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Homographic solutions in the n-body problem

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ABSTRACT

In this paper we study the classification and the existence of homographic solutions in the n-body problem.

1 Introduction

Celestial Mechanics is the field of scientific knowledge that studies the consequences of Newton's laws of gravitation. The main problem in the Celestial Mechanics is to describe the movement of n-particles where the only forces acting on the particles are the Newtonian attractions of the bodies on each other when we know the positions and velocities at a given time. This problem is called n-body problem (see the basic references on Celestial Mechanics [2], [6], [9] and [16]). The complexity of motion arising when more than two bodies move under their gravitational attractions increases quickly with the number of objects concerned. In the 2-body problem, given the initial position and velocity of one body relative to the other, we can predict the position and velocity in the space at any time and it is known that the solutions are conics with one of its focus on the center of mass. Thus, the 2-body problem is completely solved in the sense that we can describe explicitly all its solutions. When more than two bodies are involved, we have many partial results, but it is still impossible to find all its solutions and we are far from understanding the dynamics of this problem. For example, in the 3-body problem we know five explicit solutions that were discovered by Euler [4] in 1767 where the bodies are all the time at a collinear configuration and two equilateral configurations where the bodies are all the time are at an equilateral triangular configuration. They were discovered by Lagrange [5] in 1772.

The planets of the solar system constitute a classic example of the n-body problem in the first approximation. As the positions of the planets change during their orbital motion around the Sun, the gravitational forces acting on a given member of the system changes also. In the case of the Solar system, however, the Sun is the dominant centre of force. Hence the resulting planetary motions approximate closely the motions which would be observed if the Sun and each planet made up a pure two body-problem. Therefore we can consider the 2-body problem as a first approximation to understand the motion of the planet around the Sun or the motion of the Moon around the Earth.

Now, we shall outline the mathematical formulation of the n-body problem. Let n bodies with point masses $m_1, \ldots, m_n$, (or we can assume that the masses concerned are spherically symmetrical in homogeneous layers so that they attract one another like point masses) and let $\mathbf{r}_1, \ldots, \mathbf{r}_n$ be the vector position of the particles with respect to the origin $\mathbf{O}$ fixed in the space and masses m_i respectively. Then, applying Newton's law of gravitation yields that the complete description of the problem involves the solution of the second order system of n differential equations:

$$m_i \ddot{\mathbf{r}}_i = \sum_{j=1, j\neq i}^{n} \frac{m_i m_j}{\|\mathbf{r}_i - \mathbf{r}_j\|^3}(\mathbf{r}_j - \mathbf{r}_i) = \nabla_{\mathbf{r}_i} U \quad (i = 1, \ldots, n), \tag{1.1}$$

where $U = U(\mathbf{r}_1, \ldots, \mathbf{r}_n)$ is the Newtonian potential defined by

$$U = \sum_{1 \leq i < j \leq n} \frac{G m_i m_j}{\|\mathbf{r}_i - \mathbf{r}_j\|}, \tag{1.2}$$

where G is the universal gravitational constant, $G = 6.6732 \times 10^{-11} m^3/s^2 kg$, however, the units for length, mass and time may be chosen, without loss of generality, such that $G = 1$ (see [7], [16]) and $\nabla_{\mathbf{r}_i} U$ represents the gradient of U with respect to $\mathbf{r}_i$. This system of equations (1.1) define the Newtonian formulation of the n-body problem.

The Hamiltonian formulation of this problem is obtained introducing the linear momentum of the ith particle $\mathbf{p}_i = m_i \dot{\mathbf{r}}_i$, such that the system (1.1) can be rewritten as a first order system of $2n$ differential equations

$$\dot{\mathbf{r}}_i = \frac{1}{m_i}\mathbf{p}_i, \quad \dot{\mathbf{p}}_i = \nabla_{\mathbf{r}_i} U \quad (i = 1, \ldots, n), \tag{1.3}$$

with the function H defined by

$$H(\mathbf{r}, \mathbf{p}) = \sum_{i=1}^{n} \frac{\|\mathbf{p}_i\|^2}{2m_i} - U(\mathbf{r}_1, \ldots, \mathbf{r}_n), \tag{1.4}$$

the system (1.3) takes the following formulation:

$$\dot{\mathbf{r}}_i = H_{\mathbf{p}_i}, \quad \dot{\mathbf{p}}_i = -H_{\mathbf{r}_i}, \quad (i = 1, \ldots, n) \tag{1.5}$$

where $\mathbf{r} = (\mathbf{r}_1, \ldots, \mathbf{r}_n) \in \mathbb{R}^{3n}$, $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{R}^{3n}$ and $H_{\mathbf{r}_i}$, $H_{\mathbf{p}_i}$ denote the partial derivative of H with respect to the variables $\mathbf{r}_i$ and $\mathbf{p}_i$, respectively (a good reference for details on this formulation is [7]). We observe that the standard theory of differential equations yields the following result:

Theorem *Given* $(\mathbf{r}_0, \dot{\mathbf{r}}_0) \in (\mathbb{R}^{3n} \setminus \Delta) \times \mathbb{R}^{3n}$, *there exists a unique solution* $\mathbf{r}(t)$ *of (1.5) defined in a maximum interval* $t_\star^- < t < t_\star^+$, *containing* $t = t_0$, *with initial conditions* $\mathbf{r}(t_0) = \mathbf{r}_0$, $\dot{\mathbf{r}}(t_0) = \dot{\mathbf{r}}_0$. *Furthermore, all components of* $\mathbf{r}(t)$ *are analytic functions of* t *and of the coordinates of* $\mathbf{r}_0$ *and* $\dot{\mathbf{r}}_0$.

Here $\Delta = \cup_{ij, i \neq j} \Delta_{ij}$ with $\Delta_{ij} = \{\mathbf{r} = (\mathbf{r}_1, \ldots, \mathbf{r}_n) \in \mathbb{R}^{3n} \ / \ \mathbf{r}_i = \mathbf{r}_j\}$ represents the set where the potential U in (1.2) is not defined.

From this local Theorem, it is clear that the main question of the n-body problem in Celestial Mechanics is not the existence of solutions of (1.5) but it is important to know explicitly some of them and also to understand the qualitative behaviour of the solutions associated to (1.5). Firstly, the easier solution in an autonomous system of ordinary differential equations are the equilibrium solutions, but in the n-body problem it is clear by the equations of motion that (1.1) does not have equilibrium solutions. In fact, if we assume the existence of an equilibrium solution of (1.1) we have $\nabla_{\mathbf{r}_i} U = 0$ $(i = 1, \ldots, n)$ so, $\mathbf{r}_i \cdot \nabla_{\mathbf{r}_i} U = 0$ $(i = 1, \ldots, n)$, then, $0 = \sum_{i=1}^{n} \mathbf{r}_i \cdot \nabla_{\mathbf{r}_i} U = -U$, because of the Euler's relation. Therefore, we obtain a contradiction, since $U > 0$ by definition (1.2).

An important element to try to understand the n-body problem is the following definition:

Definition 1.1 An integral of motion or first integral of (1.1) is a differentiable function $F : U \subset \mathbb{R}^{3n} \times \mathbb{R}^{3n} \longrightarrow \mathbb{R}$, ($U$ is open), such that it is constant along the solutions of the equations of motion (1.1) or (1.5).

By (1.1) we have that,

$$m_1 \ddot{\mathbf{r}}_1 + \ldots + m_n \ddot{\mathbf{r}}_n = 0.$$

Integrating this expression twice with respect to t, we obtain

$$m_1 \mathbf{r}_1 + \ldots + m_n \mathbf{r}_n = \mathbf{A}t + \mathbf{B}, \tag{1.6}$$

where $\mathbf{A}$ and $\mathbf{B}$ are constant vectors depending only on the initial conditions of the problem, so A and B are integrals of the movement. Let $\mathbf{R}$ be the vector of the *center* *of mass* of $m_1, m_2, \dots m_n$, that is,

$$\mathbf{R} = \frac{1}{M}\sum_{i=1}^{n} m_i \mathbf{r}_i, \quad \text{where} \quad M = \sum_{i=1}^{n} m_i.$$

Thus from (1.6) we have

$$\mathbf{R} = \frac{\mathbf{A}}{M}t + \frac{\mathbf{B}}{M}$$

where we get that the center of mass of the particles moves uniformly on a straight line in space. Notice that $\mathbf{A} = \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i$.

The motion of each m_i relative to the center of mass is given in the following way. Let

$$\mathbf{r}_i = \mathbf{R} + \underline{\mathbf{r}}_i, \tag{1.7}$$

therefore the vectors $\underline{\mathbf{r}}_i$ denote the vector position of each body, m_i, with respect to the center of mass of the bodies. Since $\underline{\mathbf{r}}_i - \underline{\mathbf{r}}_j = \mathbf{r}_i - \mathbf{r}_i$ and $\ddot{\mathbf{R}} = 0$, the equations of motion of the bodies in the coordinates $\underline{\mathbf{r}}_i$ to stay identical those given by (1.1). Therefore, from now on we will consider the center of mass of the fixed system in the origin, i.e., $\sum_{i=1}^{n} m_i \mathbf{r}_i = 0$.

Let $L = \sum_{i=1}^{n} \mathbf{p}_i$ be the *total linear momentum* of the system. As

$$\dot{\mathbf{p}}_i = \sum_{j=1}^{n} \frac{m_i m_j}{\|\mathbf{r}_i - \mathbf{r}_j\|^3}(\mathbf{r}_j - \mathbf{r}_i),$$

then

$$\dot{L} = \sum_{i=1}^{n} \dot{\mathbf{p}}_i = 0,$$

since each term in this sum appears twice with opposite signs. Therefore the linear momentum is a first integral of the system.

Let us consider now the *total angular momentum of the system*, i.e.,

$$\mathbf{C} = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{p}_i, \tag{1.8}$$

where $\times$ denotes the canonical vectorial product in $\mathbb{R}^3$. Being $\dot{\mathbf{C}} = \sum_{i=1}^{n} \dot{\mathbf{r}}_i \times \mathbf{p}_i + \sum_{i \neq j} \frac{m_i m_j \mathbf{r}_i \times (\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_i - \mathbf{r}_j\|^3}$ and by the relation $\mathbf{r}_i \times (\mathbf{r}_j - \mathbf{r}_i) = \mathbf{r}_i \times \mathbf{r}_j - \mathbf{r}_i \times \mathbf{r}_i$ each term in the second sum repeats twice with opposite signs, then $\dot{\mathbf{C}} = 0$. Therefore, the total angular momentum is a first integral of (1.1). We have, therefore, that $\mathbf{C}, \mathbf{L}$ (six functions) and the total energy (1.4) are integrals of motion of the problem of the n-bodies.

In this work we are mainly interested in the classification and the existence of some particular solutions of the n-body problem called, homographic solutions, but we will leave open, in general, the question of the existence of such solutions. Actually we ask a question: Is the question about the existence of homographic solutions still an open problem ? The answer to this problem is affirmative (except for $n = 3$), i.e., this problem continues open. This problem is in Wintner's book [16](1941) on celestial mechanics. Afterwards, we will explain why this open question is so difficult.

Definition 1.2 An homographic solution is a solution of the n-body problem characterized by the existence of a scalar function $r = r(t)$, a matrix $\Omega(t) \in SO(3)$ and a vector $\tau = \tau(t) \in \mathbb{R}^3$, such that, $\mathbf{r}_i(t) = r(t)\Omega(t)\mathbf{r}_i^0 + \tau(t)$, $i = 1, 2, \ldots, n$, where $\mathbf{r}_i, r, \Omega, \tau$ are defined for all t where the solution is defined and $\mathbf{r}_i^0 = \mathbf{r}_i(t_0)$ (according to definition in [16]).

It follows that an homographic solution $\mathbf{r}_i(t)$ $(i = 1, \ldots, n)$ is characterized by the existence of one rotation $\Omega(t)$ and a dilatation $r(t) > 0$, such that the translation vector $\tau(t)$ must vanish identically in view of the barycentric condition, i.e., the center of mass is in the origin of the coordinate system. Therefore, the homographic solutions are given by

$$\mathbf{r}_i(t) = r(t)\Omega(t)\mathbf{r}_i^0, \quad i = 1, \ldots, n. \tag{1.9}$$

As $\mathbf{r}_i^0 = \mathbf{r}_i(t_0)$ we have that

$$r^0 = r(t_0) = 1, \quad \Omega^0 = \Omega(t_0) = I, \tag{1.10}$$

where I is the identity matrix of order 3×3. Considering the change of coordinates $\mathbf{x}_i = \Omega^{-1}(t)\mathbf{r}_i$, $i = 1, \ldots, n$, the relation (1.9) takes us to the equation

$$\mathbf{x}_i = r\mathbf{r}_i^0 \quad i = 1, \ldots n, \tag{1.11}$$

i.e., if $\mathbf{r}_i(t)$ is a solution on the inertial system, then $\mathbf{x}_i(t) = r(t)\mathbf{r}_i^0$ is the solution of the system in rotating coordinates.

There exists two particular types of homographic solutions namely:

- (i) Homothetic solution. It is the homographic solution in (1.9), characterized by:

$$\mathbf{r}_i = r\mathbf{r}_i^0, \quad \text{i.e.,} \quad (\Omega(t) = I, r = r(t) > 0). \tag{1.12}$$

  This means that the configuration is dilated without rotation.

- (ii) Solution of relative equilibrium. It is the homographic solution characterized by:

$$\mathbf{r}_i = \Omega\mathbf{r}_i^0, \quad \text{i.e.,} \quad (r(t) = 1, \quad \Omega = \Omega(t)). \tag{1.13}$$

This means that the configuration is rotated without dilatation. Notice that for the equation (1.11) every solution of relative equilibrium in the rotating system corresponds to a equilibrium solution.

In this paper we will use essentially Wintner's book [16] to do the characterization and existence of the homographic solutions, also we will show the role and importance of this solution. We intend to put in evidence the ideias used by Wintner in the proofs in a very clear and easy sequence for a reader that is not familiar or accustomed to this kind of problem.

One of the purpose of this paper is to prove the following Theorem that classifies the homothetic solutions and the solutions of relative equilibrium:

Theorem A. (a) *An homographic solution is homothetic if and only if the angular momentum C vanishes.*

(b) *An homographic solution is a solution of relative equilibrium if and only if it is planar and it rotates with constant angular velocity different from zero.*

The proof of this Theorem will be carried out in section 3.

For the question about the existence of homographic solutions we need the following definition:

Definition 1.3 The n-position vector $(\mathbf{r}_i, \dots, \mathbf{r}_n)$ of the n bodies m_i will be said to form a central configuration with respect to n fixed positive constants m_i, if the force of gravitation acting on m_i at the moment of the given configuration is proportional to the mass m_i and to the position vector $\mathbf{r}_i$, i.e.,

$$\nabla_i U(\mathbf{r}_i, \dots, \mathbf{r}_n) = \sigma m_i \mathbf{r}_i, \ (i = 1, \dots, n), \text{ for some } \sigma \in \mathbb{R}. \tag{1.14}$$

In section 4 we will prove that if a solution $\mathbf{r}_i(t)$ belonging to n given m_i of the n-body problem is homographic, then the m_i must form a central configuration at every t, i.e.,

Theorem B. *Let $\mathbf{r}_i$ $(i = 1, \dots, n)$ be an homographic solution as in* (1.2) *of* (1.1) *with masses m_i, then the initial positions $(\mathbf{r}_1^0, \dots, \mathbf{r}_n^0)$ form a central configuration.*

It is clear by the definition 1.14 that if $(\mathbf{r}_1, \dots, \mathbf{r}_n)$ forms a central configuration then $(\lambda\mathbf{r}_1, \dots, \lambda\mathbf{r}_n)$ and $(\Omega\mathbf{r}_1, \dots, \Omega\mathbf{r}_n)$ are also central configurations with $\lambda \in \mathbb{R}$ $(\lambda > 0)$ and Ω an orthogonal transformation in $\mathbb{R}^3$.

Considering $(\mathbf{r}_1(t), \dots, \mathbf{r}_n(t))$ an homographic solution of (1.1) as in (1.2) let h and $\mathbf{C}$ be its energy and angular momentum respectively and let $\phi(t)$ be the angle function associated to the rotation $\Omega(t) \in SO(3)$. Defining,

$$\mu = \frac{U(\mathbf{r}_1^0, \ldots, \mathbf{r}_n^0)}{\sum_{i=1}^{n} m_i \|\mathbf{r}_i^0\|^2}; \; h^0 = \frac{h}{\sum_{i=1}^{n} m_i \|\mathbf{r}_i^0\|^2}; \; \mathbf{C}^0 = \frac{\mathbf{C}}{\sum_{i=1}^{n} m_i \|\mathbf{r}_i^0\|^2}.$$

The next Theorem shows us how to construct homographic solutions.

Theorem C. *A solution $(\mathbf{r}_1(t), \ldots, \mathbf{r}_n(t))$ of the n bodies, with given values m_i of the masses, is homographic if and only if there exists two functions $r(t)$, $\phi(t)$ and n initial position vectors $(\mathbf{r}_1^0, \ldots, \mathbf{r}_n^0)$ such that $\mathbf{r}_i(t) = r(t)\Omega(t)\mathbf{r}_i^0$ $(i = 1, \ldots, n)$ and*

$$\Omega(t) = \begin{pmatrix} \cos\phi(t) & -\sin\phi(t) & 0 \\ \sin\phi(t) & \cos\phi(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; h^0 = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{\mu}{r}, \; \|\mathbf{C}^0\| = r^2\dot{\phi},$$

while $(\mathbf{r}_1^0, \ldots, \mathbf{r}_n^0)$ is any central configuration belonging to $(m_1, \ldots, m_n)$.

It is clear from the above Theorem that $(r(t), \phi(t))$ must satisfy the equation of the Kepler problem in polar coordinates, so its solutions are conics, depending on the energy h^0 and $\mathbf{C}^0$. Therefore, it is necessary to know the central configuration to construct all the homographic solutions. The natural question here is: how much is known about the existence of the central configurations in the n-body problem ? The answer in general is a still open. For $n = 3$ it is well known that there exists exactly five central configurations, three collinear and two equilateral central configurations arising from the Eulerian (1767) [4] and Lagrangean (1772) [5] solutions. Moulton (1910) [8] showed that there exists exactly $n!/2$ collinear central configurations in the n-body problem, one for each ordering of the masses on the line. It is not known how large is the number of central configurations for $n \geq 4$, not even if this number is finite or not. Today a very important topic of research in Celestial Mecahanics is the notion of central configurations and many excellent mathematicians dedicate their efforts to the understanding of the problems arising here, for example: Albouy [1] (he proved that for four equal masses there exist exactly fifty central configurations), Saari [12] (he studied the role and properties of n-body central configurations), Xia [17] (for $n \geq 8$ Xia's estimatives shows that the number of central configurations, in the case of equal masses increases too fast). By the way, in 1998, Smale [15] during the lecture given on the occasion of Arnold's 60th birthday at the Fields Institute, Toronto, (June 1997) published a list with the mathematical problems for the next century.

Problem 6: Finiteness of the number of relative equilibria in celestial mechanics.

> *Is the number of relative equilibria finite, in the n-body problem of celestial mechanics, for any choice of positive real numbers $m_1, \ldots, m_n$ as the masses?*

For 4-bodies the finiteness is unknown.

In Smale [14], he interpreted the relative equilibria or the solution of relative equilibrium as critical points of a function induced by the potential of the planar n-body problem. More precisely the relative equilibria correspond to the critical points of

$$\hat{\mathrm{V}} : (S_k - \Delta)/SO(2) \to \mathbb{R} \tag{1.15}$$

where $S_k = \{\mathbf{r} \in (\mathbb{R}^2)^n \ / \ \sum_{i=1}^{n} m_i \mathbf{r}_i = 0, \ \frac{1}{2}\sum_{i=1}^{n} m_i \|\mathbf{r}_i\|^2 = 1\}$,

$$\Delta = \{\mathbf{r} \in S_k \ / \ \mathbf{r}_i = \mathbf{r}_j \ \text{ some } \ i \neq j\}.$$

The rotation group $SO(2)$ acts on $S_k - \Delta$ and $\hat{\mathrm{V}}$ is induced on the quotient from the potential function

$$V(\mathbf{r}) = \sum_{i<j} \frac{m_i m_j}{\|\mathbf{r}_i - \mathbf{r}_j\|}.$$

Note that $V : S_k \to \mathbb{R}$ is invariant under the rotation group $SO(2)$ and that the quotient space $S_k/SO(2)$ is homeomorphic to complex projective space of dimension $n-2$.

Thus the question has the equivalent form:

For any choice of $m_1, \ldots, m_n$*, does* $\hat{V}$ *in (1.15) have a finite number of critical points?*

Mike Shub (1970) [13] has shown that the set of critical points is compact.

2 Preliminaries

In this section we present some important preliminaries for one better understanding of this paper, as for example the deduction of some results of the theory of Ordinary Differential Equations and Linear Algebra used with frequency along this work.

Let us denote by $\mathcal{A}(3, \mathbb{R})$ the set of the skew-symmetric matrix of order 3×3 with real coefficient and by $\mathcal{SO}(3)$ the set of the orthogonal 3×3-matrix with determinant 1.

Let us consider the curve of class C^2, $\Omega = \Omega(t) \in \mathcal{SO}(3)$ for all t. So, $\Omega^T\Omega = I$ then $(\Omega^T\Omega)^{\cdot} = 0$ and $\Omega^{-1}\dot{\Omega} = -(\Omega^{-1}\dot{\Omega})^T$, this means that $\Omega^{-1}\dot{\Omega} \in \mathcal{A}(3, \mathbb{R})$, and by the isomorphism among $\mathcal{A}(3, \mathbb{R})$ and $\mathbb{R}^3$, associated to $\Omega = \Omega(t)$, there exists a vector $\mathbf{S} = \mathbf{S}(t) \in \mathbb{R}^3$ and a matrix $\Sigma = \Sigma(t) \in \mathcal{A}(3, \mathbb{R})$, such that

$$\mathbf{S} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \quad \text{and} \quad \Sigma = \Omega^{-1}\dot{\Omega} = \begin{pmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{pmatrix}. \tag{2.1}$$

Thus,

$$\dot{\Sigma} = (\Omega^T\dot{\Omega})^{\cdot} = \Omega^T\ddot{\Omega} + \dot{\Omega}^T\dot{\Omega}. \tag{2.2}$$

As $\Sigma^T = \dot{\Omega}^T\Omega$ then $\Sigma^T\Omega^{-1} = \dot{\Omega}^T$ and it follows that $-\Sigma\Omega^T = \dot{\Omega}^T$. Substituting this expression in (2.2) we have $\dot{\Sigma} = \Omega^{-1}\ddot{\Omega} - \Sigma\Omega^T\dot{\Omega} = \Omega^{-1}\ddot{\Omega} - \Sigma^2$ i.e.,

$$\Omega^{-1}\ddot{\Omega} = \dot{\Sigma} + \Sigma^2 \tag{2.3}$$

where $\Sigma^2 = (s_i s_k - \|\mathbf{S}\|^2 e_{ik})$, $\|\mathbf{S}\|^2 = s_1^2 + s_2^2 + s_3^2$ and (e_{ik}) is the element (i,k) of the identity matrix, i.e.,

$$\Sigma^2 = \begin{pmatrix} -(s_2^2+s_3^2) & s_1s_2 & s_1s_3 \\ s_1s_2 & -(s_1^2+s_3^2) & s_2s_3 \\ s_1s_3 & s_2s_3 & -(s_1^2+s_2^2) \end{pmatrix}. \tag{2.4}$$

Reciprocally, given $S(t) \in \mathbb{R}^3$ or $\Sigma(t) \in \mathcal{A}(3,\mathbb{R})$ both of class C^2, we state that always there exists $\Omega(t) \in \mathcal{SO}(3)$ such that $\Sigma(t) = \Omega^{-1}\dot{\Omega}$ and $\Omega(t)$ is uniquely determined by $S(t)$ and by an initial condition $\Omega(t_0)$ which can be chosen as an arbitrary orthogonal matrix. In fact, this is equivalent to verify that $\frac{d\Omega}{dt} = \Sigma(t)\Omega$, so we obtain from the theory of ordinary differential equations that $\Omega(t)$ is defined by

$$\Omega(t) = \Omega(t_0)\mathrm{e}^{\int_{t_0}^{T}\Sigma(t)dt}, \tag{2.5}$$

therefore $\Omega(t) \in \mathcal{SO}(3)$ for all t since $\Omega(t_0) \in \mathcal{SO}(3)$ and as $\Sigma(t)$ is an skew-symmetric, $\int_{t_0}^{t}\Sigma(\tau)d\tau$ is also skew-symmetric and it is known that e^A is an orthogonal matrix with determinat 1 since $A \in \mathcal{A}(3,\mathbb{R})$. It is verified that $\Omega(t)$ is unique, since from the existence and uniqueness theorem of Ordinary Differential Equations, there exists a unique $\Omega(t)$ with the same initial condition $\Omega(t_0)$.

Let us observe that the property above proved is invariant under $\mathcal{SO}(3)$. In fact, let us consider $\Omega(t)$ and P in $\mathcal{SO}(3)$, where P is constant, then $\tilde{\Omega}(t) = P\Omega(t)P^{-1} \in \mathcal{SO}(3)$, because the product of orthogonal matrix is orthogonal, then by (2.1) there exists $\tilde{\Sigma}(t) = \tilde{\Omega}(t)^{-1}\dot{\tilde{\Omega}}(t) \in \mathcal{A}(3,\mathbb{R})$, satisfying

$$\tilde{\Sigma}(t) = (P\Omega^{-1}(t)P^{-1})(P\dot{\Omega}(t)P^{-1}) = P\Sigma(t)P^{-1}.$$

Reciprocally, given $\tilde{\Sigma}(t) = P\Sigma(t)P^{-1} \in \mathcal{A}(3,\mathbb{R})$, we know that there exists a unique matrix $\tilde{\Omega}(t) \in \mathcal{SO}(3)$, defined by the relation (2.5), such that $\tilde{\Omega}^{-1}(t)\dot{\tilde{\Omega}}(t) = \tilde{\Sigma}(t)$. Choosing $\tilde{\Omega}(t) = P\Omega(t)P^{-1} \in \mathcal{SO}(3)$, it is verified that $\tilde{\Omega}^{-1}(t)\dot{\tilde{\Omega}}(t) = (P\Omega^{-1}(t)P^{-1})P\dot{\Omega}(t)P^{-1} = P\Sigma(t)P^{-1} = \tilde{\Sigma}(t)$.

Let $\mathbf{E} \in \mathbb{R}^3$ and a rotation Ω defined in the following way $\mathbf{E} = (\xi,\eta,\zeta)^T$, $\Omega = (a_{ij})$, $a_{ij} \in \mathbb{R}$, and let us define the change of coordinates $\mathbf{X} = \Omega^{-1}\mathbf{E}$ with $\mathbf{X} = (x,y,z)^T$. In this case, the coordinates (ξ,η,ζ) represent a fixed coordinate system in $\mathbb{R}^3$ and (x,y,z) represent a rotating coordinates system in $\mathbb{R}^3$. Sometimes this change of coordinates is too convenient for studying the motion of a particle.

Using the previous notations and supposing that $\mathbf{E} = \mathbf{E}(t)$ represents a curve in $\mathbb{R}^3$ of class C^2, $\Omega = \Omega(t)$ a curve in $\mathcal{SO}(3)$ of class C^2 and by definition $\mathbf{X} = \mathbf{X}(t) = \Omega^{-1}(t)\mathbf{E}(t)$ is a curve of class C^2 in $\mathbb{R}^3$, the following identities are verified

$$\Sigma\mathbf{X} = \mathbf{S} \times \mathbf{X}, \quad \dot{\Sigma}\mathbf{X} = \dot{\mathbf{S}} \times \mathbf{X}, \quad \Sigma^2\mathbf{X} = (\mathbf{S}{\cdot}\mathbf{X})\mathbf{S} - (\mathbf{S}{\cdot}\mathbf{S})\mathbf{X}. \tag{2.6}$$

Computing the derivative of $\mathbf{E}$ we obtain, $\dot{\mathbf{E}} = \dot{\Omega}\mathbf{X} + \Omega\dot{\mathbf{X}}$ and $\ddot{\mathbf{E}} = \Omega\ddot{\mathbf{X}} + 2\dot{\Omega}\dot{\mathbf{X}} + \ddot{\Omega}\mathbf{X}$, thus using (2.2) and (2.3) it is proceeded that

$$\begin{aligned} \Omega^{-1}\dot{\mathbf{E}} &= \dot{\mathbf{X}} + \Sigma\mathbf{X}, \\ \Omega^{-1}\ddot{\mathbf{E}} &= \ddot{\mathbf{X}} + 2\Sigma\dot{\mathbf{X}} + (\dot{\Sigma} + \Sigma^2)\mathbf{X}. \end{aligned} \tag{2.7}$$

Finally, we have

$$\begin{aligned} \Omega^{-1}\dot{\mathbf{E}} &= \dot{\mathbf{X}} + \mathbf{S} \times \mathbf{X}, \\ \Omega^{-1}(\mathbf{E} \times \dot{\mathbf{E}}) &= \mathbf{X} \times (\dot{\mathbf{X}} + \mathbf{S} \times \mathbf{X}), \\ \Omega^{-1}\ddot{\mathbf{E}} &= \ddot{\mathbf{X}} + 2\mathbf{S} \times \dot{\mathbf{X}} + \dot{\mathbf{S}} \times \mathbf{X} + (\mathbf{S}{\cdot}\mathbf{X})\mathbf{S} - (\mathbf{S}{\cdot}\mathbf{S})\mathbf{X}. \end{aligned} \tag{2.8}$$

Let us choose now a rotation $\Omega(t)$ such that the particle stays all the time t in the (x, y)-plane of the system of rotating coordinates, that is, $z(t) = 0$, and therefore $\dot{z} = 0$. Notice that the curve $E = E(t)$ is in the(ξ, η)-plane, it is always possible to find a rotation $\Omega(t)$ convenient such that in each instant in the rotating system the particles lie in the (x, y)-plane. Thus the condition such that the (x, y)-plane of the system of rotating coordinates (x, y, z) rotates on (ξ, η)-plane of the system of non-rotating coordinates (ξ, η, ζ) and by (2.1) it is equivalent to the condition

$$\Sigma = \begin{pmatrix} 0 & -s_3 & 0 \\ s_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{or} \quad \mathbf{S} = \begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix}, \tag{2.9}$$

and also

$$\Omega = \begin{pmatrix} cos\phi & -sin\phi & 0 \\ sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.10}$$

In general, let us observe that the condition such that $\Omega(t)$ is a rotation on some fixed axis (in this case the rotation is said of invariable position) is that $\Sigma(t)$ is constant, i.e., the component s_i of the vector S are constant.

Using definition and by (2.1) we have from the first relation in (2.8) that

$$\Omega^{-1}\dot{\mathbf{E}} = \dot{\mathbf{X}} + \Sigma\mathbf{X} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \tag{2.11}$$

therefore

$$\Omega^{-1}\dot{\mathbf{E}} = \begin{pmatrix} \dot{x} - s_3 y \\ \dot{y} + s_3 x \\ -s_2 x + s_1 y \end{pmatrix}. \tag{2.12}$$

On the other hand being $\mathbf{S} \times \mathbf{X} = (-s_3 y, s_3 x, s_1 y - s_2 x)$ and $\dot{\mathbf{X}} + \mathbf{S} \times \mathbf{X} = (\dot{x} - s_3 y, \dot{y} + s_3 x, s_1 y - s_2 x)$ the second expression in (2.8) takes the form

$$\begin{pmatrix} s_1 y^2 - s_2 xy \\ s_2 x^2 - s_1 xy \\ x\dot{y} - y\dot{x} + s_3(x^2 + y^2) \end{pmatrix}. \tag{2.13}$$

We say that a matrix of 3×3, is in the normal form if, under orthogal transformation, we can make the third column or row null, i.e., $A \in \mathcal{M}_{3\times 3}(\mathbb{R})$ is in the normal form if $P \in \mathcal{S}O(3)$ such that PAP^{-1} has the third row or column null. Therefore, all skew-symmetric matrix 3×3 can be put in the normal form, because zero is an eigenvalue.

Let us observe that the rotation $\Omega(t)$ is said to be invariant position if, and only if, there exists a constant ortogonal matrix P, such that all elements of the third row and column of the matrix $\tilde{\Sigma} = P\Sigma P^{-1} \in \mathcal{A}(3, \mathbb{R})$ vanish for every t, where Σ is given by (2.1).

In fact, if $\Omega(t) \in \mathcal{S}O(3)$, $\Sigma(t) = \Omega^{-1}\dot{\Omega} \in \mathcal{A}(3, \mathbb{R})$. By what was exposed previously the affirmation follows. Reciprocally, let $\tilde{\Sigma} = P\Sigma P^{-1} \in \mathcal{A}(3, \mathbb{R})$ be a matrix with all elements of the third row or column null. Being Σ skewsymmetric there exists $\mathbf{v} \in \mathbb{R}^3$, $\mathbf{v} \neq 0$, such that $\Sigma \mathbf{v} = 0$. As $\Omega(t)$ is given by, (2.5), i.e., $\Omega(t)\mathbf{v} = \Omega(t_0)e^{\int \Sigma(t)dt}\mathbf{v} = \Omega(t_0)[I + A + \frac{A^2}{2!} + \ldots]\mathbf{v}$. Defining $A = \int \Sigma(t)dt \in \mathcal{A}(3, \mathbb{R})$ it satisfies $A\mathbf{v} = 0$. Considering $\Omega(t_0) = I$ it follows that $\Omega(t)\mathbf{v} = \mathbf{v}$, and the proof is complete.

3 Classification of the homographic solutions

Let be $(\mathbf{r}_1, \ldots, \mathbf{r}_n)$ an homographic solution of the n-body problem with initial positions given by $(\mathbf{r}_1^0, \ldots, \mathbf{r}_n^0)$ in the instant $t = t_0$. From now on we will use the following notation:

$$\mathbf{r}_i = (x_i, y_i, z_i) \in \mathbb{R}^3, \tag{3.1}$$

it follows that

$$C = \sum_{i=1}^{n} m_i(\dot{y}_i z_i - y_i \dot{z}_i, \dot{z}_i x_i - z_i \dot{x}_i, \dot{x}_i y_i - x_i \dot{y}_i). \tag{3.2}$$

Let us define the moment of inertia of the n-particles by $J = \sum_{i=1}^{n} m_i \|\mathbf{r}_i\|^2$. Then, defining, $J^0 = J(\mathbf{r}_1^0, \ldots, \mathbf{r}_n^0)$, $U^0 = U(\mathbf{r}_1^0, \ldots, \mathbf{r}_n^0)$, $U^0_{\mathbf{r_i}} = U_{\mathbf{r_i}}(\mathbf{r}_1^0, \ldots, \mathbf{r}_n^0)$, we have that

$$J = J^0 r^2,\ U = \frac{U^0}{r},\ \Omega^{-1} U_{\mathbf{r}_i} = \frac{U^0_{\mathbf{r}_i}}{r^2}. \tag{3.3}$$

Let Σ be as in (2.1). Computing the derivative in (1.9) we have $\dot{\mathbf{r}}_i = (\dot{r}\Omega + r\dot{\Omega})\mathbf{r}_i^0$, as in the preliminaries we define the skewsymmetric matrix $\Sigma = \Omega^{-1}\dot{\Omega}$, so we have

$$\Omega^{-1}\dot{\mathbf{r}}_i = (\dot{r}I + r\Sigma)\mathbf{r}_i^0. \tag{3.4}$$

On the other hand derivating twice the expression (1.9), using (2.1) and (2.3) we obtain the relation

$$\Omega^{-1}\ddot{\mathbf{r}}_i = [\ddot{r}I + 2\dot{r}\Sigma + r(\dot{\Sigma} + \Sigma^2)]\mathbf{r}_i^0. \tag{3.5}$$

Defining the constant vectors in $\mathbb{R}^3$, $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3$, by the relation $U^0_{\mathbf{r}_i} = m_i \mathbf{a}_i$, along a homographic solution $\mathbf{r}_i = \mathbf{r}_i(t)$ of the equation $m_i\ddot{\mathbf{r}}_i = U_{\mathbf{r}_i}$, we have

$$K(t)\mathbf{r}_i^0 = \mathbf{a}_i \tag{3.6}$$

where

$$r^2[\ddot{r}I + 2\dot{r}\Sigma + r(\dot{\Sigma} + \Sigma^2)] = K = (k_{pq}). \tag{3.7}$$

In fact, being $U_{\mathbf{r}_i^0} = m_i\mathbf{a}_i$ and $m_i\ddot{\mathbf{r}}_i = U_{\mathbf{r}_i}$ it follows from (3.3) that $r^2\Omega^{-1}\ddot{\mathbf{r}}_i = \mathbf{a}_i$, by (3.5) we have $r^2[\ddot{r}I + 2\dot{r}\Sigma + r(\dot{\Sigma} + \Sigma^2)]\mathbf{r}_i^0 = \mathbf{a}_i$, then taking $K(t)$ defined by (3.7), we obtain the wanted expression.

It is clear that

$$(\Sigma^2)^T = \Sigma^2, \quad \Sigma^T = -\Sigma \quad \text{and} \quad \dot{\Sigma}^T = -\dot{\Sigma}, \tag{3.8}$$

then, $K^T = r^2[\ddot{r}I + 2\dot{r}\Sigma^T + r\dot{\Sigma}^T + \Sigma^T + r(\Sigma^2)^T] = r^2[\ddot{r}I - r\dot{r}\Sigma - r\dot{\Sigma} + r\Sigma^2]$, and follows that

$$\frac{1}{2}(K + K^T) = r^2(\ddot{r}I + r\Sigma^2) \tag{3.9}$$

and

$$\frac{1}{2}(K - K^T) = r^2(r\dot{\Sigma} + 2\dot{r}\Sigma). \tag{3.10}$$

The above formulae allow us an essential simplification in the special case in which the solution $\mathbf{r}_i = \mathbf{r}_i(t)$ is planar. In this case, let us choose convenient coordinates $\mathbf{r}$ such that the third component of the vector $\mathbf{r}_i(t)$, be null for all t, i.e., Ω is given as in (2.10) and Σ is given by (2.9) where $s_3 = \dot{\phi} \geq 0$ denotes the angular velocity of the system in rotating coordinates $\mathbf{x} = \Omega^{-1}\mathbf{r}$ defined by $\dot{\phi} = 0$, in the non-planar case. Substituting (2.1) in (3.7), we obtain

$$K(t) = \begin{pmatrix} r^2(\ddot{r} - r\dot{\phi}^2) & -r^2(r\ddot{\phi} + 2\dot{r}\dot{\phi}) & 0 \\ r^2(r\ddot{\phi} + 2\dot{r}\dot{\phi}) & r^2(\ddot{r} - r\dot{\phi}^2) & 0 \\ 0 & 0 & r^2\ddot{r} \end{pmatrix}. \tag{3.11}$$

We will discuss now some identities in the planar case that will be useful in the proof of the results in the next section.

First, it is easy to verify that $\|\dot{\mathbf{r}}_i^2\| = \|\dot{r}\mathbf{r}_i^0\|^2 + \|r\dot{\phi}\mathbf{r}_i^0\|^2$. In fact, we observe that taking the dot product of (3.4) to itself, we obtain $\|\dot{\mathbf{r}}_i\|^2 = \|\dot{r}\mathbf{r}_i^0\|^2 + 2\dot{r}r\langle\mathbf{r}_i^0, \Sigma\mathbf{r}_i^0\rangle + \|r\Sigma\mathbf{r}_i^0\|^2$ as $\langle\Sigma\mathbf{r}_i^0, \mathbf{r}_i^0\rangle = \langle\mathbf{r}_i^0, \Sigma^T\mathbf{r}_i^0\rangle = -\langle\mathbf{r}_i^0, \Sigma\mathbf{r}_i^0\rangle$, the term intermediates of this expression is canceled, thus

$$\|\dot{\mathbf{r}}_i\|^2 = \|\dot{r}\mathbf{r}_i^0\|^2 + \|r\Sigma\mathbf{r}_i^0\|^2. \tag{3.12}$$

For $t = t_0$ we denoted $\mathbf{r}_i^0$ by $(x_i, y_i, 0)$. Applying Σ to $\mathbf{r}_i^0$, we obtain that $\|\Sigma\mathbf{r}_i^0\|^2 = |\dot{\phi}|^2\|\mathbf{r}_i^0\|^2$, substituting this expression in (3.12) we verify our statement. Another property is that by (3.2) the components of the vector $\mathbf{r}_i \times \dot{\mathbf{r}}_i$ in $\mathbb{R}^3$ are $(0, 0, \dot{\phi}\|r\mathbf{r}_i^0\|^2)$. In fact, let us denote $\mathbf{r}_i(t)$ by $(x_i(t), y_i(t), 0)$, is $\mathbf{r}_i \times \dot{\mathbf{r}}_i = (0, 0, x_i\dot{y}_i - y_i\dot{x}_i)$, using (1.9) we have $(x_i, y_i, 0) = r(x_i^0 cos\phi - y_i^0 sin\phi, x_i^0 sin\phi + y_i^0 cos\phi, 0)$. Developing this system and calculating $\dot{x}_i, \dot{y}_i$ we find $x_i\dot{y}_i - y_i\dot{x}_i = r^2\dot{\phi}[(x_i^0)^2 + (y_i^0)^2] = \dot{\phi}\|r\mathbf{r}_i^0\|^2$, then $\mathbf{r}_i \times \dot{\mathbf{r}}_i = (0, 0, \dot{\phi}\|r\mathbf{r}_i^0\|^2) = (0, 0, r^2\dot{\phi}\|\mathbf{r}_i^0\|^2)$.

Being $\|\dot{\mathbf{r}}_i\|^2 = \|\dot{r}\mathbf{r}_i^0\|^2 + \|r\dot{\phi}\mathbf{r}_i^0\|^2 = (\dot{r}^2 + r^2\dot{\phi}^2)\|\mathbf{r}_i^0\|^2$, we will have that

$$T = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2)J^0. \tag{3.13}$$

By definition of $\mathbf{C}$ (see (1.8)) and using the previous relations, we get

$$\mathbf{C} = (0, 0, m_1\dot{\phi}r^2\|\mathbf{r}_1^0\|^2 + \ldots + m_n\dot{\phi}r^2\|\mathbf{r}_n^0\|^2) = (0, 0, \dot{\phi}r^2\sum_{i=1}^{n} m_i\|\mathbf{r}_i^0\|^2) = (0, 0, \dot{\phi}r^2J^0), \tag{3.14}$$

then $\|\mathbf{C}\| = |\dot{\phi}|r^2J^0$. We know that $\mathbf{C}$ is a constant of motion, then we have that $\dot{\phi} = 0$ or the sign of $\dot{\phi}$ is constant for all t, so we can suppose without loss of generality, $\dot{\phi} \geq 0$, therefore

$$\dot{\phi}r^2J^0 = \|\mathbf{C}\|, \quad (J^0 > 0). \tag{3.15}$$

Now, we are going to analyse the homographic solution. Considering $\mathbf{r}_i = \mathbf{r}_i(t)$ a homothetic solution and also a solution of relative equilibrium for the n-body problem, it follows that $r\mathbf{r}_i^0 = \Omega\mathbf{r}_i^0$, i.e., $(rI - \Omega)\mathbf{r}_i^0 = 0$ for all t. It follows that $r = 1$ and $\Omega = I$. Therefore $\mathbf{r}_i(t) = \mathbf{r}_i^0$, $(i = 1, \ldots, n)$, and so it is an equilibrium solution which is absurd, because we know that there exists no equilibrium solution for this problem. Therefore, a homographic solution cannot be simultaneously homothetic and a solution of relative equilibrium. Also, it is important to observe that if a solution is homographic in a coordinate system obtained of the inercial one by a rotation of coordinates around a fixed axis, it continues to be a homographic solution in the inertial system. The same remarks are true for the homothetic solutions and the equilibrium relative solutions.

Before proving the following Lemma we introduce the definitions:

Definition 3.1

- A given solution of the n-body problem (1.1) will be called planar if there exists a plane π which contains all n bodies for every t.
- A given solution of the n-body problem (1.1) will be called flat if there exists for every t a plane $\pi(t)$ which contains all n bodies at this t.

- A given solution of the n-body problem (1.1) will be called rectilinear if there exists a line l which contains all n bodies for every t.

- A given solution of the n-body problem (1.1) will be called collinear if there exists for every t a line $l(t)$ which contains all n bodies at this t.

Remarks. 1) *It is clear that every planar solution is flat and every rectilinear solution is collinear but in both cases the reciprocal is not true.*

2) *If one solution of* (1.1) *is rectilinear or homothetic then* $\mathbf{C} = 0$.

3) *We notice that* $n > 3$ *is a necessary condition for the existence a non-flat solution of the* n*-body problem.*

4) *If* $(\mathbf{r}_1, \ldots, \mathbf{r}_n)$ *is a given non-flat homographic solution, then all* n *initial position vectors* $(\mathbf{r}_i^0, \ldots, \mathbf{r}_n^0)$ *are co-planar, and so one can select three values of* i*, say* $i = \alpha, \beta, \gamma$*, such that* $\mathbf{r}_\alpha, \mathbf{r}_\beta, \mathbf{r}_\gamma$ *are independent. This last statement deserves some explanation. The idea of the proof is to suppose the contrary, i.e., all combinations of three vectors* $\mathbf{r}_i^0, \mathbf{r}_j^0, \mathbf{r}_k^0$ *are dependent linearly, so we conclude that the solution is flat, which is a contradiction.*

5) *If a solution of the* n*-body problem is flat, then we can choose the rotation* $\Omega(t)$ *(analytic function of* t*, because every solution* $(\mathbf{r}_1, \ldots, \mathbf{r}_n)$ *of the analytic differential equations is analytic) so that the* n *bodies is, for every* t*, in* (ξ, η)*-plane of the rotating coordinate system* $\Omega(t)^{-1}(x, y, z)^T = (\xi, \eta, \zeta)^T$*, i.e.,* $\zeta(t) = 0$ *for every* t.

6) *If* $(\mathbf{r}_1, \ldots, \mathbf{r}_n)$ *(in the non-rotating coordinate system* (x, y, z)*) is collinear then it is flat, and the solution is planar.* We divide the prove of this fact in two cases. First, if the angular momentum $\mathbf{C} \neq 0$, we have $\mathbf{r}_i \times \mathbf{r}_j = 0$ where $i, j = 1, \ldots, n$. Hence, $(\mathbf{r}_i \times \mathbf{r}_j) \cdot \dot{\mathbf{r}}_i = 0$, then $(\mathbf{r}_i \times \dot{\mathbf{r}}_i) \cdot \mathbf{r}_j = 0$, and so $0 = \mathbf{C} \cdot \mathbf{r}_k = 0$ where $k = 1, \ldots, n$. Thus, the proof is complete in this case. Second, if $\mathbf{C} = 0$ we need some properties. Let $\pi(t)$ be the plane that contains the n bodies for every t. So, we can always choose, by remark (5), the rotation $\Omega(t)$ which rotates around the centre of mass, so that the particles $\mathbf{r}_i(t)$ is in the (ξ, η)-plane of the rotating coordinates $\Omega(t)^{-1}(x, , y, z)^T = (\xi, \eta, \zeta)^T$; i.e., $\zeta_i(t) = 0$ $(i = 1, \ldots, n)$. Defining,

$$\begin{aligned} &(a) \quad J^{xx} = \sum_{i=1}^{n} m_i \xi_i^2; \quad J^{yy} = \sum_{i=1}^{n} m_i \eta_i^2; \quad J^{xy} = \sum_{i=1}^{n} m_i \xi_i \eta_i \\ &(b) \quad K = \sum_{i=1}^{n} m_i (\xi_i \dot{\eta}_i - \eta_i \dot{\xi}_i), \end{aligned} \tag{3.16}$$

and considering $\mathbf{S}$ as in (2.1)(a), we have the following relations

$$(i) \quad \Omega^{-1}\begin{pmatrix} 0 \\ 0 \\ \|\mathbf{C}\| \end{pmatrix} = \begin{pmatrix} s_1 J^{yy} - s_2 J^{xy} \\ s_2 J^{xx} - s_1 J^{xy} \\ K + s_3(J^{xx} + J^{yy}) \end{pmatrix};$$

$$(ii) \quad \begin{vmatrix} J^{xx} & J^{xy} \\ J^{xy} & J^{yy} \end{vmatrix} = \sum_{1\le j<k\le n} m_j m_k \begin{vmatrix} \xi_j & \xi_k \\ \eta_j & \eta_k \end{vmatrix}^2; \tag{3.17}$$

$$(iii) \quad J^{xx} + J^{yy} = J.$$

Since $\mathbf{C} = 0$, the relation (3.17) (i), implies that we have the following homogeneous systems

$$\begin{pmatrix} J^{yy} & -J^{xy} \\ -J^{xy} & J^{xx} \end{pmatrix}\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.18}$$

As $s_1(t)$, $s_2(t)$, $s_3(t)$ and $J^{xx}(t)J^{yy}(t) - J^{xy}(t)^2$ are analytic functions the system (3.18) implies that

$$\begin{array}{ll} (a) & s_1(t) = s_2(t) = 0, \quad \text{for every} \quad t, \quad \text{or} \\ (b) & J^{xx}(t)J^{yy}(t) - J^{xy}(t)^2 = 0, \quad \text{for every} \quad t. \end{array}$$

In the case (a), we have that $\Omega(t)$ is a rotation around the ζ-axis. Then, the rotating coordinate system (ξ, ζ, η) coincides with the z-axis of the fixed coordinates system (x, y, z). So, the solution $(\mathbf{r}_1(t), \ldots, \mathbf{r}_n(t))$ is planar.

In the case (b) by (3.17) (ii) $\xi_j\eta_k - \xi_k\eta_j = 0$ for $1 \le i < k \le n$. It follows, that the area of the triangle formed by the origin and any two of the n masses vanishes identically; and so the n masses are collinear for every t. Then, the rotating coordinate system (ξ, η, ζ) can be chosen so that all n masses are in the ξ-axis for every t. Therefore, $\eta_i(t) = 0$ for every t and $i = 1, \ldots, n$. Consequently, by (3.16), $J^{yy} = J^{xy} = K = 0$ and by (3.17) (iii) $J^{xx} = J$. Because, $\mathbf{C} = 0$, (3.17)(i) implies that $s_2 J = s_3 J = 0$, for every t. As, $J \neq 0$ it follows that $s_2(t) = s_3(t) = 0$ for every t; so that the proof is complete.

Now, we can prove the Lemma:

Lemma 3.1 *If an homographic solution is not-flat, then it is homothetic.*

Proof. Let $(\mathbf{r}_1, \ldots, \mathbf{r}_n)$ be a given non-flat homographic solution, then by the above remark we can select three values $i = \alpha, \beta, \gamma$ such that $\det B \neq 0$, where $B = (\mathbf{r}^0_\alpha, \mathbf{r}^0_\beta, \mathbf{r}^0_\gamma)$ is a constant invertible matrix of order 3×3.

On the other hand by the definition of the vectors $\mathbf{a}_i$ the matrix $D = (\mathbf{a}_\alpha, \mathbf{a}_\beta, \mathbf{a}_\gamma)$ of 3×3 is a constant matrix. Using (3.6), we see that $K = DB^{-1}$ is also constant, it follows therefore from (3.9) and (3.10) that

$$r^2\ddot{r}I + r^3\Sigma^2 = \text{const.}, \tag{3.19}$$

and

$$r^3\dot{\Sigma} + 2r^2\dot{r}\Sigma = \text{const.} \tag{3.20}$$

On the other hand, $r^2\ddot{r}I$ is a diagonal matrix and, the difference of two diagonal elements of the 3×3 matrix $r^3\Sigma^2$, when compared with Σ^2, shows that $r^3 s_\mu s_\nu$ and $r^3(s_\mu^2 - s_\nu^2)$ are independent of t, where $(\mu,\nu) = \{(1,2),(2,3),(3,1)\}$ and consequently

$$r^3 s_\lambda^2 = \text{const.}, \quad (\lambda = 1,2,3). \tag{3.21}$$

In fact, by (3.19), $r^3\Sigma^2 = \text{const.} - r^2\ddot{r}I$, and therefore for elements out of the diagonal of this matrix we have $r^3 s_\mu s_\nu = \text{const.}$ Now for the diagonal elements, take the difference two by two of these elements, to obtain $r^3(s_\mu^2 - s_\nu^2) = \text{const.}$ From (2.1) we have that Σ and r depend of t and by (3.21) $\Sigma = r^{-\frac{3}{2}}\Sigma^0$, where Σ^0 is a constant matrix in $\mathcal{A}(3,\mathbb{R})$. By the remarks of the section 2, there exists a constant matrix, P_0, belonging to $\mathcal{SO}(3)$ such that $P_0\Sigma_0 P_0^{-1}$ belongs to $\mathcal{A}(3,\mathbb{R})$, where all the elements of the third column are null. Thus, from the observation done in section 2, $\Omega(t)$ is a rotation around invariable position, with reference to the system (x,y,z). Choosing Ω to be one rotation around the axis z, therefore, it is given by a matrix as in (2.10). Taking Σ as in (2.1), assuming $s_1 = s_2 = 0$ and $s_3 = \dot{\phi}$, we will show that $\dot{\phi} = 0$. For this we will use the fact that $r^3\dot{\phi}^2 = \text{const.}$, obtained when we consider $\lambda = 3$ in (3.21). Using (3.14) we obtain that the third component of $\mathbf{C}$ is $C_3 = r^2\dot{\phi}c$, with $c = \sum m_i |x_i^0|^2 + |y_i^0|^2$. If $c = 0$ then $x_i^0 = y_i^0 = 0$, it follows that all the n-bodies m_i, are on the axis z for every t, this means that the solution $\mathbf{r}_i(t)$ is flat, an absurde, it follows that $c \neq 0$. Then, $r^2\dot{\phi} = \frac{C_3}{c} = \text{const.}$, and on the other hand $r^3\dot{\phi}^2 = \text{const.}$, we see that either $\dot{\phi} = 0$ or the function r, which is positive, is independent of t. Supposing that $r = \text{const.}$, then by the choice of Ω we have that $z_i(t) = z_i^0$ for every t and for every $i = 1,\ldots,n$, i.e,

$$0 = m_i\ddot{z}_i = \sum_{j=1, j\neq i}^{n} \frac{m_i m_j}{\|\mathbf{r}_i - \mathbf{r}_j\|^3}(z_j - z_i), \quad (i = 1,\ldots,n). \tag{3.22}$$

We can take i such that $z_j - z_i \geq 0$ and there exist $j \in \{1,\ldots,n\}$ with $z_j - z_i > 0$ because the solution is not flat. Thus, the assumption $r = \text{const.}$ implies a contradiction.

Therefore, r depends on t and therefore $\dot{\phi} = 0$. So, $\phi = \text{const.}$ and $\Omega = I$, then for (1.12) the solution is homothetic, the proof is complete. ∎

Lemma 3.2 *If a homographic solution is flat, then it is planar.*

Proof. If the homographic solution is collinear then by remark (6) the solution is planar, so we will assume that $\mathbf{r}_i = \mathbf{r}_i(t)$ is a flat homographic non-collinear solution. Then, there exists, among the n initial position vectors $\mathbf{r}_i^0$, at least two, say $(\mathbf{r}_\alpha^0, \mathbf{r}_\beta^0)$, such that $\mathbf{r}_\alpha^0 \times \mathbf{r}_\beta^0 \neq 0$. Since the solution is flat, all n initial position vectors $\mathbf{r}_i^0$ lie in one and the same plane through the origin of the coordinate system $\mathbf{r} = (x,y,z)$, which can be chosen according to the preliminaries to coincide with the plane (x,y), so $z_i^0 = 0$ for every i. Hence, denoting $\mathbf{a}_i = (a_i^1, a_i^2, a_i^3)$ (defining as previously), we have by (3.6)

$$k_{\gamma 1}(t)x_i^0 + k_{\gamma 2}(t)y_i^0 = a_i^\gamma \quad (\gamma = 1, 2, 3;\ i = 1, 2, \ldots, n), \tag{3.23}$$

so that, by the definition of the vector $U^0_{\mathbf{r}_i}$ it follows that $a_i^3 = 0$. Fixing $i = \alpha, \beta$ in (3.23), we obtain two linear equations with constant coefficients

$$\begin{aligned} k_{\gamma_1}(t)x_\alpha^0 + k_{\gamma 2}(t)y_\alpha^0 = a_\alpha^\gamma \\ k_{\gamma 1}(t)x_\beta^0 + k_{\beta 2}(t)y_\beta^0 = a_\beta^\gamma, \end{aligned} \tag{3.24}$$

whose matricial form is $BX = a$, where $X = (k_{\gamma 1}(t), k_{\gamma 2}(t))$, $\quad a = (a_\alpha^\gamma, a_\beta^\gamma)$ and B is a 2×2 matrix

$$\begin{pmatrix} x_\alpha^0 & y_\alpha^0 \\ \\ x_\beta^0 & y_\beta^0 \end{pmatrix}. \tag{3.25}$$

Since $\mathbf{r}_\alpha^0 \times \mathbf{r}_\beta^0 \neq 0$, we have that $\det B \neq 0$, therefore it is easy to see using the system $X = B^{-1}a$, that for $\gamma = 1, 2$ the two scalars $k_{\gamma 1}, k_{\gamma 2}$ are constant, because they are linear combinations, with constant coefficients, of the two scalars, $a_\alpha^\gamma, a_\beta^\gamma$ and vanish for $\gamma = 3$, so

$$k_{12} + k_{21} = const, \quad k_{11} - k_{22} = \text{const.}, \quad k_{31} = 0, \quad k_{32} = 0. \tag{3.26}$$

Then, from substitution of (2.1) in (3.7) these relations become respectively

$$\begin{aligned} &r^3 s_1 s_2 = \text{const.}, \quad r^3(s_1^2 - s_2^2) = \text{const.} \\ &-2\dot{r}s_2 + r(-\dot{s}_2 + s_3 s_1) = 0 \\ &-2\dot{r}s_1 + r(\dot{s}_1 + s_3 s_2) = 0, \end{aligned} \tag{3.27}$$

therefore from the first relation above we have $r^3 s_1^2 = u, \quad r^3 s_2^2 = v$, where u, v are constant, it is follows that

$$s_1 = c_1 r^{-\frac{3}{2}}, \quad s_2 = c_2 r^{-\frac{3}{2}}, \text{where} \quad c_1 = u^{-\frac{3}{2}}, \quad c_2 = v^{-\frac{3}{2}} \quad and \quad r > 0 \tag{3.28}$$

Substituting s_1, s_2 in the two last equations in (3.27), for the expressions obtained above we find

$$\begin{aligned} &s_3 r c_1 - \frac{1}{2}\dot{r}c_2 = 0 \\ &\frac{1}{2}\dot{r}c_1 + s_3 r c_2 = 0 \end{aligned} \tag{3.29}$$

a system of linear homogeneous equations of the type $QY = 0$, where Q is the matrix of 2×2

$$\begin{pmatrix} s_3 r & -\frac{1}{2}\dot{r} \\ \frac{1}{2}\dot{r} & s_3 r \end{pmatrix}, \tag{3.30}$$

whose determinant $s_3^2 r^2 + \frac{1}{4}\dot{r}^2 = 0$ if and only if, $s_3 = \dot{r} = 0$. Let us observe that if at least one of the constants c_1 or c_2 is different from zero, the equations (3.30) show that the functions s_3 and $\dot{r}$ are null for every t, and thus one of the following conditions must be satisfied

$$\begin{array}{ll} (a) & c_1 = 0, \quad c_2 = 0 \\ (b) & s_3(t) = 0, \quad r(t) = \text{const.}. \end{array} \tag{3.31}$$

If (a) is true, it follows from (3.28) that $s_1 = s_2 = 0$ for all t, therefore the matrix Σ becomes

$$\begin{pmatrix} 0 & -s_3 & 0 \\ s_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.32}$$

As we saw previously, this means that the rotation $\Omega(t)$ is a rotation around the z-axis of the coordinate system. As the choice of the z-axis it done so that $z = 0$, then the system (1.9) show that $z_i^0(t) = 0$ for all t, and the movement happens in the plane (x, y) and the solution is planar. This proves the theorem in the case (a).

Let us suppose now that (b) is true, then clearly (3.28) shows that s_1, s_2, s_3 are constant, with $s_3 = 0$. This means that the matrix $\Sigma(t) = \text{const.}$, thus $\Omega(t)$ is a rotation on a fixed axis with reference to the system (x, y, z), which it is located in the plane (x, y), because $s_3 = 0$. As each $z_i^0(t_0) = 0$, for $i = 1, \ldots, n$ it follows from (1.9) that the system of rotating coordinates system $X = \Omega^{-1}\mathbf{r}$ cannot rotate around a fixed axis contained in the (x, y)-plane. Therefore, there is no rotation at all, i.e., $\Omega(t) = \text{const.}$ and so this solution is homothetic.

This completes the proof. ■

Now, we can prove:

Theorem 3.1 *(a) An homographic solution is homotethic if and only if the angular momentum C is null.*
(b) An homographic solution is a solution of relative equilibrium if and only if it is planar and rotates with a constant angular velocity different from zero.

Proof. We are going to divide the proof in two cases:

(I) **Planar case**. If the homographic solution is planar, we have for (3.15) that $\dot{\phi}^2 r^2 = \frac{\|\mathbf{C}\|}{J^0}$ and then:
i) $\mathbf{C} = 0$ if and only if, $\dot{\phi}(t) \equiv 0$ if and only if, $\dot{\phi}(t) = \text{const.}$ if and only if, the homographic solution is homothetic.

ii) The homographic solution is a solution of relative equilibrium if and only if, $r = r(t) = \text{const.} > 0$ if and only if, $\dot{\phi} = \text{const.} \neq 0$, if and only if, $\mathbf{C} \neq 0$.

Therefore, this proves (a) and (b) for the planar case.

(II) **Non-planar case**. If a homographic solution is not planar, then by Lemma 3.2 the solution is not flat and by Lemma 3.1 the solution is homothetic, so it is not a solution of relative equilibrium. Therefore, if the homographic solution is a solution of relative equilibrium it is planar and by case (I) we have that it rotates with constant angular velocity. The converse of (b) is clear by the case (I). This completes the proof in (b).

If the homographic solution is homothetic then $\mathbf{C} = 0$. So in order to complete the proof in (a), it is sufficient to prove that $\mathbf{C} = 0$ for every non-planar homographic solution. Because in this situation we have $\mathbf{C} \neq 0$ then the homographic solution is planar. Let a non-planar homographic solution, then by Lemma 3.2 it is not flat and by Lemma 3.1 it is homothetic and therefore $\mathbf{C} = 0$.

This completes the proof of (a) and of the Theorem. ∎

Remark. *The Theorem* 3.1 *is sometimes called Lagrange-Pizzeti Theorem because Lagrange had already used it in the three body problem and this result is due to Pizzetti* [11].

Since (3.13) holds in the planar case and $U = U^0/r$ in every case, it follows that the energy integral $T - U = h$ of every homographic solution may be written in the form

$$\frac{1}{2}(\dot{r}^2 + r^2\dot{\phi})J^0 - \frac{1}{r}U^0 = h$$

if $\dot{\phi} = \dot{\phi}(t)$, which is defined as the angular velocity of the rotating coordinate system $(\xi, \eta, \zeta)^T = \Omega^{-1}(x, y, z)^T$ in the planar case, and it is defined by $\dot{\phi}(t) \equiv 0$ in the non-planar case. In this sense (3.13) holds in the non-planar case also, since then $\mathbf{C} = 0$, by Theorem 3.1, Lemma 3.1 and Lemma 3.2. Finally, we see from (3.7) that (3.11) holds with $\dot{\phi} \equiv 0$ if $\Sigma \equiv 0$, because from (3.7) we have $r^2\ddot{r} = 0$, which means, by the preliminaries that $\Omega \equiv 0$. Since Lemma 3.1 and Lemma 3.2 show that $\Omega(t) =$ const. is satisfied in the non-planar case, it follows that (3.11) becomes valid for this case by placing again $\dot{\phi} \equiv 0$.

4 Existence of the homographic solutions

In the section 3 the results there obtained contain a classification of all possible homothetic solutions and solutions of relative equilibrium but they leave open the question about the existence of such solutions. The first result in this section is:

Theorem 4.1 *If a solution $\mathbf{r}_i = \mathbf{r}_i(t)$ of the n-body problem with masses m_i is homographic, then the bodies m_i must form a central configuration at every t.*

Proof. If the solution is planar, we can choose the coordinate system (x, y, z) such that $z(t) = 0$ for every t, and $\dot{\phi}(t) \geq 0$ will denote the angular velocity of the rotating plane (ξ,η) where $(\xi,\eta,\zeta)^T = \Omega^{-1}(x,y,z)^T$. On the other hand, if the solution is non-planar, let $\dot\phi(t)$ be defined by $\dot\phi \equiv 0$. Then, as show in the remark at the end of section 3, all the formulae in the planar case are true in the non-planar case.

We define the constants:

$$\begin{array}{ll} (a) & \mu = \frac{U^0}{J^0} \\ (b) & h^0 = \frac{h}{J^0} \\ (c) & \mathbf{C}^0 = \frac{\mathbf{C}}{J^0}, \quad (U^0, J^0 > 0). \end{array} \tag{4.1}$$

then the energy relation is $\frac{1}{2}(\dot r^2 + r^2\dot\phi^2)J^0 - r^{-1}U^0 = h$ and the expression for the angular momentum (3.15) are transformed, using (4.1), in:

$$\begin{array}{ll} (i) & \frac{1}{2}(\dot r^2 + r^2\dot\phi^2) - \frac{\mu}{r} = h^0 \\ (ii) & r^2\dot\phi = \|\mathbf{C}^0\|. \end{array} \tag{4.2}$$

The equivalent formulation $\ddot J = 2U + 4h$ of the energy integral may be written, as

$$(r\ddot r + \dot r^2)J^0 - r^{-1}U^0 = 2h$$

or equivalently,

$$\dot r^2 = -r\ddot r + \frac{\mu}{r} + 2h^0 \tag{4.3}$$

We see from (4.2) i) and (4.3) that

$$\ddot r - r\dot\phi^2 = -\frac{\mu}{r^2}. \tag{4.4}$$

Since the expression on the right hand side in (4.2) (ii) is constant, then computing the derivative, we obtain $(r^2\dot\phi)^{\cdot} = 0$, so

$$r\ddot\phi + 2\dot r\dot\phi = 0. \tag{4.5}$$

On the other hand, substituing (4.4) in (3.11) K is a matrix given by diag$(-\mu, -\mu, -\mu + r^3\dot\phi)$ and as $\dot\phi = 0$, then

$$K(t) = \begin{pmatrix} -\mu & 0 & 0 \\ 0 & -\mu & 0 \\ 0 & 0 & -\mu \end{pmatrix}.$$

Therefore, from (3.6) we have $K(t)\mathbf{r}_i^0 = -\mu\mathbf{r}_i^0 = \mathbf{a}_i$ but, by definition $U^0_{\mathbf{r}_i} = m_i\mathbf{a}_i$ if only if, $\mathbf{a}_i = m_i^{-1}U^0_{\mathbf{r}_i}$, thus $m_i^{-1}U^0_{\mathbf{r}_i} = -\mu\mathbf{r}_i^0$, in both the planar and non-planar cases. Thus, $U^0_{\mathbf{r}_i} = -\mu m_i\mathbf{r}_i^0$ and since the initial date t_0 may be chosen arbitrary, and remembering the definition 1.14 the proof is complete. ∎

Remark. *This Theorem* 4.1 *shows that homographic solutions and central configuration belong to the same group of mathematical objects and that their properties are strongly related. It was proved by Lagrange* [5] *for* $n = 3$*, and a modern version by Elmabsout* [3] *for central configurations.*

Since $\dot{\phi} = 0$, in the non-planar case, we can write a homographic solution in (1.9) not only in the planar case but also in the non-planar case in the form

$$\begin{aligned} &\mathbf{r}_i = r\Omega\mathbf{r}_i^0, \quad r = r(t) \\ &\Omega(t) = \begin{pmatrix} \cos\phi(t) & -\sin\phi(t) & 0 \\ \sin\phi(t) & \cos\phi(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{4.6}$$

The identities (1.7) implies that $\phi^0 = \phi(t_0) = 0$, on the other hand (4.2) (ii) for $t = t_0$ implies $\dot{\phi} = \frac{\|\mathbf{C}^0\|}{r^2(t_0)}$, but $r(t_0) = 1$ and $\|\mathbf{C}^0\| \geq 0$ then $\dot{\phi} \geq 0$. Thus, the following conditions are valid:

$$r^0 = 1, \quad \phi^0 = 0, \quad \dot{\phi} \geq 0. \tag{4.7}$$

The functions $r(t), \phi(t)$ satisfying the conditions above and defining $\mathbf{r}_i(t) = r(t)\Omega(t)\mathbf{r}_i^0$ are solutions of (4.4) and (4.5) if and only if they are solutions of the Kepler problem. In fact, by (4.3) $\ddot{r} = -\frac{\dot{r}^2}{r} + \frac{\mu}{r^2} + \frac{2h^0}{r}$ and substituing this expression in (4.4) we obtain $-\frac{\dot{r}^2}{2} + \frac{\mu}{r^2} + \frac{2h_0}{r} - r\dot{\phi}^2 = -\frac{\mu}{r^2}$. Multiplying this last equation by $\frac{r}{2}$ we get $h_0 = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{\mu}{r}$ which is the energy associated to the Kepler problem.

The next result shows that is easy to construct homographic solutions, once we know the central configurations.

Theorem 4.2 *A solution $(\mathbf{r}_1(t), \ldots, \mathbf{r}_n(t))$ of the n-body problem is homographic if and only if, there exist functions $r(t), \phi(t)$ satisfying (4.4) and ((4.5), and initial conditions $(\mathbf{r}_1^0, \ldots, \mathbf{r}_n^0)$ forming a central configurations with masses m_i and $\mathbf{r}_i(t) = r(t)\Omega(t)\mathbf{r}_i^0 \ \ (i = 1, \ldots, n)$.*

Proof. The necessary conditions follows by Theorem 4.1 and the above remarks. In order to prove the sufficient condition, it is necessary to prove that $\mathbf{r}_i(t) = r(t)\Omega(t)\mathbf{r}_i^0$ $(i = 1, \ldots, n)$ is a solution of the n-body problem with masses m_i. We need to prove that if $r(t), \phi(t)$ are solutions of (4.4), (4.5) with initial conditions as in (4.7). Using the notations in (4.1) and as $(\mathbf{r}_1^0, \ldots, \mathbf{r}_n^0)$ form a central configuration (as in (1.14)), then $\sigma = -\frac{U^0}{J^0}$, thus we have

$$U_{\mathbf{r}_i}^0 = -\mu m_i \mathbf{r}_i^0. \tag{4.8}$$

From (3.3) we obtain $U_{\mathbf{r}_i} = \frac{1}{r^2}\Omega U_{\mathbf{r}_i}^0$, it follows from the motion equation (1.1) that $\mathbf{r}_i(t)$ $(i = 1, \ldots, n)$ is a solution if and only if $-\frac{1}{r^2}\Omega U_{\mathbf{r}_i}^0 = m_i\ddot{\mathbf{r}}_i$ so by (4.8) $\frac{1}{r^2}\Omega\mu\mathbf{r}_i^0 = \ddot{\mathbf{r}}_i$ and

$$r^2\Omega^{-1}\ddot{\mathbf{r}}_i = -\mu\mathbf{r}_i^0. \tag{4.9}$$

Consequently, we have only to show that (4.9) is an identity in t.

To this end, let $r(t), \phi(t)$ be any given pair of functions of class C^2. Let $\Omega(t)$ be defined in terms of $\phi(t)$ in (4.6) and $K(t)$ be defined by (3.11) in terms of r and ϕ. Since, $r^2\Omega^{-1}(r\Omega)^{\cdot\cdot} = r^2\Omega^{-1}(\ddot{r}\Omega + 2\dot{r}\dot{\Omega} + r\ddot{\Omega}) = r^2\ddot{r}I + 2r^2\dot{r}\Omega^{-1}\dot{\Omega} + r^3\Omega^{-1}\ddot{\Omega}$, we have using (2.1) and (2.3) that $r^2\Omega^{-1}(r\Omega)^{\cdot\cdot} = r^2[\ddot{r}I + 2\dot{r}\Sigma + r(\dot{\Sigma} + \Sigma^2)]$, so by (3.7) $r^2\Omega^{-1}\,(r\Omega)^{\cdot\cdot} = K(t)$. But, from (4.6) $\ddot{\mathbf{r}}_i = (r\Omega)^{\cdot\cdot}\mathbf{r}_i^0$, $i = 1, \ldots, n$. Comparing we have

$$r^2\Omega^{-1}\ddot{\mathbf{r}}_i = K(t)\mathbf{r}_i^0. \tag{4.10}$$

Being $a_i = m_i^{-1}U^0_{\mathbf{r}_i}$, the relation (3.6) implies that $K(t)\mathbf{r}_i^0 = m_i^{-1}U^0_{\mathbf{r}_i}$ and comparing this expression with (4.10) it follows that $r^2\Omega^{-1}\ddot{\mathbf{r}}_i = -\mu\mathbf{r}_i^0$. Thus, (4.9) is satisfied and $\mathbf{r}_i(t)$ is a homographic solution of the n-body problem. ∎

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