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An introduction to the Fractional Fourier Transform and friends

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ABSTRACT

In this survey paper we introduce the reader to the notion of the fractional Fourier transform, which may be considered as a fractional power of the classical Fourier transform. It has been intensely studied during the last decade, an attention it may have partially gained because of the vivid interest in time-frequency analysis methods of signal processing, like wavelets. Like the complex exponentials are the basic functions in Fourier analysis, the chirps (signals sweeping through all frequencies in a certain interval) are the building blocks in the fractional Fourier analysis. Part of its roots can be found in optics and mechanics. We give an introduction to the definition, the properties and approaches to the continuous fractional Fourier transform.

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RESUMEN

En este artículo de prospección introducimos al lector en la noción de la transformada de Fourier fraccional, que puede ser considerada como una potencia fraccional de la transformada de Fourier clásica. Ha sido objeto de intensos estudios durante la última década, que puede deberse parcialmente al interés respecto de los métodos de análisis tiempo-frecuencia en el proceso de señales, como es el caso de los wavelets. Tal como las exponenciales complejas son las funciones básicas del análisis de Fourier, los llamados chirps (señales que barren todas las frecuencias en un intervalo dado) son los elementos básicos del análisis de Fourier fraccional. Parte de sus orígenes se pueden encontrar en la óptica y la mecánica. Damos una introducción a la definición, las propiedades y acercamientos a la transformada de Fourier fraccional.

Key words and phrases: *Fourier transform, fractional transforms, signal processing, chirp, phase space*

Math. Subj. Class.: *42A38, 65T20*

1 Introduction

The idea of fractional powers of the Fourier operator appears in the mathematical literature as early as 1929 [32, 8, 11]. It has been rediscovered in quantum mechanics [19, 16], optics [17, 21, 2] and signal processing [3]. The boom in publications started in the early years of the 1990's and it is still going on. A recent state of the art can be found in [22].

The outline of the paper is as follows. Section 2 gives a motivation for our definition of the fractional Fourier transform (FrFT) given in the next section. Whereas in the classical Fourier transform, the harmonics and the delta functions play a prominent role, these are for the FrFT replaced by a more general class of chirp functions introduced in Section 4. The Wigner distribution is a function that essentially gives the distribution of the energy of the signal in a time-frequency or phase plane. The effect of a FrFT can be effectively visualized with the help of this function. This is described in Section 5. Relations with the windowed or short time Fourier transform, with wavelets and chirplets can be found in Section 6. The FrFT may be seen as a special case of a more general linear canonical transform (LCT). Whereas the FrFT corresponds to a rotation of the Wigner distribution in the time-frequency plane, the LCT will correspond to any linear transform that can be represented by a unimodular 2×2 matrix. This is the subject of Section 7. Thus everything that is explained in this section will also hold for the fractional Fourier transform. This includes computational aspects, filtering in the transform domain, generalization to higher dimension, etc. To define the LCT in higher dimensions, we give a brief introduction to a group theoretic approach in Section 8. We conclude by a section giving a quick review of some closely related transforms. Because of page limitations, we shall have to refer the reader for all the details to the literature.

2 The classical Fourier Transform

We recall some of the definitions and properties that are related to the classical continuous Fourier transform (FT) so that we can motivate our definition of the fractional Fourier transform (FrFT) later.

On an appropriate function space $\mathcal{L}$ like e.g., $L^2(\mathbb{R})$, the classical FT operator $\mathcal{F} : f \to F$ and its inverse are defined as

$$F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} \mathrm{d}x, \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi) e^{i\xi x} \mathrm{d}\xi. \tag{1}$$

In signal processing applications, f is often a time depending signal so that x denotes time and ξ frequency. Therefore $f(x)$ is a time domain description of the signal and $F(\xi)$ a frequency domain description.

Furthermore, it is immediately verified that $(\mathcal{F}^2 f)(x) = f(-x)$, $(\mathcal{F}^3 f)(\xi) = F(-\xi)$, and $(\mathcal{F}^4 f)(x) = f(x)$. This means that for $a \in \mathbb{Z}$ we may identify $\mathcal{F}^a$ with a rotation in the (x, ξ)-plane over an angle $\alpha = a\pi/2$. The idea of the FrFT is to define $\mathcal{F}^a$ for any $a \in \mathbb{R}$.

It will be useful to introduce some notation. Let R_a denote the rotation matrix

$$R_a = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = e^{J\alpha}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and suppose that $(x_a, \xi_a)^T = R_a(x, \xi)^T$, or switching to complex variables $z = x - i\xi$, then $z_a = e^{i\alpha} z$. Note that with this notation $\xi = x_1$, and in general $\xi_a = x_{a+1}$.

The notation $\mathcal{R}_a$ will also be used as an operator working on a function of two variables to mean $\mathcal{R}_a f(x, \xi) = f(x_a, \xi_a)$ and to indicate that $\mathcal{R}_a(x, \xi) = (x_a, \xi_a)$.

3 The fractional Fourier transform

In [22] the authors give 6 different possible definitions of the FrFT and others can be found elsewhere. We prefer to follow an intuitive approach and define it as an extension of $\mathcal{F}^a$ for $a \in \mathbb{Z}$ to $a \in \mathbb{R}$.

3.1 Eigenfunctions

How to define $\mathcal{F}^a$ for $a \in \mathbb{R}$? The key is the eigenvalue decomposition of $\mathcal{F}$. It is known that $\mathcal{F}$ has a complete set of eigenvectors that span $L^2(\mathbb{R})$. Since $\mathcal{F}^4 = \mathcal{I}$, the different eigenvalues are $\{1, -i, -1, i\}$ each with an infinite dimensional eigenspace. The eigenvectors are thus not unique, but a possible choice of orthonormal eigenfunctions is given by the set of normalized Hermite-Gauss functions:

$$\phi_n(x) = \frac{2^{1/4}}{\sqrt{2^n n!}} e^{-x^2/2} H_n(x), \quad \text{where} \quad H_n(x) = (-i)^n e^{x^2} \mathcal{D}^n e^{-x^2}, \quad \mathcal{D} = -i\frac{\mathrm{d}}{\mathrm{d}x},$$

is an Hermite polynomial of degree n. We have $\mathcal{F}\phi_n = \lambda_n\phi_n$ with $\lambda_n = e^{-in\pi/2}$. So, provided we properly define λ^a for $a \in \mathbb{R}$, we may set $\mathcal{F}^a\phi_n = \lambda_n^a\phi_n$, and since $\{\phi_n\}$ is a complete set, this defines $\mathcal{F}^a$ on $\mathcal{L}$.

If we define the analysis operator $\mathcal{T}_\phi$, the synthesis operator $\mathcal{T}_\phi^*$ and the scaling operator $\mathcal{S}_\lambda$ as

$$\mathcal{T}_\phi : f \mapsto \{c_n = \langle f, \phi_n\rangle_2\}, \quad \mathcal{S}_\lambda : \{c_n\} \mapsto \{\lambda_n c_n\}, \quad \mathcal{T}_\phi^* : \{d_n\} \mapsto \sum_{n=0}^{\infty} d_n\phi_n,$$

($\langle\cdot,\cdot\rangle_2$ is the inner product in $L^2(\mathbb{R})$) then it is clear that we may write

$$\mathcal{F} = \mathcal{T}_\phi^*\mathcal{S}_\lambda\mathcal{T}_\phi \quad \text{and} \quad \mathcal{F}^a = \mathcal{T}_\phi^*\mathcal{S}_\lambda^a\mathcal{T}_\phi. \tag{2}$$

Note that the operator $\mathcal{T}_\phi$ is unitary on $L^2(\mathbb{R})$ and that $\mathcal{T}_\phi^*$ is its adjoint.

The formula (2) gives a general procedure to define the fractional power of any operator that has a complete set of eigenfunctions.

This definition implies that $\mathcal{F}^a$ can be written as a operator exponential $\mathcal{F}^a = e^{-i\alpha\mathcal{H}} = e^{-ia\pi\mathcal{H}/2}$ where the Hamiltonian operator $\mathcal{H}$ is given by $\mathcal{H} = \frac{1}{2}(\mathcal{D}^2 + \mathcal{U}^2 - \mathcal{I})$ with $\mathcal{D} = -i\mathrm{d}/\mathrm{d}x$ and $\mathcal{U}$ the shift operator of $L^2(\mathbb{R})$ defined as $(\mathcal{U}f)(x) = xf(x)$ or $\mathcal{U} = \mathcal{F}\mathcal{D}\mathcal{F}^{-1}$ (see [16, 19, 22]). The form of the operator $\mathcal{H}$ can be readily checked by differentiating the relation

$$e^{-i\alpha\mathcal{H}}\left(e^{-x^2/2}H_n(x)\right) = e^{-in\alpha}\left(e^{-x^2/2}H_n(x)\right)$$

with respect to α, setting $\alpha = 0$ and then using the differential equation $(\mathcal{D} + 2i\mathcal{U})\mathcal{D}H_n = 2nH_n$ satisfied by the Hermite polynomials.

Note that this form identifies $\mathcal{F}^a$ as a unitary operator, and hence the Parseval equality holds in $L^2(\mathbb{R})$.

Several simple properties can now be derived, the most glamorous one being $\mathcal{F}^a\mathcal{F}^b = \mathcal{F}^{a+b}$, which reflects the group structure of the rotations.

3.2 Integral representation

Any function $f \in L^2(\mathbb{R})$ can be expanded as $f = \sum_n \langle f, \phi_n\rangle_2 \phi_n$, so that after application of $\mathcal{F}^a$ we have $(\mathcal{F}^a f)(\xi) = \langle f(x), \sum_n \phi_n(x)\lambda_n^a\phi_n(\xi)\rangle_2$, which identifies $\mathcal{F}^a$ as an integral transform with kernel $K_a(\xi, x) = \sum_n \phi_n(x)\lambda_n^a\phi_n(\xi)/\sqrt{2\pi}$. For $a = \pm 1$ this reduces to the FT kernel $K_{\pm 1}(\xi, x) = e^{\mp ix\xi}/\sqrt{2\pi}$. For $a \neq \pm 1$, this is not so simple. Using the eigenvalues and eigenfunctions for the transform $\mathcal{F}^a$, we obtain

$$\begin{aligned} K_a(\xi, x) &= \sum_{n=0}^{\infty} \frac{e^{-ina\pi/2}H_n(\xi)H_n(x)}{2^n n!\sqrt{\pi}} e^{-(x^2+\xi^2)/2} \\ &= \frac{1}{\sqrt{\pi}\sqrt{1 - e^{-2i\alpha}}} \exp\left\{\frac{2x\xi e^{-i\alpha} - e^{-2i\alpha}(\xi^2 + x^2)}{1 - e^{-2i\alpha}}\right\} \exp\left\{-\frac{\xi^2 + x^2}{2}\right\} \end{aligned}$$

where in the last step we used Mehler's formula ([19, p. 244] or [4, eq. (6.1.13)])

$$\sum_{n=0}^{\infty} \frac{e^{-in\alpha} H_n(\xi) H_n(x)}{2^n n! \sqrt{\pi}} = \frac{\exp\left\{\frac{2x\xi e^{-i\alpha} - e^{-2i\alpha}(\xi^2+x^2)}{1-e^{-2i\alpha}}\right\}}{\sqrt{\pi(1-e^{-2i\alpha})}}.$$

To rewrite this expression, we observe that the following identities hold (they are easily checked)

$$\frac{2x\xi e^{-i\alpha}}{1-e^{-2i\alpha}} = -ix\xi \csc\alpha$$

$$\frac{1}{\sqrt{\pi}\sqrt{1-e^{-2i\alpha}}} = \frac{e^{-\frac{i}{2}(\frac{\pi}{2}\hat{\alpha}-\alpha)}}{\sqrt{2\pi|\sin\alpha|}}$$

$$\frac{e^{-2i\alpha}}{1-e^{-2i\alpha}} + \frac{1}{2} = -\frac{i}{2}\cot\alpha$$

where $\hat{\alpha} = \text{sgn}\,(\sin\alpha)$. Obviously, such relations only make sense if $\sin\alpha \neq 0$, i.e., if $\alpha \notin \pi\mathbb{Z}$ or equivalently $a \notin 2\mathbb{Z}$. The branch of $(\sin\alpha)^{1/2}$ we are using for $\sin\alpha < 0$ is the one with $0 < |\alpha| < \pi$. With these expressions, we obtain a more tractable integral representation of $\mathcal{F}^a$ for $a \notin 2\mathbb{Z}$ viz.

$$f_a(\xi) := (\mathcal{F}^a f)(\xi) = \frac{e^{-\frac{i}{2}(\frac{\pi}{2}\hat{\alpha}-\alpha)} e^{\frac{i}{2}\xi^2\cot\alpha}}{\sqrt{2\pi|\sin\alpha|}} \int_{-\infty}^{\infty} \exp\left\{-i\frac{x\xi}{\sin\alpha} + \frac{i}{2}x^2\cot\alpha\right\} f(x)\mathrm{d}x, \tag{3}$$

where $\hat{\alpha} = \text{sgn}\,(\sin\alpha)$ and $0 < |\alpha| < \pi$.

Previously we defined $(\mathcal{F}^a f)(\xi) = f(\xi)$, if $\alpha = 0$, and $(\mathcal{F}^a f)(\xi) = f(-\xi)$, if $\alpha = \pm\pi$. That is consistent with this integral representation because for these special values, it holds that $\lim_{\epsilon\to 0} f_{a+\epsilon} = f_a$. Thus, with this limiting property, we can assume that the integral representation holds on the whole interval $|\alpha| \leq \pi$. When $|\alpha| > \pi$, the definition is taken modulo 2π and reduced to the interval $[-\pi, \pi]$.

Defining the FrFT via this integral transform, we can say that the FrFT exists for $f \in L^1(\mathbb{R})$ (and hence in $L^2(\mathbb{R})$) or when it is a generalized function. Indeed, in that case, the integrand in (3) is also in $L^1(\mathbb{R})$ (or $L^2(\mathbb{R})$) or is a generalized function. Thus the FrFT exists under exactly the same conditions as under which the FT exists. Thus we have proved

Theorem 3.1 *Assume $\alpha = a\pi/2$ then the FrFT has an integral representation*

$$f_a(\xi) := (\mathcal{F}^a f)(\xi) = \int_{-\infty}^{\infty} K_a(\xi, x) f(x)\mathrm{d}x.$$

The kernel is defined as follows: For $a \notin 2\mathbb{Z}$, then with $\hat{\alpha} = sgn\,(\sin\alpha)$,

$$K_a(\xi, x) = C_\alpha \exp\left\{-i\frac{x\xi}{\sin\alpha} + \frac{i}{2}(x^2+\xi^2)\cot\alpha\right\}$$

with

$$C_\alpha = \frac{e^{-\frac{i}{2}(\frac{\pi}{2}\hat{\alpha}-\alpha)}}{\sqrt{2\pi|\sin\alpha|}} = \sqrt{\frac{1-i\cot\alpha}{2\pi}}.$$

For $a \in 4\mathbb{Z}$ the FrFT becomes the identity, hence $K_{4n}(\xi,x) = \delta(\xi - x)$, $\quad n \in \mathbb{Z}$ and for $a \in 2+4\mathbb{Z}$, it is the parity operator: $K_{2+4n}(\xi,x) = \delta(\xi + x)$, $\quad n \in \mathbb{Z}$.
If we restrict a to the range $0 < |a| < 2$, then $\mathcal{F}^a$ is a homeomorphism of $L^2(\mathbb{R})$ (with inverse $\mathcal{F}^{-a}$).

The last statement is proved in [16, p. 162].

It is directly verified that the kernel K_a has the following properties.

Theorem 3.2 *IF $K_a(x,t)$ is the kernel of the FrFT as in Theorem 3.1, then*

1. $K_a(\xi,x) = K_a(x,\xi)$ *(diagonal symmetry)*
2. $K_{-a}(\xi,x) = \overline{K_a(\xi,x)}$ *(complex conjugate)*
3. $K_a(-\xi,x) = K_a(\xi,-x)$ *(point symmetry)*
4. $\int_{-\infty}^{\infty} K_a(\xi,t)K_b(t,x)\mathrm{d}t = K_{a+b}(\xi,x)$ *(additivity)*
5. $\int_{-\infty}^{\infty} K_a(t,\xi)\overline{K_a(t,x)}\mathrm{d}t = \delta(\xi - x)$ *(orthogonality)*

4 The chirp function

A chirp function (or chirp for short) is a signal that contains all frequencies in a certain interval and sweeps through it while it progresses in time. The interval can be swept in several ways (linear, quadratic, logarithmic,...), but we shall restrict us here to the case where the sweep is linear.

The complex exponential $e^{i\omega t}$ contains just one frequency: ω. This type of functions is essential in Fourier analysis. In fact, they form a basis for the space of functions treated by the FT. Indeed, the relation $f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\mathrm{d}\omega$ can be seen as a decomposition of f into a (continuous) combination of the basis functions $\{e_\omega(t) = e^{i\omega t}\}_{\omega\in\mathbb{R}}$.

On the other hand, if the frequencies of the signal sweeps linearly through the frequency interval $[\omega_0,\omega_1]$ in the time interval $[t_0,t_1]$, then we should have $\omega = \omega_0 + \frac{\omega_1-\omega_0}{t_1-t_0}(t-t_0)$. Thus, a chirp will have the form $\exp\{i(\chi t+\gamma)t\}$. The parameter χ is called the sweep rate. Now consider the FrFT kernel $K_a(\xi,x)$, then, seen as a function of x and taking ξ as a parameter, this is a chirp with sweep rate $\frac{1}{2}\cot\alpha$. So, by rearranging the kernel (see also Section 7), it can be seen that one way of describing a FrFT is

1. multiply by a chirp
2. do an ordinary FT
3. do some scaling
4. multiply by a chirp.

The inverse FrFT can be written as $f(x) = \int_{-\infty}^{\infty} f_a(\xi)\psi_\xi(x)\mathrm{d}\xi$ where $\psi_\xi(x) = K_{-a}(\xi, x)$ is a chirp parameterized in ξ with sweep rate $-\frac{1}{2}\cot\alpha$. Thus we see that the role played by the harmonics in classical FT, is now taken by chirps, and the latter relation is a decomposition of $f(x)$ into a linear combination of chirps with a fixed sweep rate determined by α. Note also that in this expansion in chirp series, the basis functions are orthogonal by property 5 of the previous theorem. However, there is more. The chirps are in between harmonics and delta functions. Indeed, up to a rotation in the time-frequency plane, the chirps *are* delta functions and harmonics. To see this, take the FrFT of a delta function $\delta(x-\gamma)$. That is $(\mathcal{F}^a\delta(\cdot-\gamma))(\xi) = K_a(\xi,\gamma)$, which is a chirp with sweep rate $\frac{1}{2}\cot\alpha$. Thus, given a (linear) chirp with sweep rate $\frac{1}{2}\cot\alpha$, we can transform it by a FrFT $\mathcal{F}^{-a}$ into a delta function and hence by taking the FT of the delta function, we can take the chirp by a FrFT $\mathcal{F}^{1-a}$ into an harmonic function.

5 The Wigner distribution and the FrFT

The relation between the multiplication operator $\mathcal{U}$ and the complex differentiation operator $\mathcal{D}$, in the case of the classical Fourier transform is $\mathcal{UF} = \mathcal{FD}$, which can be generalized as follows

$$\mathcal{F}^a \begin{bmatrix} \mathcal{U} \\ \mathcal{D} \end{bmatrix} = \begin{bmatrix} \mathcal{U}_a \\ \mathcal{D}_a \end{bmatrix} \mathcal{F}^a \quad \text{where} \quad \begin{bmatrix} \mathcal{U}_a \\ \mathcal{D}_a \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \mathcal{U} \\ \mathcal{D} \end{bmatrix}.$$

Thus $\mathcal{U}_a$ and $\mathcal{D}_a$ correspond to multiplication and complex differentiation in the variable of the FrFT domain. It are rotations of the usual $\mathcal{U}$ and $\mathcal{D}$: $(\mathcal{U}_a, \mathcal{D}_a) = \mathcal{R}_a(\mathcal{U}, \mathcal{D})$. This property is intuitively clear: by first applying $\mathcal{U}$ or $\mathcal{D}$ (i.e., multiplication in the x, respectively ξ direction) followed by a rotation in the (x,ξ)-plane must be the same as the rotation followed by the same operations applied to the rotated variables. Because the rotation is an orthogonal transformation, we also have $\mathcal{D}_a^2 + \mathcal{U}_a^2 = \mathcal{D}^2 + \mathcal{U}^2$, so that the Hamiltonian is rotation invariant: $\mathcal{H}_a = \frac{1}{2}(\mathcal{D}_a^2 + \mathcal{U}_a^2 - \mathcal{I}) = \frac{1}{2}(\mathcal{D}^2 + \mathcal{U}^2 - \mathcal{I}) = \mathcal{H}$.

The rotation property of the FrFT that has been mentioned several times now, can be visualised by the Wigner distribution which is what will be defined next. Let f be in $L^2(\mathbb{R})$, then its *Wigner distribution* or *Wigner transform* $\mathcal{W}f$ is defined as

$$(\mathcal{W}f)(x,\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x+u/2)\overline{f(x-u/2)}e^{-i\xi u}\mathrm{d}u.$$

Its meaning is roughly speaking one of energy distribution of the signal in the time-frequency plane. Indeed, setting $f_1 = \mathcal{F}f$, we have

$$\int_{-\infty}^{\infty} (\mathcal{W}f)(x,\xi)\mathrm{d}\xi = |f(x)|^2 \quad \text{and} \quad \int_{-\infty}^{\infty} (\mathcal{W}f)(x,\xi)\mathrm{d}x = |f_1(\xi)|^2,$$

so that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\mathcal{W}f)(x,\xi)\mathrm{d}\xi\mathrm{d}x = \|f\|^2 = \|f_1\|^2,$$

which is the energy of the signal f.

An important property of the FrFT is the following.

Theorem 5.1 *The Wigner distribution of a signal and its FrFT are related by a rotation over an angle $-\alpha$:*

$$(\mathcal{W}f_a)(x,\xi) = \mathcal{R}_{-a}(\mathcal{W}f)(x,\xi)$$

where $\alpha = a\pi/2$, $f_a = \mathcal{F}^a f$. Equivalently

$$\mathcal{R}_a(\mathcal{W}f_a)(x,\xi) = (\mathcal{W}f_a)(x_a,\xi_a) = (\mathcal{W}f)(x,\xi)$$

with $(x_a,\xi_a) = \mathcal{R}_a(x,\xi)$.

This theorem says that if we have the Wigner distribution of f, then the Wigner distribution of f_a is obtained by rotating it clockwise over an angle α in the (x,ξ)-plane. The proof is tedious but straightforward. For details see [3, p. 3087]. Looking at Figure 1, the result is in fact obvious since it just states that before and after a rotation of the coordinate axes, the Wigner distribution is computed in two different ways taking the new variables into account, and that should of course give the same result.

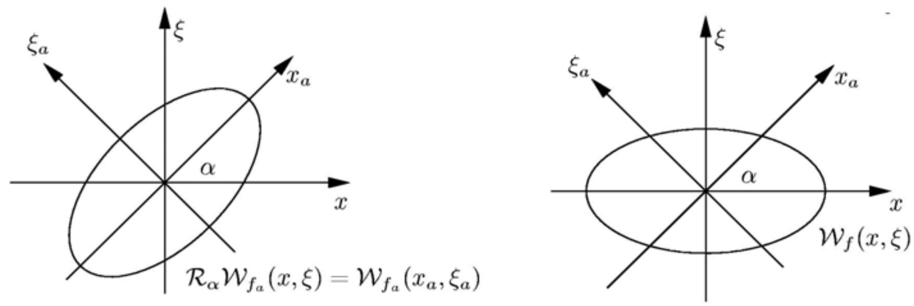

Figure 1: Wigner distribution of a signal f and the Wigner distribution of its FrFT are related by a rotation.

This implies for example

$$\int_{-\infty}^{\infty} (\mathcal{W}f_a)(x,\xi)\mathrm{d}\xi = |f_a(x)|^2 \quad \text{and} \quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\mathcal{W}f_a)(x,\xi)\mathrm{d}x\mathrm{d}\xi = \|f\|^2.$$

The *ambiguity function* is closely related to the Wigner distribution. Its definition is

$$(\mathcal{A}f)(x,\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u+x/2)\overline{f(u-x/2)}e^{-iu\xi}\mathrm{d}u.$$

Thus it is like the Wigner distribution, but now the integral is over the other variable. The ambiguity function and the Wigner distribution are related by what is essentially a 2-dimensional Fourier transform. Whereas the Wigner distribution gives an idea about how the energy of the signal is distributed in the (x,ξ)-plane, the ambiguity function will have a correlative interpretation. Indeed $(\mathcal{A}f)(x,0)$ is the autocorrelation function of f and $(\mathcal{A}f)(0,\xi)$ is the autocorrelation function of $f_1 = \mathcal{F}f$.

6 Windowed transform, wavelets and chirplets

The *short time Fourier transform* or *windowed Fourier transform* (WFT) is defined as

$$(\mathcal{F}_w f)(x,\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\overline{w(t-x)}e^{-i\xi t}\mathrm{d}t$$

where w is a window function. It is a *local* transform in the sense that the window function more or less selects an interval, centered at x to cut out some filtered information of the signal. So it gives information that is local in the time-frequency plane in the sense that we can find out which frequencies appear in the time intervals that are parameterized by their centers x.

It can be shown that

$$(\mathcal{F}_w f)(x,\xi) = e^{-ix\xi}(\mathcal{F}_{w_1} f_1)(\xi,-x) = e^{-ix\xi}(\mathcal{F}_{w_1} f_1)(x_1,\xi_1)$$

where $w_1 = \mathcal{F}w$ and $f_1 = \mathcal{F}f$. Because of the asymmetric factor $e^{-ix\xi}$, it is more convenient to introduce a modified WFT defined by

$$(\tilde{\mathcal{F}}_w f)(x,\xi) = e^{ix\xi/2}(\mathcal{F}_w f)(x,\xi).$$

Then it can be shown that

$$(\tilde{\mathcal{F}}_w f)(x,\xi) = (\tilde{\mathcal{F}}_{w_a} f_a)(x_a,\xi_a).$$

For more information on windows applied in the FrFT domain see [27].

From its definition $f_a(\xi) = \int K_a(\xi,u)f(u)\mathrm{d}u$, we get by setting $x = \xi \sec\alpha$ and $g(x) = f_a(x/\sec\alpha)$

$$g(x) = C(\alpha)e^{-i4x^2\sin(2\alpha)} \int_{-\infty}^{\infty} \exp\left[\frac{i}{2}\left(\frac{x-u}{\tan^{1/2}\alpha}\right)^2\right] f(u)\mathrm{d}u.$$

$C(\alpha)$ is a constant that depends on α only. Although, there are some characteristics of a wavelet transform, this can not exactly be interpreted as a genuine wavelet transform. We do have a scaling parameter $\tan^{1/2}\alpha$ and a translation by u of the basic function $\psi(t) = e^{it^2}$ but since $\int_{-\infty}^{\infty}\psi(x)dx \neq 0$ and it has no compact support, this is not really a wavelet.

Multiscale chirp functions were introduced in [5, 15]. A. Bultan [6] has developed a so called chirplet decomposition which is related to wavelet package techniques. It is especially suited for the decomposition of signals that are chirps, i.e., whose Wigner distribution corresponds to straight lines in the (x,ξ)-plane.

The idea is that a dictionary of chirplets is obtained by scaling and translating an atom whose Wigner distribution is that of a Gaussian that has been stretched and rotated. So, we take a Gaussian $\tilde{g}(t) = \pi^{-1/4}e^{-x^2/2}$ with Wigner distribution $(\mathcal{W}\tilde{g})(x,\xi) = (2/\pi)^{1/2}\exp\{-(x^2+\xi^2)\}$. Next we stretch it as $g(x) = s^{-1/2}\tilde{g}(x/s)$ giving $(\mathcal{W}g)(x,\xi) = (\mathcal{W}\tilde{g})(x/s, s\xi)$. Finally we rotate $(\mathcal{W}g)$ to give $(\mathcal{W}c)(x,\xi)$ with $c = \mathcal{F}^a g$. The chirplet c depends on two parameters s and a and its main support in the (x, ξ)-plane can be thought of as a stretched (by s) and rotated (by a) ellipse centered at the origin. To cover the whole (x, ξ)-plane, we have to tile it with shifted versions of this ellipse, i.e., we need the shifted versions $(\mathcal{W}g)(x-u, \xi-\nu)$ corresponding to the functions $c(x-u)e^{i\nu x}$. With these four-parameters $\rho = (s, a, u, \nu)$ we have a redundant dictionary $\{c_\rho\}$.

The next step is to find a discretization of these 4 parameters such that the dictionary is complete when restricted to that lattice. It has been shown [31] that such a system can be found for $a = 0$ that is indeed complete, and the rotation does not alter this fact.

If the discrete dictionary is $\{c_n\}$ with $c_n = c_{\rho_n}$, then a chirplet representation of the signal f has to be found of the form $f(x) = \sum_n a_n c_n(x)$. Such a discrete dictionary for a signal with N samples has a discrete chirplet dictionary with $O(N^2)$ elements. Therefore a matching pursuit algorithm [14] can be adapted from wavelet analysis. The main idea is that among all the atoms in the dictionary the one that matches best the data is retained. This gives a first term in the chirplet expansion. The approximation residual is then again approximated by the best chirplet from the dictionary, which gives a second term in the expansion etc. This algorithm has a complexity of the order $O(MN^2 \log N)$ to find M terms in the expansion. This is far too much to be practical. A faster $O(MN)$ algorithm based on local optimization has been published [10].

This approach somehow neglects the nice logarithmic and dyadic tiling of the plane that made more classical wavelets so attractive. So this kind of decomposition will be most appropriate when the signal is a composition of a number of chirplets. Such signals do exist like the example of a signal emitted by a bat which consists of 3 nearly parallel chirps in the (x, ξ)-plane. Other examples are found in seismic analysis. For more details we refer to [6]. An example in acoustic analysis was given in [10].

7 The linear canonical transform

As we have seen, the FrFT is essentially a rotation in the (x, ξ)-plane. So, it can be characterized by a 2×2 rotation matrix which depends on one parameter, namely the rotation angle. It is a subgroup $SO(2)$ of the group $GL(2)$ of 2×2 real invertible matrices. Most of what has been said can be generalized to a more general linear transform, which is characterized by a general matrix M in the subgroup $SL(2) = \{M \in \mathbb{R}^{2\times 2} : \det(M) = 1\}$. These generalizations are called linear canonical transforms (LCT).

7.1 Definition

Consider a 2×2 unimodular matrix (i.e., whose determinant is 1). Such a matrix has 3 free parameters u, v, w which we shall arrange as follows

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{w}{v} & \frac{1}{v} \\ -v + \frac{uw}{v} & \frac{u}{v} \end{bmatrix} = \begin{bmatrix} \frac{u}{v} & -\frac{1}{v} \\ v - \frac{uw}{v} & \frac{w}{v} \end{bmatrix}^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}^{-1}.$$

The parameters can be recovered from the matrix by

$$u = \frac{d}{b} = \frac{1}{a}\left(\frac{1}{b} + c\right), \quad v = \frac{1}{b}, \quad w = \frac{a}{b} = \frac{1}{d}\left(\frac{1}{b} + c\right)$$

A typical example is the rotation matrix associated with $\mathcal{R}_\alpha$ where $a = d = \cos\alpha$ and $b = -c = \sin\alpha$. Let us call this matrix R_α. Although $M \in \mathbb{R}^{2\times 2}$ is a matrix, we shall for typographical reasons often write $M = (a, b, c, d)$.

The linear canonical transform $\mathcal{F}^M$ of a function f is an integral transform with kernel $K_M(\xi, x)$ defined by

$$K_M(\xi, x) = \sqrt{\frac{v}{2\pi i}}\ e^{\frac{i}{2}(u\xi^2 - 2v\xi x + wx^2)} = \frac{1}{\sqrt{2\pi i b}}\ e^{\frac{i}{2b}(d\xi^2 - 2\xi x + ax^2)}.$$

Note that, just like in the case of the FrFT, there is some ambiguity since we have to choose the branch of the square root in the definition of the kernel.

7.2 Effect on Wigner distribution and ambiguity function

Note that if M is the rotation matrix R_α, then the kernel K_M reduces almost to the FrFT kernel because $M = R_\alpha$ implies $u = w = \cot\alpha$ while $v = \csc\alpha$. Hence $\mathcal{F}^{R_\alpha} = e^{-i\alpha/2}\mathcal{F}^a$. If f denotes a signal, and f_M its linear canonical transform, then the Wigner transform gives

$$(\mathcal{W}f_M)(ax + b\xi, cx + d\xi) = (\mathcal{W}f)(x, \xi). \tag{4}$$

The latter equation can be directly obtained from the definition of linear canonical transform and the definition of Wigner distribution. Thus if $\mathcal{R}_M$ is the operator defined by $\mathcal{R}_M f(\mathbf{x}) = f(M\mathbf{x})$, then $\mathcal{W} = \mathcal{R}_M \mathcal{W}\mathcal{F}^M$. Note that this generalizes Theorem 5.1, since (up to a unimodular constant factor which does not influence the Wigner distribution) $\mathcal{F}^{R_\alpha}$ and $\mathcal{F}^a$ are the same. Similarly for the ambiguity function: $\mathcal{A} = \mathcal{R}_M \mathcal{A}\mathcal{F}^M$. The group structure can be used to show that $\mathcal{F}^M$ is unitary in $L^2(\mathbb{R})$ and it holds that $\mathcal{F}^A\mathcal{F}^B = \mathcal{F}^C \quad$ if and only if $\quad C = AB$.

7.3 Special cases

When we restrict ourselves to real matrices M, there are several interesting special cases, the FrFT being one of them. Others are

- *The Fresnel transform:* This is defined by

$$g_z(\xi) = \frac{e^{i\pi z/l}}{\sqrt{ilz}} \int_{-\infty}^{\infty} e^{i(\pi/lz)(\xi - z)^2} f(x)\mathrm{d}x.$$

  This corresponds to the choice $M = (1, b, 0, 1)$ with $b = \frac{zl}{2\pi}$, because, with this M we have $g_z(\xi) = e^{i\pi z/l}(\mathcal{F}^M f)(\xi)$.

- *Dilation:* The operation $f(x) \mapsto g_s(\xi) = \sqrt{s} f(s\xi)$, can be also obtained as a LCT because with $M = (1/s, 0, 0, s)$ we have $g_s(\xi) = \sqrt{\text{sgn } (s)}(\mathcal{F}^M f)(\xi)$.

- *Gauss-Weierstrass transform or chirp convolution:* This is obtained by the choice $M = (1, b, 0, 1)$:

$$(\mathcal{F}^M f)(\xi) = \frac{1}{\sqrt{2\pi i b}} \int_{-\infty}^{\infty} \exp\{i(x-\xi)^2/2b\} f(x) \mathrm{d}x.$$

- *Chirp (or Gaussian) multiplication:* Here we take $M = (1, 0, c, 1)$ and get

$$(\mathcal{F}^M f)(\xi) = \exp\{ic\xi^2/2\} f(\xi).$$

7.4 On the computation of the LCT

To compute the LCT, it is only in exceptional cases that the integral can be evaluated exactly. So in most practical cases, the integral will have to be approximated numerically. Two forms depending on different factorizations of the M matrix are interesting for the fast computation or the LCT and thus also for the FrFT.

The first one reflects the decomposition

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ (d-1)/b & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (a-1)/b & 1 \end{bmatrix} \tag{5}$$

which means (see section 7.3) that the computation can be reduced to a chirp multiplication, followed by a chirp convolution, followed by a chirp multiplication. Taking into account that the convolution can be computed in $O(N \log N)$ operations using the fast Fourier transform (FFT), the resulting algorithm is a fast algorithm.

Another interesting decomposition is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ db^{-1} & 1 \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b^{-1}a & 1 \end{bmatrix} \tag{6}$$

and this is to be interpreted as a chirp multiplication, followed by an ordinary Fourier transform (which can be obtained using FFT), followed by a dilation, followed eventually by another chirp multiplication. Again it is clear that this gives a fast way of computing the FrFT or LCT.

In Figure 2, the effect of the LCT on a unit square is illustrated showing the different steps when the matrix M, which is for this example $M = (2, 0.5, 0, 1, 0.525)$, is decomposed as in (5) or as in (6). As we can see the two methods compute quite different intermediate results. In the example given there, it is clear that the second decomposition on the right stretches the initial unit square much more and shifts it over larger distances compared to first decomposition on the left. This is an indication that more severe numerical rounding errors are to be expected with the second way of computing than with the first one.

The straightforward implementation of these steps may be a bit naive because for example in the FrFT case, the kernel may be highly oscillating for certain values of

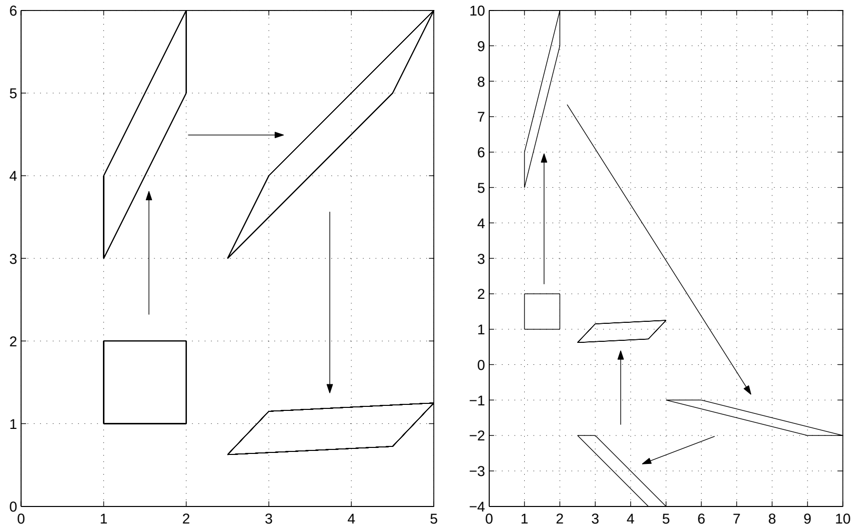

Figure 2: The effect of a LCT on a square. Left when the matrix M is decomposed as in (5) and right when it is decomposed as in (6).

a. It is clear that those values should be avoided. Therefore it is best to evaluate the FrFT only for a in the interval $[0.5, 1.5]$ and to use the relation $\mathcal{F}^a = \mathcal{F}\mathcal{F}^{1-a}$ for $a \in [0, 0.5) \cup (1.5, 2]$. A discussion in [20] follows the approach given by the first decomposition (5).

7.5 Filtering in the LCT domain

One may now set up a program of generalizing all the properties that were given in the case of the FrFT to the LCT. Usually this does not pose a big problem and the generalization is smoothly obtained. We just pick one general approach to what could be called canonical filtering operations. For its definition, we go back to the classical Fourier transform. If we want to filter a signal, then we have to compute a convolution of the signal and the filter. However, as we know, the convolution in the x-domain corresponds to a multiplication in the ξ-domain. Thus the filtering operation is characterized by $f * g = \mathcal{F}^{-1}[(\mathcal{F}f)(\mathcal{F}g)]$. The natural fractional generalization would then be to define a *fractional convolution* $f *_a g = (\mathcal{F}^a)^{-1}[(\mathcal{F}^a f)(\mathcal{F}^a g)]$ and the *canonical convolution* would be $f *_M g = (\mathcal{F}^M)^{-1}[(\mathcal{F}^M f)(\mathcal{F}^M g)]$. Clearly, if $M = I$ or $a = 1$, the classical convolution is recovered. This definition has been used in many papers. See for example [18] and [22, p. 420]. Similar definitions can be given in connection with correlation instead of convolution operations. The essential difference between convolution and correlation is a complex conjugate, so that a canonical correlation can be defined as $f \star_M g = (\mathcal{F}^M)^{-1}[(\mathcal{F}^M f)(\mathcal{F}^M g)^*]$. One could generalize even more and define for example an operation like $\mathcal{F}^{M_3}[(\mathcal{F}^{M_1} f)(\mathcal{F}^{M_2} g)]$ (see [24]).

If we consider the convolution in the x-domain and the multiplication in the ξ-

domain as being *dual* operations, then we can ask for the notion of dual operations in the fractional or the canonical situation. A systematic study of dual operations has been undertaken in [12], but we shall not go into details here.

The windowed Fourier transform can be seen as a special case. Indeed, as we have seen, applying a window in the x-domain corresponds to applying a transformed window in the x_a-domain. So it may well be that in some fractional domain, it may be easier to design a window that will separate different components of the signal, or that can better catch some desired property of the signal because its spread is smaller in the transform domain [27].

Also the Hilbert transform which is defined as

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(x)}{x-x'}\mathrm{d}x' \tag{7}$$

(integral in the sense of principal value) corresponds to filtering in the x domain with a filter $g(x) = 1/x$.

A somewhat different approach to the definition of a canonical convolution is taken in [1]. It is based on the fact that a classical convolution $f * g = f *_0 g$ is an inner product of f with a time-inverted and shifted version of g:

$$(f *_0 g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x')g(x-x')\mathrm{d}x' = \langle f(\cdot), g^*(x-\cdot)\rangle_2 .$$

If we denote a shift in the x-domain as $(\mathcal{T}_0(x')f)(x) = f(x-x')$ and recalling that time-inversion is obtained by the parity operator $\mathcal{F}^2$, it is clear that $g^*(x-x') = (\mathcal{F}^2\mathcal{T}_0(x)g^*)(x')$. So $f *_0 g = \left\langle f, \mathcal{F}^2\mathcal{T}_0(x)g^*\right\rangle_2$. If we now define a *canonical shift* as $\mathcal{T}_M(x_M) = (\mathcal{F}^M)^{-1}\mathcal{T}_0(x_M)\mathcal{F}^M$ that is, we transform the signal, shift it in the transform domain, and then transform back, then another definition of a canonical convolution could be $(f *_M g)(x) = \left\langle f, \mathcal{F}^2\mathcal{T}_M(x)g^*\right\rangle_2$. It still has the classical convolution as a special case when $M = I$, but it is different from the previous definition.

8 Groups and generalization to higher dimensions

There is a nice interpretation of the LCT as a group representation. The purpose of [28] is to find a unitary operator $\mathcal{V}$ on $L^2(\mathbb{R}^n)$ such that it has the effect that the Wigner transform of $\mathcal{V}f$ is the Wigner transform of f subject to a general linear transformation. The n-dimensional Wigner transform is defined as

$$(\mathcal{W}f)(\mathbf{x},\boldsymbol{\xi}) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} f(\mathbf{x}+\mathbf{u}/2)\overline{f(\mathbf{x}-\mathbf{u}/2)}e^{-i\boldsymbol{\xi}\cdot\mathbf{u}}\mathrm{d}\mathbf{u}.$$

The dot represents the standard inner product in $\mathbb{R}^n$. Thus we want to find the unitary operator $\mathcal{V} = \mathcal{F}^{\mathbf{M}}$ on $L^2(\mathbb{R}^n)$ for which a matrix $\mathbf{M} \in GL(2n) = \{\mathbf{M} \in \mathbb{R}^{2n\times 2n} : \det\mathbf{M} \neq 0\}$ can be found such that $\mathcal{W} = \mathcal{R}_{\mathbf{M}}\mathcal{W}\mathcal{V}$, where as before $(\mathcal{R}_{\mathbf{M}}f)(\mathbf{x}) = f(\mathbf{M}\mathbf{x})$. $GL(2n)$ is a Lie group and some subgroups are $SL(2n) = \{\mathbf{M} \in GL(2n) : \det\mathbf{M} = 1\}$, $O(2n) = \{\mathbf{M} \in GL(2n) : \mathbf{M}^T\mathbf{M} = \mathbf{1}\}$ ($\mathbf{1}$ is the identity in $\mathbb{R}^{n\times n}$) and $SO(2n) = SL(2n) \cap O(2n)$. A symplectic form on $\mathbb{R}^{2n}$ can be defined as a Lie bracket: $[\mathbf{f}, \mathbf{g}] = \mathbf{f}^T \mathbf{J}\mathbf{g}$ with $\mathbf{f}, \mathbf{g} \in \mathbb{R}^{2n}$ and

$$\mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{bmatrix}.$$

The symplectic group $Sp(n)$ is the group of real $2n \times 2n$ matrices that leave a symplectic form invariant, i.e., that satisfy $\mathbf{M}^T\mathbf{J}\mathbf{M} = \mathbf{J}$. This implies that a symplectic matrix has determinant ± 1. We have in fact $Sp(n) \subset SL(2n)$ with equality if $n = 1$.

The Heisenberg group H_n is identified with $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with group law

$$(\mathbf{x}_1, \boldsymbol{\xi}_1, t_1)(\mathbf{x}_2, \boldsymbol{\xi}_2, t_2) = (\mathbf{x}_1 + \mathbf{x}_2, \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2, t_1 + t_2 + (\boldsymbol{\xi}_1 \cdot \mathbf{x}_2 - \mathbf{x}_1 \cdot \boldsymbol{\xi}_2)/2).$$

A representation of a topological group G on a Hilbert space H is a mapping μ from G to the space $B(H)$ of bounded operators on H such that $\mu(x)\mu(y) = \mu(xy)$, $\mu(e) = \mathcal{I}$ with e the identity in G and $\mathcal{I}$ the identity operator in $B(H)$ and $x \to \mu(x)f$ is a continuous mapping for all $f \in H$. The representation μ is unitary if $B(H)$ can be replaced by $U(H)$, the unitary operators on H. And μ is called irreducible if $\{0\}$ and H are the only invariant subspaces of H under the group action $\mu(x)$ for all $x \in G$. It can be shown that a unitary irreducible representation of H_n in the space $L^2(\mathbb{R}^n)$ is the Schrödinger representation defined as

$$(\mu(\mathbf{x}, \boldsymbol{\xi}, t)f)(\mathbf{u}) = e^{\mathbf{x}\cdot\mathbf{u}} e^{i(t+\mathbf{x}\cdot\boldsymbol{\xi}/2)} f(\mathbf{u} + \boldsymbol{\xi}).$$

It takes a couple of lines to show that the relation with the Wigner distribution is that we can write

$$(\mathcal{W}f)(\mathbf{x}, \boldsymbol{\xi}) = [\mathcal{F}\langle \mu(\cdot, \cdot, 0)f, f\rangle_2](\mathbf{x}, \boldsymbol{\xi})$$

where $\langle \cdot, \cdot \rangle_2$ is the inner product in $L^2(\mathbb{R}^n)$ and $\mathcal{F}$ the $2n$-dimensional FT acting on the variables indicated by a dot.

Given a unitary $\mathcal{V} \in U(L^2(\mathbb{R}^n))$, another equivalent unitary representation would be given by $\rho(\mathbf{h}) = \mathcal{V}^*\mu(\mathbf{h})\mathcal{V}$ for all $\mathbf{h} \in \mathrm{H}_n$ so that

$$\begin{aligned}(\mathcal{W}\mathcal{V}f)(\mathbf{x}, \boldsymbol{\xi}) &= [\mathcal{F}\langle \rho(\cdot, \cdot, 0)f, f\rangle_2](\mathbf{x}, \boldsymbol{\xi}) \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \langle \rho(\mathbf{u}, \mathbf{v}, 0)f, f\rangle_2 \; e^{-i\mathbf{u}\cdot\mathbf{x}} e^{-i\mathbf{v}\cdot\boldsymbol{\xi}} \mathrm{d}\mathbf{u}\,\mathrm{d}\mathbf{v}.\end{aligned}$$

This implies that if there is a matrix $\mathbf{M} \in \mathbb{R}^{2n\times 2n}$ such that $\mu(\mathbf{g}, 0) = \rho(\mathbf{M}\mathbf{g}, 0)$ for all $\mathbf{g} \in H_n' = \{\mathbf{g} \in \mathbb{R}^{2n} : (\mathbf{g}, t) \in \mathrm{H}_n\}$, then by a change of variables in the last expression, we get

$$(\mathcal{W}\mathcal{V}f)(\mathbf{g}) = |\det \mathbf{M}|(\mathcal{W}f)(\mathbf{M}\mathbf{g}).$$

Now consider the subgroup of $U(L^2(\mathbb{R}^n))$

$$G = \{\mathcal{V} \in U(L^2(\mathbb{R}^n)) : \forall (\mathbf{g}, t) \in \mathbb{R}^{2n+1}, \exists\, \mathbf{g}' \in \mathbb{R}^{2n} \text{ s.t. } \mathcal{V}^*\mu(\mathbf{g}, t)\mathcal{V} = \mu(\mathbf{g}', t)\}$$

The $\mathbf{g}'$ is uniquely defined so that there is a homomorphism $\nu(\mathcal{V}) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ given by $\nu(\mathcal{V})\mathbf{g} = \mathbf{g}'$. This ν is a continuous mapping from G onto $Sp(n)$ in the subspace topology of $G \subset U(L^2(\mathbb{R}^n))$ with kernel $\{c\mathcal{I} : |c| = 1\}$. This means that ν^{-1} is only defined up to a unimodular factor. We obtain the metaplectic group which is a twofold covering of the symplectic group. This shows up in the formulas in the form of a square root for which the sign has to be chosen.

With these tools, our original problem of finding a unitary $\mathcal{V}$ that causes an arbitrary linear transform of the Wigner distribution, can be solved. It requires some more lines to show that $\mathcal{W} = \mathcal{R}_{\mathbf{M}}\mathcal{W}\mathcal{V}$, if and only if $\mathcal{V} \in G$ and $\mathbf{M} = \nu(\mathcal{V})^{-1} \in Sp(n)$.

Several simple examples from the group G can be found.

- **Fourier transform**
  For example, the n-dimensional FT $\mathcal{F}$ satisfies all the properties and $\nu(\mathcal{F}) = \mathbf{J}^T$.

- **dilation**
  A second example is the dilation operator: $\mathcal{D}^*_{\mathbf{b}}\mu(\mathbf{x}, \boldsymbol{\xi}, t)\mathcal{D}_{\mathbf{b}} = \mu(\mathbf{b}^T\mathbf{x}, \mathbf{b}^{-1}\boldsymbol{\xi}, t)$ with $\mathbf{b} \in GL(n)$. We have now

$$\nu(\mathcal{D}_{\mathbf{b}}) = \begin{bmatrix} \mathbf{b}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{b}^T \end{bmatrix}.$$

  Note that if $\mathbf{b}$ is symmetric then $(\mathcal{D}_{\mathbf{b}}f)(\mathbf{x}) = (\det \mathbf{b})^{-1/2} f(\mathbf{b}^{-1}\mathbf{x})$.

- **chirp multiplication**
  A third example is a chirp multiplication $\mathcal{C}^*_{\mathbf{s}}\mu(\mathbf{x}, \boldsymbol{\xi}, t)\mathcal{C}_{\mathbf{s}} = \mu(\mathbf{x} + \mathbf{s}\boldsymbol{\xi}, \boldsymbol{\xi}, t)$.

$$\nu(\mathcal{C}_{\mathbf{s}}) = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{s} & \mathbf{1} \end{bmatrix}.$$

  With an n-dimensional chirp defined as $c_{\mathbf{s}}(\mathbf{x}) = \exp\{\frac{i}{2}\mathbf{x}^T\mathbf{s}\mathbf{x}\}$, and the effect is that $(\mathcal{C}_{\mathbf{s}}f)(\mathbf{x}) = c_{\mathbf{s}}(\mathbf{x})f(\mathbf{x})$. It is clearly no restriction if we assume that $\mathbf{s}$ is symmetric.

In view of the decomposition of the LCT in the scalar case, it is natural to define the n-dimensional LCT as $\mathcal{F}^{\mathbf{M}} = c\,\mathcal{C}_{\mathbf{d}\mathbf{b}^{-1}}\mathcal{D}_{\mathbf{b}}\mathcal{F}\mathcal{C}_{\mathbf{b}^{-1}\mathbf{a}}$ with $|c| = 1$. It is represented by a matrix

$$\nu(\mathcal{C}_{\mathbf{d}\mathbf{b}^{-1}}\mathcal{D}_{\mathbf{b}}\mathcal{F}\mathcal{C}_{\mathbf{b}^{-1}\mathbf{a}}) = \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix}.$$

The special case of the separable n-dimensional FrFT corresponds to $\mathbf{a} = \mathbf{d} = \operatorname{diag}(\cos\alpha_1, \ldots, \cos\alpha_n)$ and $\mathbf{b} = -\mathbf{c} = \operatorname{diag}(\sin\alpha_1, \ldots, \sin\alpha_n)$. Then $\mathbf{M} \in Sp(n) \cap SO(2n)$, the orthogonal symplectic group. For more details on this approach see e.g. [9, 28] and for the integral representation [29].

9 Other transforms

Probably motivated by the success of the FrFT and the LCT, quite some effort has been put in the design of fractional versions of related classical transforms.

9.1 Radial canonical transforms

It should be clear that for problems with circular symmetry, this symmetry should be taken into account when defining the transforms. Take for example the 2-dimensional case. Instead of Cartesian (x, y) coordinates, one should switch to polar coordinates so that, because of the symmetry, the transform will only depend on the radial distance. For example, it is well known that the Hankel transform appears naturally as a radial form of the (2-sided) Laplace transform [34, sec. 8.4]. Giving directly the n-dimensional formulation, we shall switch from the n-dimensional variables $\mathbf{x}$ and $\boldsymbol{\xi}$ to the scalar variables $x = \|\mathbf{x}\|$ and $\xi = \|\boldsymbol{\xi}\|$, and the n-dimensional LCT will become *canonical Hankel transforms* [33, 30]. It is a one-sided integral transform $\int_0^\infty K_M(\xi, x)f(x)\mathrm{d}x$ with kernel

$$K_M(\xi, x) = x^{n-1}\frac{e^{-\frac{\pi}{2}(\frac{n}{2}+\nu)}}{b}(x\xi)^{1-n/2}\exp\left\{\frac{i}{2b}(ax^2 + d\xi^2)\right\}J_{n/2+\nu-1}\left(\frac{x\xi}{b}\right),$$

where J_ν is the Bessel function of the first kind of order ν. The *fractional Hankel transform* is a special case of the canonical Hankel transform when the matrix M is replaced by a rotation matrix.

9.2 Fractional Hilbert transform

The definition of the Hilbert transform has been given before in (7). Note that the convolution defining the transform can be characterized by a multiplication with $-i\mathrm{sgn}\ (\xi)$ in the Fourier domain. Since $-i\mathrm{sgn}\ (\xi) = e^{-i\pi/2}h(\xi) + e^{i\pi/2}h(-\xi)$ with h the Heaviside step function: $h(\xi) = 1$ for $\xi \geq 0$ and $h(\xi) = 0$ for $\xi < 0$, we can now define a fractional Hilbert transform as

$$(\mathcal{F}^M)^{-1}[(e^{-i\phi}h(\xi) + e^{i\phi}h(-\xi))(\mathcal{F}^M f)(\xi)],$$

with M the rotation matrix $M = R_a$. For further reading see [35, 13, 23, 7].

9.3 Cosine, sine and Hartley transform

While in the classical Fourier transform, the integral is taken of $f(x)e^{-i\xi x}$, one shall in the cosine, sine, and Hartley transform replace the complex exponential by $\cos(\xi x)$, $\sin(\xi x)$ or $\mathrm{cas}(\xi x) = \cos(\xi x) + \sin(\xi x)$ respectively. Since cos and sin are the real and imaginary part of the complex exponential, one might think of defining the fractional cosine and sine transforms by replacing the kernel in the FrFT by its real or imaginary part. However, this will not lead to index additivity for the transforms. We could however use the general fractionalization procedure given in (2). We just have to note that the Hermite-Gauss eigenfunctions are also eigenfunctions of the cosine and sine transform, except that for the cosine transform, the odd eigenfunctions will correspond to eigenvalues zero and for the sine transform, the even eigenfunctions will correspond to eigenvalue zero. This implies that the odd part of f will be killed by the cosine transform. So, the cosine transform will not be invertible unless we restrict ourselves to the set of even functions. A similar observation holds for the sine transform: it can only be invertible when we restrict the transform to the set of odd functions. This motivates the habit to define sine and cosine transforms by one sided integrals over $\mathbb{R}^+$. See [26]. The bottom line of the whole fractionalization process is that to obtain the good fractional forms of these operators we essentially have to replace in the definition of the FrFT the factor $e^{i\xi x}$ in the kernel of the transform by $\cos(\xi x)$, $\sin(\xi x)$ or $\mathrm{cas}(\xi x)$ to obtain the kernel for the fractional cosine, sine or Hartley transforms respectively. In the case of the cosine or sine transform, the restriction to even or odd functions implies that we need only to transform half of the function, which means that the integral over $\mathbb{R}$ can be replaced by two times the integral over $\mathbb{R}^+$. Besides the fractional forms of these operators there are also canonical forms for which we refer to [26]. Also here simplified forms exist [25, 26].

9.4 Other transforms

The list of transforms that have been fractionalized is too long to be completed here. Some examples are: Laplace, Mellin, Hadamard, Haar, Gabor, Radon, Derivative, Integral, Bragman, Barut-Girardello,... The fractionalization procedure of (2) can be used in general. This means the following. If we have a linear operator $\mathcal{T}$ in a complex separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_2$ and if there is a complete set of orthonormal eigenvectors ϕ_n with corresponding eigenvalues λ_n, then any element in the space can be represented as $f = \sum_{n=0}^{\infty} a_n\phi_n$, $a_n = \langle f, \phi_n\rangle_2$, so that $(\mathcal{T}f) = \sum_{n=0}^{\infty} a_n\lambda_n\phi_n$. The fractional transform can the be defined as

$$(\mathcal{T}^a f)(\xi) = \sum_{n=0}^{\infty} a_n\lambda_n^a\phi_n(\xi) = \sum_{n=0}^{\infty} \lambda_n^a \langle f, \phi_n\rangle_2 \phi_n(\xi) = \langle f, K_a(\xi,\cdot)\rangle_2 ,$$

where

$$K_a(\xi, x) = \sum_{n=0}^{\infty} \overline{\lambda}_n^a \overline{\phi_n(\xi)}\phi_n(x).$$

Of course a careful analysis will require some conditions like for example if it concerns the Hilbert space $L^2_\mu(I)$ of square integrable functions on an interval I with respect to a measure μ, then we need $K_a(\xi,\cdot)$ to be in this space, which means that $\sum_{n=0}^{\infty} |\lambda_n|^{2a}|\phi_n(\xi)|^2 < \infty$ for all ξ.

In view of the general development for the construction of fractional transforms, it is clear that the main objective is to find a set on orthonormal eigenfunctions for the transform that one wants to “fractionalize”. There were several papers that give eigenvalues and eigenvectors for miscellaneous transforms.

Zayed [36] has given an alternative that uses instead of the kernel $K_a(\xi, x) = \sum_n \lambda_n^a \phi_n^*(x)\phi_n(\xi)$ the kernel

$$K_a(\xi, x) = \lim_{|\lambda|\to 1^-} \sum_n |\lambda| e^{in\alpha}\phi_n(\xi)^*\phi_n(x).$$

Thus λ_n^a is replaced by $|\lambda|^n e^{in\alpha}$ and the ϕ_n can be any (orthonormal) set of basis functions. In this way he obtains fractional forms of the Mellin, and Hankel transforms, but also of the Riemann-Liouville derivative and integral, and he defines a fractional transform for the space of functions defined on the interval $[-1, 1]$ based on Jacobi-functions which play the role of the eigenfunctions.

To the best of our knowledge, a further generalization by taking a biorthogonal system spanning the Hilbert space, which is very common in wavelet analysis, has not yet been explored in this context.

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