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A note on discrete monotonic dynamical systems

Dongsheng Liu [1]
Department of Applied Mathematics, Nanjing University of Science & Technology
Nanjing, 210094, Jiangsu, Peoples R China
d.liu@lancaster.ac.uk

ABSTRACT

We give a upper bound of Lebesgue measure $V(S(f,h,\Omega))$ of the set $S(f,h,\Omega)$ of points $q \in Q_h^d$ for which the triple (h,q,Ω) is dynamically robust when f is monotonic and satisfies certain condition on some compact subset $\Omega \in \mathbb{R}^d$.

RESUMEN

Damos una cota superior de la medida de Lebesgue $V(S(f,h,\Omega))$ del conjunto $S(f,h,\Omega)$ de puntos $q \in Q_h^d$ para los cuales el trío (h,q,Ω) es dinámicamente robusto cuando f es monótona y satisface ciertas condiciones en algunos subconjuntos compactos $\Omega \in \mathbb{R}^d$.

Key words and phrases: *Roundoff operator, dynamical robustness, dynamical system.*
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1 Introduction

A discrete dynamical system on the state space $\mathbb{R}^d$ is generated by the iteration of a mapping $f : \mathbb{R}^d \to \mathbb{R}^d$, that is $x_{n+1} = f(x_n), n = 0, 1, 2, \cdots$.

[1]Current address: Department of Physics, Lancaster University, Lancaster, LA1 4YB, UK.

Let Q_h^d denote the h-cube in $\mathbb{R}^d$ centered at origin, that is

$$Q_h^d = \{x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d : -\frac{h}{2} < x_i \leq \frac{h}{2}, i = 1, 2, \cdots, d\}$$

and for each $q \in Q_h^d$, let $L_{h,q} = \{q + hz : z \in \mathbb{Z}^d\}$ be the uniform h-lattice in $\mathbb{R}^d$ centered at q.

For $q \in Q_h^d$, we define the roundoff operator $[.]_{h,q}$ from $\mathbb{R}^d$ into $L_{h,q}$ by $[x]_{h,q} = L_{h,q} \cap (x + Q_h^d)$ for $x \in \mathbb{R}^d$, or equivalently by

$$[x]_{h,q} = ([x_1 - q_1]_h + q_1, \cdots, [x_d - q_d]_h + q_d)$$

where $x = (x_1, x_2, \cdots .x_d), q = (q_1, q_2, \cdots, q_d)$ and $[y]_h$ is scalar roundoff operator defined by

$$[y]_h = kh \qquad \text{if } (k - \frac{1}{2})h \leq y < (k + \frac{1}{2})h.$$

Let f be a dynamical system in $\mathbb{R}^d$. The map $f_{h,q} : L_{h,q} \to L_{h,q}$ defined by

$$f_{h,q}(x) = [f(x)]_{h,q}, \qquad x \in L_{h,q}$$

is called $L_{h,q}$-discretization of f.

Now we give the definition of dynamical robustness [2]:

Given $h > 0$, $q \in Q_h^d$, and a compact set $\Omega \subset \mathbb{R}^d$, we say the triple (h, q, Ω) is dynamically robust if the discretization $f_{h,q}$ has a single equilibrium $x_{h,q} = f_{h,q}(x_{h,q}) \in \Omega \cap L_{h,q}$ and

$$\lim_{n\to\infty} |f_{h,q}^n(x) - x_{h,q}| = 0 \qquad \forall x \in L_{h,q} \cap \Omega.$$

In [2] the following question was raised: given f and a compact set $\Omega \in \mathbb{R}^d$, what is the Lebesgue measure $V(S(f, h, \Omega))$ of the set $S(f, h, \Omega)$ of points $q \in Q_h^d$ for which the triple (h, q, Ω) is dynamically robust?

They answered this question partially: when Ω is a parallel-polyhedron in $\mathbb{R}^d$, f is monotonic on Ω and satisfies some condition, they give a lower bound for $V(S(f, h, \Omega))$. In this paper we give a upper bound of $V(S(f, h, \Omega))$ for f is monotonic and satisfies certain condition on some compact subset $\Omega \in \mathbb{R}^d$.

2 Main Results

We give the semi-ordering in $\mathbb{R}^d$: for $x, y \in \mathbb{R}^d$, we say $x \leq y$ if $x_i \leq y_i$ for $i = 1, 2, \cdots, d$ and $x < y$ if $x_i < y_i$ for $i = 1, 2, \cdots, d$. We shall say f is monotonically increasing on a set $S \in \mathbb{R}^d$ if $f(x) \leq f(y)$ for all $x, y \in S$ with $x \leq y$.

In the following we restrict attention to monotonically increasing functions, noting that the monotonic decreasing case is handled similarly.

By the definition of dynamically robust, we have

Proposition 1 *Let f be monotonic on the compact set $\Omega \subset \mathbb{R}^d$ and suppose that (h, q, Ω) is a dynamically robust, $x_{h,q}$ is the single equilibrium. $\forall x \in L_{h,q} \cap \Omega$, if $x \geq x_{h,q}$, $f_{h,q}(x) = x_{h,q}$; if $x \leq x_{h,q}$, there exists $k \in \mathbb{N}$ such that $f_{h,q}^k(x) = x_{h,q}$.*

Proof. Because Ω is a compact subset of $\mathbb{R}^d$, $L_{h,q}\cap\Omega$ is a finite set. For $x\in L_{h,q}\cap\Omega$, if $x\le x_{h,q}$, let $f_{h,q}(x)=x^{(1)}$, then $x^{(1)}=f_{h,q}(x)\le f_{h,q}(x_{h,q})=x_{h,q}$. If $x^{(1)}=x_{h,q}$ it is proved with $k=1$. Otherwise we consider $x^{(2)}:=f_{h,q}(x^{(1)})\le f_{h,q}(x_{h,q})=x_{h,q}$. If $x^{(2)}=x_{h,q}$ it is proved with $k=2$. If $x^{(2)}\neq x_{h,q}$ we can continue this process. But $L_{h,q}\cap\Omega$ is finite set there exists a $k\in\mathbb{N}$ such that $f^k_{h,q}(x)=f_{h,q}(x^{(k-1)})=x_{h,q}$. If $x\ge x_{h,q}$, and $f_{h,q}(x)\neq x_{h,q}$, by the monotonicty, $f_{h,q}(x)\ge f_{h,q}(x_{h,q})=x_{h,q}$. so $|f^n_{h,q}(x)-x_{h,q}|\ge|f_{h,q}(x)-x_{h,q}|>0$ for any $n\in\mathbb{N}$, It is contradiction to the definition of dynamically robust of (h,q,Ω). So we have $f_{h,q}(x)=x_{h,q}$.

In fact, $\forall x\in L_{h,q}\cap\Omega$ there exists $k\in\mathbb{N}$ such that $f^k_{h,q}(x)=x_{h,q}$. ■

Now we can estimate $V(S(f,h,\Omega))$.

Theorem 1 *Ω is a compact subset of $\mathbb{R}^d$ and satisfies: $\forall q\in Q^d_h$, there exist $u_{1,q},u_{2,q}\in L_{h,q}\cap\Omega$ such that $\forall x\in L_{h,q}\cap\Omega$, $u_{1,q}\le x\le u_{2,q}$. Let f be monotonic on the compact set $\Omega\subset\mathbb{R}^d$ and $f(\Omega)\subset\Omega'$ where $\Omega'\subset\Omega$ and satisfies $\forall x=(x_1,\cdots,x_d)\in\partial\Omega$, the boundary of Ω, and $\forall x'=(x'_1,\cdots,x'_d)\in\Omega'$, $|x_i-x'_i|\ge\frac{h}{2}$, $i=1,2,\cdots,d$. we have*

$$V(S(f,h,\Omega))\le\frac{L}{L-1}h^d-\frac{1}{L-1}V(\{x\in\Omega:x-\frac{h}{2}\le f(x)<x+\frac{h}{2}\})$$

where $L=L_1\times L_2\times\cdots\times L_d$ and L_i is determined by following: let $l_i=|max\{x_i:x=(x_1,\cdots,x_i,\cdots,x_d)\in\Omega\}-min\{x_i:x=(x_1,\cdots,x_i,\cdots,x_d)\in\Omega\}|$ and $l_i=rh+p$, $0\le p<h$ then $L_i=r+1$.

Proof. The method of this proof is following that in [2]. Let $F(h,q)=L_{h,q}\cap\{x: x-\frac{h}{2}\le f(x)<x+\frac{h}{2}\}$ and $k(h,q)=\#\{F(h,q)\}$. In order to carry on proof, we need following

Lemma 1 *[3].*

$$\int_{Q^d_h}k(h,q)dq=V(\{x:x-\frac{h}{2}\le f(x)<x+\frac{h}{2}\}).$$

We also need the following special case of the Birkhoff-Tarski Theorem

Lemma 2 *[1]. Let g be a monotonic map of a finite set $\Gamma\in\mathbb{R}^d$ into itself. If g satisfies $g(x)\ge x$ or $g(x)\le x$ for $x\in\Gamma$, then the iterative sequence $x_{n+1}=g(x_n)$ with $x_0=x$ converge to the fixed point $g(x^*)=x^*\in\Gamma$.*

Remark:
(1). We can get the fixed point by following: take any $x\in\Gamma$ with $g(x)\ge x$ or $g(x)\le x$ and iterate $x_{n+1}=g(x_n)$ with $x_0=x$, because Γ is finite, after a finite number of steps we can get a fixed point.
(2). If $f_{h,q}$ has only one fixed point x^*, then $x^*=f^k_{h,q}(u_{1,q})$ and $x^*=f^l_{h,q}(u_{2,q})$ for some $k,l\in\mathbb{N}$. Since $u_{1,q}\le x\le u_{2,q}$ it is easy to see $f^n_{h,q}(x)=x^*$, $\forall x\in L_{h,q}\cap\Omega$ for large $n\in\mathbb{N}$.

The condition on f guarantees that $f_{h,q}$ is a mapping of $L_{h,q}\cap\Omega$ into itself. The elements of $F_{h,q}$ are precisely the fixed points of $f_{h,q}$. So it is easy to see $k(h,q)\ge 1$

from the lemma 2 because $f_{h,q}(u_{1,q}) \geq u_{1,q}$. By the definition of dynamically robust and remark (2), we have $q \in S(f,h,\Omega)$ if and only if $k(h,q) = 1$. So $V(S(f,h,\Omega)) = V(\{q : k(h,q) = 1\})$. But

$$V(\{q : k(h,q) = 1\}) + V(\{q : k(h,q) > 1\}) = h^d,$$

and $k(h,q)$ at most equal to $L = L_1 \times \cdots \times L_d$. By lemma 1, we have

$$V(\{x : x - \frac{h}{2} \leq f(x) < x + \frac{h}{2}\}) = \int_{Q_h^d} k(h,q)dq$$

$$\begin{aligned}
&= V(\{q : k(h,q) = 1\}) + \sum_{i=2}^{L} i \times V(\{q : k(h,q) = i\}) \\
&\leq V(\{q : k(h,q) = 1\}) + L \times V(\{q : k(h,q) > 1\}) \\
&= V(\{q : k(h,q) = 1\}) + Lh^d - L \times V(\{q : k(h,q) = 1\}) \\
&= Lh^d - (L-1)V(\{q : k(h,q) = 1\}) \\
&= Lh^d - (L-1)V(S(f,h,\Omega)).
\end{aligned}$$

So,

$$V(S(f,h,\Omega)) \leq \frac{L}{L-1}h^d - \frac{1}{L-1}V(\{x \in \Omega, x - \frac{h}{2} \leq f(x) < x + \frac{h}{2}\}).$$

Under the condition of Theorem 2, the result of Theorem 1 in [2] still holds. Combining with the Theorem 1 in [2], we get

Corollary 1 *Under the condition of Theorem 2 below we have*
$Max\{0, 2h^d - V(\{x \in \Omega : x - \frac{h}{2} \leq f(x) < x + \frac{h}{2}\})\} \leq V(S(f,h,\Omega))$
$\leq \frac{L}{L-1}h^d - \frac{1}{L-1}V(\{x \in \Omega, x - \frac{h}{2} \leq f(x) < x + \frac{h}{2}\})$.

Remark: It is easy to see $h^d \leq V(\{x \in \Omega : x - \frac{h}{2} \leq f(x) < x + \frac{h}{2}\}) \leq Lh^d$.

If f is not monotonic, the situation is complex. Following we give a special example.

For g is a map from Ω into itself, we say x is a periodic point of g, if there exist $n \in \mathbb{N}$ such that $g^n(x) = x$. The least n which satisfies $g^n(x) = x$ is called period of g at x.

Now we give the example.

Theorem 2 *Let f be a map from a compact set Ω into Ω'. where $\Omega' \subset \Omega$ and satisfies $\forall x = (x_1, \cdots, x_d) \in \partial\Omega$, the boundary of Ω, and $\forall x' = (x'_1, \cdots, x'_d) \in \Omega'$, $|x_i - x'_i| \geq \frac{h}{2}$, $i = 1, 2, \cdots, d$. If $\forall q \in Q_h^d$, $f_{h,q}$ has no periodic point with period more than 1, then*

$$V(S(f,h,\Omega)) \leq \frac{L}{L-1}h^d - \frac{1}{L-1}V(\{x \in \Omega : x - \frac{h}{2} \leq f(x) < x + \frac{h}{2}\})$$

where $L = L_1 \times L_2 \times \cdots \times L_d$ and L_i is determined by following: let $l_i = |max\{x_i : x = (x_1, \cdots, x_i, \cdots, x_d) \in \Omega\} - min\{x_i : x = (x_1, \cdots, x_i, \cdots, x_d) \in \Omega\}|$ and $l_i = rh + p$, $0 \leq p < h$ then $L_i = r + 1$.

Proof. We note

$$V(\{q : k(h,q) = 0\}) + V(\{q : k(h,q) = 1\}) + V(\{q : k(h,q) > 1\}) = h^d,$$

so

$$V(\{q : k(h,q) > 1\}) \leq h^d - V(\{q : k(h,q) = 1\}).$$

■

Now we only need to prove following.

Lemma 3

$$q \in S(f,h,\Omega) \qquad \textit{if and only if} \quad k(h,q) = 1.$$

Proof. Let $q \in S(f,h,\Omega)$, but $k(h,q) \neq 1$. Then $k(h,q) = 0$ or $k(h,q) > 1$. That means dynamical system $f_{h,q}$ has no equilibrium or has at least two distinct equilibria, it is contradition to $q \in S(f,h,\Omega)$.

If $k(h,q) = 1$, let $x_{h,q}$ is the unique fixed point of $f_{h,q}$ in $L_{h,q} \cap \Omega$. $\forall x_1 \in L_{h,q} \cap \Omega$, the condition of f guarantee $f_{h,q}(x_1) \in L_{h,q} \cap \Omega$. Let $F_{h,q}(x_1) := x_2$, if $x_2 \neq x_1$, we consider $f_{h,q}(x_2) := x_3$. If $x_3 \neq x_2$ then $x_3 \neq x_1$ since $f_{h,q}$ has no periodic point with period more than 1. We continue this process and get $x_1, x_2, \cdots, \in L_{h,q} \cap \Omega$, which are pairwise distinct. But $L_{h,q} \cap \Omega$ is finite, so after finite number of steps, say N steps, we have $f^N_{h,q}(x_1) = f_{h,q}(x_N) = x_N$. But $x_{h,q}$ is the unique fixed point, we get $x_N = x_{h,q}$ and $f^N_{h,q}(x_1) = x_{h,q}$. So $f^m_{h,q}(x_1) = x_{h,q}$ for any $m \geq N$, that is

$$\lim_{n\to\infty} f^n_{h,q}(x) = x_{h,q}, \qquad \forall x \in L_{h,q} \cap \Omega.$$

i.e., $q \in S(f,h,\Omega)$.

The next step of the proof is the same as that in Theorem 2. ■

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