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On a pencil of K_3 surfaces *

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Resumen.

Una superficie K_3 es una superficie analítica compleja $S(dim_{\mathbb{C}} S = 2)$ que es simplemente conexa y cuyo fibrado canónico K_S es trivial. Para las superficies K_3 que son algebraicas (es decir, inmersas en P_n), existe un espacio de módulos $\mathcal{M}(K_3)$ que es 19 dimensional.

En esta nota se construye una subvariedad 1-dimensional de $\mathcal{M}(K_3)$ con un grupo de simetrías fijo $(\mathbb{Z}/4\mathbb{Z})^2 \times G_4$) y se describen explícitamente sus degeneraciones.

Let P_3 be the complex projective space with coordinates $(x_1, x_2, x_3, x_4) \in P_3$. The section $F \in H^o(P_3, O_{P_3}(4))$ given by $F(x_1, x_2, x_3, x_4) = x_1^4 + x_2^4 + x_3^4 + x_4^4$, defines the classical Fermat's quartic. Consider the section $T \in H^o(P_3, O_{P_3}(4))$ given by $T(x_1, x_2, x_3, x_4) = x_1 x_2 x_3 x_4$, as F and T are linearly independent the 2-dimensional vector subspace $S = \langle F, T \rangle \subset H^o(P_3, O_{P3}(4))$ defines a pencil of quartic surfaces that we will denote by:

$$S_\alpha = \{F + \alpha T / \alpha \in P_1\}$$

Each generic quartic S_α of the pencil is smooth, by Lefschetz's theorem on hyperplane sections, S_α is simply-connected, furthermore as $K_{S_\alpha} \cong \mathcal{O}_{S_\alpha}(4 - (3 + 1)) \cong \mathcal{O}_{S_\alpha}$ the canonical sheaf is trivial and S_α is a K_3 surface.

We begin describing the base locus of our pencil.

Proposition 1 *The base locus of $(S_\alpha)_{\alpha \in P_1}$ denoted by $B(S_\alpha)$ consists of a stable curve of genus $g = 33$ with associated graph given by Fig 1.*

Proof We consider the four hyperplanes $H_i = \{x_i = 0\}$ with $i \in \{1, ..., 4\}$. The intersection of $F = 0$ with each hyperplane $H_i, i \in \{1, ..., 4\}$, gives a Fermat's quartic of genus $g = 3$, that we denote by $F_i, i \in \{1, ..., 4\}$. The intersection $F_i \cap F_j$ is contained in each line $F_i \cap F_j = L_{ij}$ and consists of four differents points.

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Then the base locus is a stable curve, that consists of four curves F_i of genus $g = 3$, where each F_i cuts each of the other three F_j at four points and the graph associated corresponds to Fig. 1, the genus $g = 33$ follows from the well known formula [1] for graph curves. ■

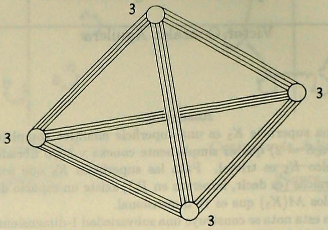

Fig. 1

It is well known that the biggest group of projective automorphisms of $F = 0$ (Fermat's quartic surface) is isomorphic to $G \cong (\mathbb{Z}/4\mathbb{Z})^3 \times \mathcal{G}_4$, where $\mathcal{G}_4$ is the symmetric group. The group G can be represented in $PGL(4, \mathbb{C})$ as $g_{ijk}(x_1, x_2, x_3, x_4) = (\xi^i x_1, \xi^j x_2, \xi^k x_3, x_4)$ with ξ a primitive rooth of the unity of order 4 and $\mathcal{G}_4$ acting by permutation of coordinates.

As the section $T \in H^o(P_3, Q_{P3}(4))$ is also invariant by G, then $G \subseteq Aut(T)$ and the 2-dimensional vector subspace $S = \langle F, T \rangle \subset H^o(P_3, O_{P3}))$ is G-invariant.

We have the following proposition.

Proposition 2 *The group G acts on the pencil by multiplication by i and the subgroup $H \cong (\mathbb{Z}/4\mathbb{Z})^2 \times \mathcal{G}_4$ acts trivially.*

Proof By definition the subspace $S \subset H^o(P_3, O_{P3}(4))$ is invariant by the action of G then the group G acts on the pencil.

We consider the epimorphism:

$$\mu : \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z}$$

given by $\mu(\xi^i, \xi^j, \xi^k) = \xi^i \xi^j \xi^k$, ker μ is isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ and each $h \in H = (\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z})^2$ acts trivially on T, as $\mathcal{G}_4$ acts also trivially on this section, we have the result. ■

An elementary calculation allows to obtain the values of $\alpha \in P_1$ where S_α is singular and we have the following proposition.

Proposition 3 *For each projective K_3 surface in the pencil $(S_\alpha)_{\alpha \in P_1}$ we have that $H \subseteq Aut\ (S_\alpha)$. There are only five singular fibers in the pencil, they correspond to the values $\alpha = \{i, -1, -i, 1\}$ and $\alpha = \infty$. For $\alpha \neq \infty$ the singular fibers are Kummer's surface, for $\alpha = \infty$ the singular fiber is of type*

III, a rational surface meeting along rational curves its dual graph is the tetrahedron.

Proof For each $\alpha \in \{i, -1, -i, 1\}$ an easy calculation proves that S_α has the maximum of ordinary double points (16 double points) then by [2] S_α is a Kummer surface.

For $\alpha = \infty$ it is clear that S_∞ consists of the union of four rational surfaces (the hyperplanes $H_i = \{x_i = 0\}, i \in \{1, ...4\}$) where two of them intersects along a rational curve. ■

Remark. As every Kummer surface is isomorphic to a surface of the form $\mathcal{J}ac(C)/\{\pm 1\}$, where C is a curve of genus 2, and $\mathcal{J}ac(C)$ denotes the Jacobi variety. Then in our case $\mathcal{J}ac(C)$ must admit $\mathcal{G}_4$ as reduced group of automorphisms.

Finally it would be interesting to study the variation of the Picard number of S_α as in [4].

References

[1] Miranda, R., *Graph Curves and Curves on K_3 Surfaces*, Lectures on Riemann Surfaces, I.C.T.P., **1987**. World Scientific.

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H_i a rational surface meeting along rational curves the dual graph of the tetrahedron.

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References

[1] Miranda, R., *Graph Curves and Curves on K_3 Surfaces*, Lectures on Riemann Surfaces, I.C.T.P., 1987. World Scientific.

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