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On Quasi orthogonal Bernstein Jordan algebras.*

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Abstract.

Bernstein algebras were introduced by P.Holgate in [1] to deal with the problem of populations which are in equilibrium after the second generation. In [3] we work with Weak Bernstein Jordan algebras, i.e. a class of commutative algebras with idempotent element and defined by relations. In [3, section 4] we prove that if $A = Ke \oplus U \oplus V$ is the Pierce decomposition of A relative to the idempotent e, then the situations $U^3 = \{0\}$ and $U^2(UV) = \{0\}$ are independents of the different Pierce decompositions of A, then they are invariants of A.

We say that A is orthogonal if $U^3 = \{0\}$ and quasi-orthogonal if $U^2(UV) = \{0\}$. The orthogonality case was treated in [2].

In this paper we prove that every Bernstein-Jordan algebra of dimension less than 11 is quasi-orthogonal. Moreover we prove that there exists only one non quasi-orthogonal Bernstein-Jordan algebra of dimension 11.

1 Preliminaries

In what follows K denotes an infinite field of characteristic different from 2 and A a finite dimensional, commutative, not necessarily associative algebra over K.

If $w : A \to K$ is a non zero algebra homomorphism, then the ordered pair (A, w) is called a baric algebra and w the weight function of A. A Bernstein algebra is a

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baric algebra (A, w) such that $(x^2)^2 = w(x)^2 x^2$ for every $x \in A$.

A is a Jordan algebra if the identity $x^2(yx) = (x^2y)x$ holds in A.

It is known (see [4]) that if (A, w) is a baric algebra then A is a Bernstein and a Jordan algebra if and only if the identity $x^3 = w(x)x^2$ holds in A. Moreover every Bernstein-Jordan algebra is a Weak Bernstein Jordan algebra and in [3] we prove that if $A = Ke \oplus U \oplus V$, is the Pierce decomposition of a Weak Bernstein Jordan algebra A relative to the idempotent e, then $U^2 \subseteq V$, $UV \subseteq U$ and $V^2 \subseteq Ke$. In the case $V^2 \neq \{0\}$, the concepts of orthogonality and quasi-orthogonality are the same and in the case $V^2 = \{0\}$, A is a Bernstein-Jordan algebra. Moreover since $U^2 \subseteq V$ and $V^2 = \{0\}$ the following relations are satisfied (see [3], Corollary 3.3 and Proposition 3.5) For every $x \in U \oplus V$

$$x^3 = 0 \tag{1}$$

For every $u, u' \in U, v \in V$ and $v' \in U^2$

$$(uu')v = 0 \tag{2}$$

$$u(u'v) + u'(uv) = 0 \tag{3}$$

$$v(uv') + v'(uv) = 0 \tag{4}$$

$$v'(uv') = 0 \tag{5}$$

$$v(uv) = 0 \tag{6}$$

$$(uv')^2 = 0 \tag{7}$$

$$(uv)^2 = 0 \tag{8}$$

$$(uv)(uv') = 0 \tag{9}$$

By linearizing relation(1) we have for every $x, y, z \in U \oplus V$

$$(xy)z + (yz)x + (zx)y = 0 \tag{10}$$

2 Quasi-orthogonality

The following result will be used in Theorem 2.2

Proposition 2.1 *Let A be a non quasi-orthogonal Bernstein-Jordan algebra. Then there exist u, $u_1 \in U$, $v \in V$ such that $u^2(u_1v) \neq 0$. So, $u^2u_1 \neq 0$ and $\{u, u_1\}$ is a linearly independent subset of U. Moreover we can choose u_1 such that $u_1^2(uv) \neq 0$.*

Proof: Since A is non quasi-orthogonal then there exist u_1, u_2, $u_3 \in U$, $v \in V$ such that $(u_2u_3)(u_1v) \neq 0$. Then if $u_2^2(u_1v) = 0 = u_3^2(u_1v)$ we have $(u_2 + u_3)^2(u_1v) \neq 0$. Thus in any case there exist u, $u_1 \in U$, $v \in V$ such that $u^2(u_1v) \neq 0$.

Relation (4) implies $v(u^2u_1) \neq 0$ and $u^2u_1 \neq 0$. So $\{u, u_1\}$ is a linearly independent subset of U.

Moreover we can choose an element u_1 such that $u_1^2(uv) \neq 0$. In fact if $u_1^2(uv) = 0$ let us consider the element $u' = u_1 + u$. Then $u' \neq 0$, because $\{u, u_1\}$ is a linearly independent subset of U. Moreover $u^2(u'v) \neq 0$ and by using relations (4), (3) and (10) we have $u'^2(uv) = -u^2(vu_1) \neq 0$. ∎

EXAMPLE: Let $A = Ke \oplus U \oplus V$, where $U =< u_0, ..., u_7 >$ and $V =< v_0, ..., v_4 >$ be a commutative real algebra with multiplication table given by $e^2 = e$, $eu_i = \frac{1}{2}u_i$, $u^2 = v_2$, $u_0u_1 = v_1$, $u_0u_3 = v_4$, $u_0v_0 = u_2$, $u_0v_1 = -\frac{1}{2}u_4$, $u_0v_3 = u_7$, $u_0v_4 = -\frac{1}{2}u_5$, $u_1^2 = v_3$, $u_1u_2 = -v_4$, $u_1v_0 = u_3$, $u_1v_1 = -\frac{1}{2}u_7$, $u_1v_2 = u_4$, $u_1v_4 = \frac{1}{2}u_6$, $u_2v_1 = -\frac{1}{2}u_5$, $u_2v_3 = u_6$, $u_3v_1 = -\frac{1}{2}u_6$, $u_3v_2 = u_5$, $u_4v_0 = -u_5$, $u_7v_0 = -u_6$, all other products being zero. Then A is a baric algebra with weight homomorphism $w : A \to K$ defined by $w(e) = 1$, $w(u_i) = 0$, $w(v_j) = 0$, $i = 0, ..., 7$ and $j = 0, ..., 4$. Moreover the identity $x^3 = w(x)x^2$ holds in A. Therefore A is a Bernstein-Jordan algebra. Finally $u^2(u_1v_0) = v_2u_3 = u_5 \neq 0$. Thus A is non quasi-orthogonal.

Theorem 2.2 *Every Bernstein Jordan algebra of dimension less than* 11 *is quasi-orthogonal.*

Proof: By proposition 2.1, if A is non quasi-orthogonal, there exist elements u, $u_1 \in U$, $v \in V$ such that $u \neq u_1$, $u^2u_1 \neq 0$, $u^2(u_1v) \neq 0$, $u_1^2 \neq 0$ and $u_1^2(uv) \neq 0$.

Moreover $\{u.u_1, uv, u_1v, u^2u_1, u^2(u_1v)\}$ is a linearly independent subset of U. In fact if (*) $\alpha u + \beta u_1 + \gamma uv + \delta u_1v + \eta u^2u_1 + \theta u^2(u_1v) = 0$, then multiplying by u and by using relations (10), (3) and (1) we have

$$\alpha u^2 + \beta uu_1 + \delta(u_1v)u = 0 \tag{11}$$

Multiplying (11) by u and by using $V^2 = \{0\}$ and relations (1) and (10) we have

$$\beta u_1u^2 + \delta(u_1v)u^2 = 0 \tag{12}$$

Now multiplying (*) by v and after that by u^2 and using relations (4), (10) and (1) we have $\beta u^2(u_1v) + \eta u^2((u_1u^2)v) = 0$. Relations (4) and (5) imply $\eta u^2((u_1u^2)v) = 0$. Therefore $\beta u^2(u_1v) = 0$ and $\beta = 0$. By using relation (12) we obtain $\delta = 0$, and relation (11) implies $\alpha = 0$. Then we have (**) $\gamma uv + \eta u^2u_1 + \theta u^2(u_1v) = 0$.

Multiplying (**) by v and by using $V^2 = \{0\}$ and relations (10) and (4) we obtain $\eta v(u^2u_1) = 0$ and $\eta = 0$. Finally, multiplying (**) by u_1^2 and by using $\eta = 0$, relations (4) and (1) imply $\gamma(uv)u_1^2 = 0$ and $\gamma = 0$. Thus $dim_K(U) \geq 6$.

On the other hand $\{v, u^2, u_1^2, u(u_1v)\}$ is a linearly independent subset of V. In fact if $\alpha v + \beta u^2 + \gamma u_1^2 + \delta u(u_1v) = 0$ then multiplying this relation by u_1u^2 and using relation (5), (4) and (1) we obtain $\alpha v(u_1u^2) + \delta(u(u_1v))(u_1u^2) = 0$.

Relation (10) together with (3), (9) and (1) imply $(u(u_1v)(u_1u^2) = 0$. Then $\alpha v(u_1u^2) = 0$ and $\alpha = 0$.

Multiplying $\beta u^2 + \gamma u_1^2 + \delta u(u_1v) = 0$ by u_1v and by using relations (1), (10) and (8) we have $\beta u^2(u_1v) = 0$ and $\beta = 0$.

Now multiplying $\gamma u_1^2 + \delta u(u_1v) = 0$ by u_1 then relations (1), (3) and (10) imply $\delta u_1^2(uv) = 0$ then $\delta = 0$. Finally $\gamma u_1^2 = 0$, then $\gamma = 0$. Thus $dim_K(V) \geq 4$.

Therefore if A is non quasi-orthogonal then $dim_K(A) \geq 11$. ∎

Theorem 2.3 *There exists only one non quasi-orthogonal Bernstein Jordan algebra of dimension* 11

Proof: Let $A = Ke \oplus U \oplus V$, $U =< u_0, ..., u_5 >$, $V =< v_0, ..., v_3 >$ be an algebra with the following multiplication table: $e^2 = e$, $eu_i = \frac{1}{2}u_i$, $i = 0, ..., 5$, $u_0^2 = v_1$, $u_0u_1 = -\frac{1}{2}v_1 - \frac{1}{2}v_2$, $u_0u_3 = v_3$, $u_0v_0 = u_2$, $u_0v_2 = u_4$, $u_0v_3 = -\frac{1}{2}u_5$, $u_1^2 = v_2$, $u_1u_2 = -v_3$, $u_1v_0 = u_3$, $u_1v_1 = u_4$, $u_1v_3 = \frac{1}{2}u_5$, $u_2v_2 = u_5$, $u_3v_1 = u_5$, $u_4v_0 = -u_5$, all other products being zero. Moreover A is a baric algebra with weight homomorphism $w : A \to K$ defined by $w(e) = 1$, $w(u_i) = 0$, $w(v_j) = 0$, $i = 0, ..., 5$ and $j = 0, ..., 3$ and for every $x \in A$, $x^3 = w(x)x^2$. Then A is a Bernstein-Jordan algebra. Finally, since $u_1^2(u_0v_0) = v_2(u_0v_0) = v_2u_2 = u_5 \neq 0$, A is non quasi-orthogonal.

Next we give the way of built this algebra.

If $dim_K(U) = 6$ and $dim_K(V) = 4$, then the proof of Theorem 2.2 implies $\{v, u^2, u_1^2, u(u_1v)\}$ and $\{u, u_1, uv, u_1v, u^2u_1, u^2(u_1v)\}$ are basis of V and U respectively. Let us consider the elements $uu_1 \in V$ and $uu_1^2 \in U$. Then

$$uu_1 = \alpha v + \beta u^2 + \gamma u_1^2 + \delta u(u_1v) \quad (13)$$

$$uu_1^2 = \varepsilon u + \zeta u_1 + \eta uv + \theta u_1v + \lambda u^2u_1 + \mu u^2(u_1v) \quad (14)$$

First we shall prove that $\lambda \neq 0$. In fact if $\lambda = 0$, then multiplying (14) by u^2 and by using relations (1), (4) and (6) we have $0 = \zeta u^2u_1 + \theta u^2(u_1v)$. Since $\{u^2u_1, u^2(u_1v)\}$ is a linearly independent subset of U we have $\zeta = 0$ and $\theta = 0$. Therefore

$$uu_1^2 = \varepsilon u + \eta uv + \mu u^2(u_1v) \quad (15)$$

Multiplying (15) by u_1 and by using relations (3) and (9) we obtain

$$0 = \varepsilon uu_1 + \eta(uv)u_1 \quad (16)$$

Now multiplying (16) by u, using relations (10) and (4) and that $\{u^2u_1, u^2(u_1v)\}$ is a linearly independent subset of U we obtain $\varepsilon = \eta = 0$ and $uu_1^2 = \mu u^2(u_1v)$. Multiplying this relation by v and by using relations (4) and (6) we obtain $(uu_1^2)v = 0$ i.e. $u_1^2(uv) = 0$, a contadiction. Therefore $\lambda \neq 0$.

Next we shall prove that we can choose $\lambda = 1$.

In fact, by setting $u_\lambda = \lambda u$ we have $u_\lambda \neq 0$, $\{u_\lambda, u_1, u_\lambda v, u_1v, u_\lambda^2u_1, u_\lambda^2(u_1v)\}$ and $\{v, u_\lambda^2, u_1^2, u_\lambda(u_1v)\}$ are basis of U and V respectively. Moreover relations (13) and (14) become

$$u_\lambda u_1 = \lambda\alpha v + \frac{\beta}{\lambda}u_\lambda^2 + \lambda\gamma u_1^2 + \delta u(\lambda)(u_1v) \quad (17)$$

$$u_\lambda u_1^2 = \varepsilon u_\lambda + \lambda\zeta u_1 + \eta u_\lambda v + \lambda\theta u_1v + u_\lambda^2u_1 + \frac{\mu}{\lambda}u_\lambda^2(u_1v) \quad (18)$$

Now we shall prove that $\varepsilon = \eta = \theta = \mu = \zeta = \alpha = 0$.

Multiplying (17) by u_1, replacing $x = u_\lambda$, $y = z = u_1$ in relation (10) together with relation (3) imply $-\frac{1}{2}u_\lambda u_1^2 = \alpha\lambda vu_1 + \frac{\beta}{\lambda}u_\lambda^2u_1 - \delta(u_1(u_\lambda v))u_1$. Now multiplying by u_λ^2 and by using relations (4), (1) and (5) we have $0 = \alpha\lambda(vu_1)u_\lambda^2 - \delta(u_1(u_\lambda v)u_1)u_\lambda^2$.

Since $UV \subseteq U$ and $U^2 \subseteq V$, relations (10), (4) and (1) imply $(u_1(u_\lambda v)u_1)u_\lambda^2 = -\frac{1}{2}(u_1^2(u_\lambda v))u_\lambda^2 = \frac{1}{2}(u(\lambda)^2(u_\lambda v))u_1^2 = -\frac{1}{2}(vu_\lambda^3)u_1^2 = 0$. Then $\alpha(vu_1)u_\lambda^2 = 0$ i.e. $\alpha v(u_1u^2) = 0$ and $\alpha = 0$. Thus

$$u_\lambda u_1 = \frac{\beta}{\lambda}u_\lambda^2 + \gamma\lambda u_1^2 + \delta u_\lambda(u_1v) \tag{19}$$

Multiplying (19) by u_λ and by using $UV \subseteq U$ and relation (10) we have

$$-\frac{1}{2}u_\lambda^2u_1 = \gamma\lambda u_\lambda u_1^2 - \frac{1}{2}\delta u_\lambda^2(u_1v) \tag{20}$$

Replacing (18) in relation (20) and by using $\{u_\lambda, u_1, u_\lambda v, u_1v, u_\lambda^2u_1, u_\lambda^2(u_1v)\}$ a linearly independent subset of U we have $0 = \varepsilon = \zeta = \eta = \theta$, $-\frac{1}{2} = \gamma\lambda$, and $\gamma\mu = \frac{1}{2}\delta$. Then $\mu = -\lambda\delta$ and

$$u_\lambda u_1 = \frac{\beta}{\lambda}u_\lambda^2 - \frac{1}{2}u_1^2 + \delta u_\lambda(u_1v) \tag{21}$$

$$u_\lambda u_1^2 = u_\lambda^2u_1 - \delta u_\lambda^2(u_1v) \tag{22}$$

Multiplying relation (22) by v and by using relations (4) and (6) we obtain $(u_\lambda u_1^2)v = (u_\lambda^2u_1)v$ and

$$u_1^2(u_\lambda v) = u_\lambda^2(u_1v) \tag{23}$$

Now multiplying (19) by u_1 and by using relations (1), (10) and (3) we have $-\frac{1}{2}u_\lambda u_1^2 = \frac{\beta}{\lambda}u(\lambda)^2u_1 + \frac{1}{2}\delta u_1^2(u_\lambda v)$. Therefore relations (22) and (23) imply $-\frac{1}{2}u_\lambda^2u_1 = \frac{\beta}{\lambda}u_\lambda^2u_1$. Then $\frac{\beta}{\lambda} = -\frac{1}{2}$ and

$$u_\lambda u_1 = -\frac{1}{2}u_\lambda^2 - \frac{1}{2}u_1^2 + \delta u_\lambda(u_1v) \tag{24}$$

Finally by setting $u = u_\lambda$ we have

$$uu_1 = -\frac{1}{2}u^2 - \frac{1}{2}u_1^2 + \delta u(u_1v) \tag{25}$$

Next we prove that always it is possible to find elements $u, u_1 \in U$ and $v \in V$ satisfying Proposition 2.1 and $(u + u_1)^2 = 0$. If $(u + u_1)^2 \neq 0$ let us consider the elements $u' = u - \delta uv$ and $u_1' = u_1 - 2\delta u_1v$. Then $(u' + u_1')^2 = 0$ if and only if $(u + u_1)^2 = \delta u(u_1v)$, but this is true because $uu_1 = \frac{1}{2}((u + u_1)^2 - u^2 - u_1^2)$ and we use relation (25). Moreover the elements u', u_1 and v satisfy Proposition 2.1.

Thus

$$uu_1 = -\frac{1}{2}u^2 - \frac{1}{2}u_1^2 \tag{26}$$

Now if $A = Ke \oplus U \oplus V$, identifying $u_0 = u$, $u_2 = uv$, $u_3 = u_1v$, $u_4 = u^2u_1$, $u_5 = u^2(u_1v)$, $v_0 = v$, $v_1 = u^2$, $v_2 = u_1^2$, and $v_3 = u(u_1v)$ we have the algebra given at the beginning of the proof of this Theorem. ∎

In a forthcoming paper the same authors study orthogonality in Weak Bernstein Jordan algebras which are not baric algebras.

References

[1] Holgate P.,*Genetic algebras satisfying Bernstein's stationnarity principle*,J. London Math. Soc. (2) **9 612-624 (1975).**

[2] Labra A.,*On Orthogonal Weak Bernstein Jordan Algebras*, Atas de la XI Escola de Algebra, Brasil **(1990).**

[3] Labra A., Mallol C., Micali A., Varro R.,*Sur les algèbres de Bernstein V: Les algèbres de Bernstein Jordan Faibles*,Proc. Edinburg Math. Soc. (**in press**)

[4] Walcher S.,*Bernstein algebras which are Jordan algebras*,Arch. der Math. **50 218-222 (1988).**

[5] Wörz-Busekros A.,*Algebras in Genetics*,Lecture Notes in Biomathematics 36, Springer, New York **1980.**

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