of means at the t-th moment, obtained by the k-th company, is

$$\sum_{j, j\neq k} p_{jk}^t s_j^t + s_k^t - s_k^t \sum_{i, i\neq k} p_{ki}^t = s_k^{t+1}. \tag{3}$$

We suggest in this model that the j-th client makes a payment to the k-th client and the k-th client pays back all debts at the same moment of time, namely, at the $t+1$-th moment of time.

Equation (3) can be written in the following matrix form

$$(P^t)' s^t - \begin{pmatrix} \sum_{i, i\neq 1} p_{1i}^t & 0 & \dots & 0 \\ 0 & \sum_{i, i\neq 2} p_{2i}^t & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sum_{i, i\neq n} p_{ni}^t \end{pmatrix} \begin{pmatrix} s_1^t \\ s_2^t \\ \vdots \\ s_n^t \end{pmatrix} =$$

$$= ((P^t)' - I^t) s^t = s^{t+1}. \tag{4}$$

where

$$I^t = \begin{pmatrix} \sum_{i,i\neq 1} p^t_{1i} & 0 & \dots & 0 \\ 0 & \sum_{i,i\neq 2} p^t_{2i} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sum_{i,i\neq n} p^t_{ni} \end{pmatrix}$$

We will denote by

$$Q^t = (P^t)' - I^t.$$

We will have from above that

$$Q^t s^t = s^{t+1} \tag{5}$$

Let us prove that Q^t is a stochastic matrix. According to the definition we need to prove that

1. all elements of the matrix Q^t are nonnegative;
2. the sum $\sum_i q^t_{ij} = 1 \ \forall i, j \in \overline{1:n}$,

where is the q^t_{ij} -th element of the matrix Q^t.

Really, as soon as

$$\sum_{j,i\neq j} p^t_{ij} \leq 1,$$

then

$$1 - \sum_{j,i\neq j} p^t_{ij} \geq 0.$$

Consequently, all diagonal elements of the matrix Q^t are the nonnegative elements. The other elements are nonnegative obviously, as far as the elements of the matrix P^t are nonnegative.

Let us prove the second quality. For this we should find the sum of the elements of the matrix Q^t staying in the columns. We have for the j-th column

$$\sum_{i=1,i\neq j} p^t_{ji} + 1 - \sum_{i=1,i\neq j} p^t_{ji} = 1.$$

We will take into consideration that the columns of the matrix Q^t are the lines of the matrix P^t. Consequently, the matrix Q^t is really stochastic. So we proved the following lemma.

Lemma 2.1 *The matrix Q^t is the stochastic matrix.*

We have from here

Corollary 2.1 *The sum of all money of all companies or organizations is constant at any moment t i.e.*

$$\sum_{j=1}^{n} s^t_j = const \ \ \forall t > 0. \tag{6}$$

Proof. According to equation (5) we have

$$Q^t s^t = s^{t+1}.$$

Let us write this equation for all coordinates. We have then

$$\sum_{j=1}^{n} q_{ij}^t s_j^t = s_i^{t+1}, \tag{7}$$

where q_{ij}^t– are the elements of the matrix Q^t. The sum on i of both sides of (7) is equal to the following number

$$\sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij}^t s_j^t = \sum_{i=1}^{n} s_i^{t+1}.$$

Let us change the order of the sum above. We will have then

$$\sum_{j=1}^{n} s_j^t \sum_{i=1}^{n} q_{ij}^t = \sum_{i=1}^{n} s_i^{t+1}.$$

As soon as

$$\sum_{i=1}^{n} q_{ij}^t = 1,$$

then

$$\sum_{j=1}^{n} s_j^t = \sum_{i=1}^{n} s_i^{t+1},$$

The conclusion is proved.△

Remark 2.1 *The Conclusion 2.1 follows from the general properties of the stochastic matrices.*

Remark 2.2 *If s_j^t is a part of the common sum of means which the company j has at the t-th moment of time, then equality (6) is satisfied with the consideration of inflation. We have in general case the following equation*

$$\sum_{j=1}^{n} s_j^t = const = 1 \quad \forall t > 0.$$

Stability of any economical system can be defined correspondingly to stability of (5). It is known from the theory of the stochastic matrices that stability of (5) is defined by means of the multiplicity of the root equal to the one. We will say in this case that absolute stability has place.

The explanation of this fact is the following.

If the matrix Q^t has the several roots, equal to the one, then the matrix Q^t can be decomposed and has the following form after several transpositions

$$Q^t = \begin{pmatrix} Q_1^t & S_{12} & S_{13} & \dots & S_{1k} \\ 0 & Q_2^t & S_{21} & \dots & S_{2k} \\ \dots & \dots & \dots\dots & \dots & \dots \\ 0 & 0 & 0 & \dots & Q_k^t \end{pmatrix}, \tag{8}$$

where the matrices $Q_i^t, i \in \overline{1:k}$, have the eigenvalue $\lambda = 1$ with multiplicity 1. Then any solution of the system

$$Q^t x = x$$

tends to the set

$$X = \mathrm{co}\,\{x_1, x_2, \dots, x_k\},$$

when $t \to \infty$, where x_i is a solution of the system

$$Q_i^t x_i = x_i.$$

It points to instability of system (5). We will call this kind of instability as relative instability as far as stability and instability are defined by quality of control of any economical system in every case.

If we take into consideration additional subsidies or transmissions of money from one branch to another, then equation (5) can be rewritten in the form

$$Q^t s^t + B^t u^t = s^{t+1}, \tag{9}$$

where u^t is a control-vector and B^t is a matrix of effective control elements. The vector u^t is usually chosen in the form

$$u^t = C^t s^t.$$

The additional condition on the matrix C^t can be given for equation (9) to describe a stochastic process. We will study (9) later.

Let us rewrite equation (5) into a system of differential equations. Write (5) in the form

$$s(t+h) = Q(t+h)s(t).$$

Denote by $Q_0 = E_n$ an initial value of the stochastic matrix $Q(t)$. Then

$$\frac{s(t+h) - s(t)}{h} = \frac{Q(t+h) - E_n}{h} s(t).$$

Take the limit on $h \to +0$. We will get the following differential equation

$$s'(t) = A(t)s(t),$$

where

$$A(t) = \lim_{h \to +0} h^{-1} [Q(t+h) - E_n].$$

The matrix $A(.)$ exists if the matrix $Q(.)$ is differentiable. The obtained matrix $A(.)$ has the following qualities

1. $a_{ii}(t) < 0$;

2. $a_{ij}(t) > 0$ for $i \neq j$;

3. $\sum_i a_{ij}(t) = 0$.

The question about stability of an economical system is reduced to the question about stability of the differential equation system

$$\dot{x}(t) = A(t)x(t).$$

This problem can be solved with the help of Laypunov's functions [1]-[3]. The same problem about effective methods of control can be formulated by means of the control vector for the following system of differential equations

$$\dot{x}(t) = A(t)x(t) + B(t)u(t). \tag{10}$$

It is not difficult to see that the system (10) is stochastic if the matrix $A(t) + B(t)C(t)$ is stochastic for all t in terms of the definition given above. The problem about possibility of transference of an economical system from one state to a more beneficial state is equivalent in mathematical terms to the problem about total controllability of system (10). System (10) is well studied and the necessary and sufficient conditions of total controllability of system (10) are found [1]-[3].

If we have a delay of a tax payment or a loan in k years then model (5) can be rewritten in the form

$$s^{t+1} = Q_0^t s^t + Q_1^{t-1} s^{t-1} + Q_2^{t-2} s^{t-2} + ... + Q_k^{t-k} s^{t-k}. \tag{11}$$

The condition of stability of system (11) is

Theorem 2.1 *It is necessary and sufficient for system (11) to be absolute stable that the matrix*

$$Q = Q_0 + Q_1 + Q_2 + ... + Q_k$$

has the eigen-value equal to the one with the multiplicity equal to the one where for any sequence $t_m \to \infty$

$$\lim_{t_m \to \infty} Q_k^{t_m - k} = Q_k.$$

The goal of any government is to make its economy stable so that all basic companies would develop proportionally. It means in the mathematical terms that it is necessary to find a stationary point of a set valued mapping $C^t(.) : \Re \to 2^{\Re^n}$ with the images

$$C^t(s) = \{As \mid A \in \Xi^t\}$$

where Ξ^t is a set of the stochastic matrices at the moment t with the help of which the government tries to stabilize economy i.e. it is necessary to solve the problem

$$s^t \in C^t(s^t). \tag{12}$$

The conditions are well known when the set-valued mapping $C^t(.)$ has a stationary point. The future studies are to use these conditions in order to find the stationary points of the set-valued mapping $C^t(.)$. It is important for economics.

As soon as the matrices A in the definition of the set valued mapping $C(.)$ are the stochastic matrices, then the mapping $C(.)$ transforms the compact

$$K^n = \{x = (x_1, x_2, ..., x_n) \in \Re^n \mid \sum\nolimits_1^n x_1 = 1\}$$

in itself. Since $C(.)$ is continuous and, consequently, upper semi-continuous, $C(.)$ has a stationary point according to Caratheodory's theorem [5]. The following step is construction of a numerical method of looking for a stationary point s^* and a matrix $A \in \Xi^t$ for some t, for which

$$s^* = As^*.$$

The methods of looking for the stationary points are described in [6].

The problem (12) can be paraphrased as the following optimization problem

$$F(A, x) = Ax - x = 0_n \tag{13}$$

under constraints

$$x \in K^n = \{z \in \Re^n \mid z = (z_1, z_2, ..., z_n), \sum_i z_i = 1, \ z_i \geq 0 \forall i \in 1:n\},$$

$$\sum_i^n a_{ij} = 1 \ \forall j \in 1:n, \ a_{ij} \geq 0, \ a_{ij} \leq 1 \ \forall i, j \in 1:n.$$

There are sets of matrices $\triangle_1$ and $\triangle_2$ for credit and tax politics correspondingly that the following constraints $A \in \triangle_1$ and $A \in \triangle_2$ have to be satisfied. It is possible to find the best economical politics changing the sets $\triangle_1$ and $\triangle_2$ or, by other words, changing the lines and columns of the matrix A, corresponding governmental organizations, and solving (12), or (13).

The problem (13) can be solved using developed algorithms (see, for example, [7]). We are looking for the elements of the matrix $A = (a_{ij})$ and all coordinates of the vector $x = (x_1, x_2, ..., x_n)$. The formulas for gradients are the following

$$\frac{\partial F}{\partial a_{ij}} = x_j \ \forall i, j \in 1:n,$$

and

$$\frac{\partial F}{\partial x} = \begin{pmatrix} a_{11} - 1 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} - 1 & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} - 1 \end{pmatrix},$$

The last one is the matrix of dimension $n \times n$.

Thus, it is necessary for efficient control of economics that, the first, the economical system was controllable totally in some region and, the second, all basic companies would work stably.

To fulfill to the last one we will introduce two operations. The first one is a combination and the second one is a separation of the companies. We will prove that these two operations are sufficient to make the companies working stably.

Let us introduce a matrix Q^t in the form (8). If $S_{12} \neq 0$, then as it is known from [4] the matrix Q_2^t can not have the eigen-values equal to the one that means instability of the second company. If we unite the first two companies the matrix Q^t will have the form

$$Q^t = \begin{pmatrix} Q_1^t & S_{12} & S_{13} & \dots & S_{1k} \\ \bar{S}_{12} & Q_2^t & S_{23} & \dots & S_{2k} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & Q_k^t \end{pmatrix}, \qquad (14)$$

The united company 1-2 will have the eigen-value equal to the one that means its stability.

If we use the separation operation of the companies 1 and 2 we will get the matrix Q^t that has $S_{12} = 0$ and $\bar{S}_{21} = 0$. The matrix Q^t will have the form

$$Q^t = \begin{pmatrix} Q_1^t & 0 & S_{13} & \dots & S_{1k} \\ 0 & Q_2^t & S_{23} & \dots & S_{2k} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & Q_k^t \end{pmatrix}, \qquad (15)$$

The matrices Q_1^t and Q_2^t of (15) have the eigen-values equal to the one that means stability of the companies 1 and 2.

Thus, the following theorem is proved.

Theorem 2.2 *The two operations: combinations and separations of the companies lead to the situation when the new organized companies will be absolute stable.*

The numerical experiments were done for a model economical system. The conclusions were done about efficiency of the fiscal and credit politics. The results say that development of small and medium businesses is more efficient politics for reviving of economy. The last one is not difficult to understand. The elements of the matrix Q^t , corresponding to the small and medium businesses, are much more than the elements of the same matrix, responsible for the government's economical politics. The more elements we can change the more efficiently we can change the eigen-values of the matrix. That is right if we have widely developed small and medium business. The examples can be given.

3 Conclusion

A situation is possible when the matrix Q has the several eigen-values equal to the one. We saw above that the matrix Q can be presented in this situation in the form of matrix (8) where the matrices Q_i^t, $i \in \{1 : k\}$, have the eigen-values $l = 1$ with the multiplicity equal to the one. Let us denote by s_{sum} the summary money that the basic companies that correspond to the matrices Q_i^t, $i \in \{1 : k\}$, have.

It is not difficult to prove that some critical number $0 \leq s_{crit} \leq 1$ corresponds to each economical system that if $s_{crit} > s_{sum}$ then the economical system falls into a deep economical crisis.

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