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Some Problems in Functional Differential Equations *

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Abstract.
Functional differential systems close to ordinary differential systems, which are an h-system in variations, are studied. We obtain existence results and asymtotic formulae for their solutions. Several explicite examples and applications are shown.

1 Systems of functional differential equations close to systems of ordinary differential equations.

In [1], Bellman proposed to investigate conditions on the lag r to know the behavior of solutions of the functional differential equation

$$u'(t) + au(t - r(t)) = 0, \quad a \text{ constant} \tag{1}$$

when $r(t) \to 0$ as $t \to \infty$. In [2], Cooke proves that if $r \in L_1([0,\infty))$ then any solution u of (1) satisfies

$$u(t) = e^{at}[c + o(1)], \quad t \to \infty$$

for some constant c. In [3], Cooke generalizes this result to linear systems of functional differential equations asymptotically autonomous. Grossman and Yorke [4] consider the one-dimensional functional differential equation

$$u'(t) = a(t)u(t - r(t)).$$

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We consider systems of functional differential equations which behave asymptotically as an ordinary h-system [5, 6]. That is to say, let $P : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ a continuous function with derivative $P_x = P_x(t,x)$ continuous for which the system

$$x' = P(t,x) \tag{2}$$

is an h-system in variation. We recall that (2) or the null-solution of (2) is an h-system in variation [5,6] if there exist a continuous function $h : [0,\infty) \to (0,\infty)$ and constants $K \geq 1$, and $\delta > 0$ such that for $0 \leq| x_0 |< \delta$ we have

$$| \Phi(t,t_0,x_0) |\leq Kh(t)h(t_0)^{-1} \quad t \geq t_0 \geq 0$$

where $\Phi(t,t_0,x_0)$ is the fundamental matrix of the variational system

$$z' = P_x(t,x(t,t_0,x_0))z$$

with $\Phi(t_0,t_0,x_0) = \mathrm{Id}$ (the identity matrix)

Further, let $F : [0,\infty) \times C_0 \to \mathbb{R}^n$ for which the system

$$y' = F(t,y_t) \tag{3}$$

verifies

$$| F(t,y_t) - P(t,y) |\leq r(t) \parallel y_t' \parallel, \tag{4}$$

where $r \in C([0,\infty),\mathbb{R})$ and $0 \leq r(t) \leq q$. Here $C_0 = C([-q,0],\mathbb{R}^n)$ and if $y \in C([t-q,t],\mathbb{R}^n)$, we denote y_t the element in C_0 defined by

$$y_t(s) = y(t+s), \quad -q \leq s \leq 0.$$

We define also for $y \in C([t-2q,t],\mathbb{R}^n)$:

$$y_t(s) = y(t+s), \quad -2q \leq s \leq 0.$$

Moreover, we define

$$\parallel y \parallel = \sup_{-q \leq s \leq 0} | y(s) |,$$

and

$$\parallel y \parallel_2 = \sup_{-2q \leq s \leq 0} | y(s) |$$

Theorem 1.1 *Assume*

(i) The ordinary differential system (2) is an h-system, with radius of attraction δ

(ii) There exists a continuous and nonnegative function $c(t)$ *such that*

$$| F(t,g) |\leq c(t) \parallel g \parallel$$

for all $t \geq 0$ *and all* $g \in C_0$.

(iii) There exists a continuous and nonnegative function $r = r(t)$ such that for all continuously differentiable $g \in C_0$ and all $t \geq 0$:

$$| F(t,g) - P(t,g(0)) | \leq r(t) \| g'_t \|$$

iv) $\beta(t)r(t) \| c_t \| \in L_1([0,\infty))$, *where* $\beta(t) = h(t)^{-1} \| t^h \|_2$.

Then for any solution $y = y(t;t_0,y_{t_0})$ of (3) with $\| y_{t_0} \| \leq \delta$ there exists a solution x of (1) such that $y = x + h \cdot \bar{o}$ (1), where $\bar{o}$ (1) is a function defined on $[t_0,\infty)$ which converges as $t \to \infty$.

Proof If $| y(t_0) | \leq \delta$, then the solution $x = x(t;t_0,y(t_0))$ is well defined and verifies $| x(t,t_0,y(t_0)) | \leq K | y(t_0) | h(t)h(t_0)^{-1}$ for $t \geq t_0 \geq 0$ and $K \geq 1$ a constant. Now, by (ii) $y = y(t,t_0,y_{t_0})$ is defined on $[t_0 - q, \infty)$. By the formula of variation of the constants, we have for $t \geq t_1 \geq t_0$

$$y(t) = x(t;t_1,y(t_1)) + \int_{t_1}^{t} \Phi(t,s,y(s))[F(s,y_s) - P(s,y(s))]ds. \tag{5}$$

Then, by (i) and (ii)

$$| y(t) | \leq K | y(t_1) | h(t)h(t_1)^{-1} + Kh(t) \int_{t_1}^{t} h(s)^{-1}r(s) \| y'_s \| ds$$

or

$$h(t)^{-1} | y(t) | \leq Kh(t_1)^{-1} | y(t_1) | + K \int_{t_1}^{t} r(s)h(s)^{-1} \| y'_s \| ds.$$

Thus $z(t) = h(t)^{-1} | y(t) |$ satisfies

$$z(t) \leq Kz(t_1) + \int_{t_1}^{t} Kr(s)h(s)^{-1} \| y'_s \| ds. \tag{6}$$

For $u \in I = [-q, 0]$ and $s \geq t_1$, by (ii), we have

$$| y'_s(u) | = | F(s+u, y_{s+u}) | \leq c_s(u) \| y_{s+u} \| = c_s(u) | y(v) |$$

for some $v = v(s) \in [s - 2q, s]$. Further,

$$c(s+u)h(s)^{-1} | y(v) | = c(s+u)h(s)^{-1}h(v)z(v) \leq \beta(s)c(s+u)z(v).$$

Thus denoting $m(t) = max\{z(u) : \ t_0 - 2q \leq u \leq t\}$ we get

$$h(s)^{-1} \| y'_s \| \leq \beta(s) \| c_s \| m(s). \tag{7}$$

Substituting this into (6) we obtain

$$z(t) \leq Kz(t_1) + \int_{t_1}^{t} Kr(s)\beta(s) \| c_s \| m(s)ds. \tag{8}$$

Since the right member of (8) is increasing as a function in t, we have $m(t) \leq Kz(t_1) + \int_{t_1}^{t} Kr(s)\beta(s) \| c_s \| m(s)ds$. Then by (iv), Gronwall's inequality implies that m and hence z are bounded. Moreover, for any t fixed $\Phi(t,s,y(s))[F(s,y_s) - P(s,y(s))] \in L_1([0,\infty)$ as a function of s because by (i), (iii), (iv) and (7) we get

$$| \Phi(t,s,y(s))[F(s,y_s) - P(s,y(s))] |\leq Kh(t)h(s)^{-1}r(s) \parallel y_s' \parallel \leq$$

$$\leq K_1 h(t) \parallel c_s \parallel r(s)\beta(s)m(s) \leq K_2 h(t) r(s)\beta(s) \parallel c_s \parallel \in L_1([0,\infty)).$$

Then the integral in (5) can be written as $h(t)\cdot \tilde{o}(1)$, where $\tilde{o}(1)$ denotes a function of t which has a limit as $t \to \infty$. ■

Theorem 1 includes the interesting type of equations as:

$$y' = F(t, y(t) - y(t - r(t))) \tag{9}$$

For this equation, system (2) becomes $x' = 0$ and (iii) becomes

$$| F(t,g) |\leq r(t) \parallel g' \parallel \tag{10}$$

Thus here $h \equiv 1, \beta \equiv 1$ and we have

Corollary 1.2 *Let us assume (ii), (iv) with $\beta = 1$ and (iii) with (9) instead of (3). Then for any solution $y = y(t;t_0,y_{t_0})$ of (9) there exists a constant vector such that*

$$y = y(t_o) + v + o(1)$$

as $t \to \infty$. In particular, any solution of (9) is asymptotically constant.

■

For equations $y' = y^3(t) - y^3(t - r(t))$ or $y' = [y(t) - y(t - r(t))]^3$ condition (4) becomes

$$| F(t,g) |\leq Kr(t)w(\parallel g' \parallel) \tag{11}$$

Thus from lemma 1, [5] we obtain:

Corollary 1.3 *Assume (ii), (iv) and (iii) with (11) instead of (4). Then there exists a constant $\delta > 0$ such that any solution $y = y(t;t_0,y_{t_0})$ with $\parallel y_{t_0} \parallel \leq \delta$ is defined on $[t_0 - q, \infty)$ and*

$$y = y(t_o) + v(t_o) + o(1), \ t \to \infty$$

where $v = v(t_0)$ is a constant vector such that $v(t_0) \to 0$ as $t_0 \to \infty$. Moreover, $\delta = \delta(t_0)$ verifies $\delta(t_0) \to \infty$ as $t_0 \to \infty$. Then if t_0 is chosen large enough for any initial function φ there exists t_0 large enough such that the solution $y = y(t,t_0,\varphi)$ verifies the above property.

■

Corollary 1.4 *If $\int_t^s a(u)du \leq K$, K constant, for $s - 2q \leq t \leq s$ and $a \parallel a_t \parallel \cdot r \in L_1([0,\infty))$, then the solutions of the scalar equation $y'(t) = a(t)y(t - r(t))$, satisfy*

$$y(t) = exp(\int_0^t a(s)ds)[c + o(1)], \quad c \text{ constant.}$$

Thus, in particular, the solutions of

$$y'(t) = -ty(t - e^{-3t}) \text{ and}$$
$$y'(t) = e^{t}y(t - e^{-2t})$$

satisfy respectively

$$y = e^{-\frac{t^2}{2}}[c + 0(1)], \quad c \text{ constant and}$$

$$y = e^{t}[c + o(1)], \quad c \text{ constant}$$

■

Corollary 1.5 *If A is an stable matrix, then any solution of*

$$y' = Ay(t - r(t)),\ r \in L_1([0,\infty))$$

satisfies

$$y = e^{tA}x_0 + e^{-\alpha t} \cdot \tilde{o}(1)$$

where x_0 is a constant vector, $0 > \alpha > maxRe\lambda$ with λ an eigenvalue of A and $\tilde{o}(1)$ is a convergent vector as $t \to \infty$. ■

Corollary 1.6 *If the system*

$$x' = A(t)x$$

is an h-system strong and $r \cdot \parallel A \parallel\parallel A_t \parallel \in L_1([0,\infty))$, then any solution y of

$$y' = A(t)y(t - r(t))$$

satisfies

$$y = \Phi[y_0 + o(1)] \text{ as } t \to \infty$$

where y_0 is a constant vector and Φ is a fundamental matrix of (12). ■

2 Asymptotic formulae for the solutions of $y'' + c(t)y(t - r(t)) = 0$

Consider the functional differential equation

$$y'' + c(t)y(t - r(t)) = 0 \tag{1}$$

where $c : [0,\infty) \to \mathbb{R}$ and $r : [0,\infty) \to [0,\infty)$ are continuous functions. For $r = r(t)$ small, in some sense which will be precised, we hope that the solutions y of (1) behave asymptotically as the solutions z of the ordinary differential equation.

$$z'' + c(t)z(t) = 0 \tag{2}$$

In fact, we will prove that any solution y of (1) are defined on all of $I = [0, \infty)$ and it satisfies as $t \to \infty$:

$$y = (\delta_1 + o(1))z_1 + (\delta_2 + o(1))z_2$$
$$y' = (\delta_1 + o(1))z_1' + (\delta_2 + o(1))z_2' \qquad (3)$$

where $\{z_1, z_2\}$ is a fundamental system of solutions of Eq(2) and $\{\delta_1, \delta_2\}$ are constants.

Suppose $r(t) \leq q$ and consider the Banach space $C_o = C([-q, 0], \mathbb{R})$ with the norm

$$\| \varphi \| = \sup_{-q \leq s \leq 0} | \varphi(s) |, \ \varphi \in C_o.$$

Furthermore, for $y \in C([0, \infty), \mathbb{R})$, we define y_t the useful element in C_0 given by

$$y_t(s) = y(t+s), \ -q \leq s \leq 0$$

Let

$$y(t) = A(t)z_1(t) + B(t)z_2(t) \qquad (4)$$

under the condition

$$A'z_1 + B'z_2 = 0 \qquad (5)$$

Then, we have $y' = Az_1' + Bz_2'$ and $y'' = A'z_1' + B'z_2' + Az_1'' + Bz_2''$. Thus $y'' = A'z_1' + B'z_2' - c(Az_1 + Bz_2)$. Therefore

$$A'z_1' + B'z_2' = c(t)[y(t) - y(t - r(t))] \qquad (6)$$

Solving Eqs. (5) and (6), we get

$$A' = -w^{-1}z_2.c(t)[y(t) - y(t - r(t))]$$
$$B' = w^{-1}z_1.c(t)[y(t) - y(t - r(t))] \qquad (7)$$

where w is the Wronskian of system $\{z_1, z_2\}$. Now, we have

$$\begin{aligned} | y(t) - y(t - r(t)) | &= | \textstyle\int_{t-r(t)}^{t} y'(s)ds | = | \int_{-r(t)}^{0} y'(t+s)ds | \\ &= | \textstyle\int_{-r(t)}^{0} y_t'(s)ds | = | \int_{-r(t)}^{0} (Az_1' + Bz_2')_t(s)ds | . \end{aligned}$$

Thus

$$| y(t) - y(t - r(t)) | \leq r(t) \max_{i=1,2} \| z_{it}' \| \cdot (\| A_t \| + \| B_t \|)$$

Then, by system (7), the vector $x = (A, B)$ satisfies a system of functional differential equations of the type.

$$x' = F(t, x_t) \qquad (8)$$

satisfying the conditions (i) $F : I \times C_0 \to \mathbb{R}$ is a continuous function, (ii) $| F(t, \varphi) | \leq \lambda(t) \| \varphi \|, (t, \varphi) \in I \times C_0$.

In this point, we need the following Theorem concerning the asymptotic behavior of system (8).

Theorem 2.1 *Assume conditions (i), (ii) where* $\lambda \in C(I, \mathbb{R})$ *satisfy* $\lambda(t) \in L_1(I)$. *Then the solutions of Eq(8) are defined on all of* I *and they converge as* $t \to \infty$.

The proof of this theorem is omitted because it is similar to Theorem 1.1. ∎
Thus, we get:

Theorem 2.2 *Assume that* $r(t) \mid c(t) \mid \cdot \mid z_i(t) \mid \cdot \parallel z_{it} \parallel \in L_1(I) \quad i = 1, 2$. *Then formulae (3) hold.*

Proof The application of Theorem 1 implies that A and B converge as $t \to \infty$. The formulae (3) follow by (4) and (5). ∎

Corollary 2.3 *If* $r \in L_1(I)$, *then any solution* y *of the functional differential equation*

$$y'' + ay(t - r(t)) = 0, \ a > 0 \ constant$$

satisfies for $t \to \infty$,

$$y = (\delta_1 + o(1))sin\ at + (\delta_2 + o(1))cos\ at$$

$$y' = a(\delta_1 + o(1))cos\ at - a(\delta_2 + o(1))sin\ at.$$

∎

Corollary 2.4 *If* $c(t) \in C^2(I)$, $c > 0$ *and* $c^{-3/2}c''$, $r(t). \mid c^{-3/4}(t) \mid\parallel c_t^{-1/4} \parallel \in L_1(I)$ *then any solution* y *of the functional differential equation*

$$y'' + c(t)y(t - r(t)) = 0$$

satisfies for $t \to \infty$

$$y = c(t)^{-1/4}[(\delta_1 + o(1))exp(i\int^t c^{1/2}(s)ds) + (\delta_2 + o(1))exp(-i\int^t c^{1/2}(s)ds)]$$

$$y' = c(t)^{1/4}[i(\delta_1 + o(1))exp(i\int^t c^{1/2}(s)ds) - i(\delta_2 + o(1))exp(-i\int^t c^{1/2}(s)ds)].$$

∎

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