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Conjectures in Inverse Boundary Value Problems for Quasilinear Elliptic Equations

Ziqi Sun
Department of Mathematics and Statistics Wichita State university
Wichita, KS 67226, USA
ziqi.sun@wichita.edu

ABSTRACT

Inverse boundary value problems originated in early 80's, from the contribution of A.P. Calderon on the inverse conductivity problem [C], in which one attempts to recover the electrical conductivity of a body by means of boundary measurements on the voltage and current. Since then, the area of inverse boundary value problems for linear elliptic equations has undergone a great deal of development [U]. The theoretical growth of this area contributes to many areas of applications ranging from medical imaging to various detection techniques [B-B][Che-Is].

In this paper we discuss several conjectures in the inverse boundary value problems for quasilinear elliptic equations and their recent progress. These problems concern anisotropic quasilinear elliptic equations in connection with nonlinear materials and the nonlinear elasticity system.

RESUMEN

Problemas inversos a valores en la frontera se desarrollaron a comienzos de la década de los 80, a partir de contribuciones de A.P. Calderon en el problema de conductividad inversa [C], en el cual se intenta recuperar las conductividad eléctrica de un cuerpo mediante mediciones de voltaje y corriente en la frontera. Desde entonces, el área de problemas a valores en la forntera inversos para ecuaciones lineales elípticas ha sido objeto de mucho desarrollo [U]. El crecimiento de la teoría en esta área tiene aplicaciones en muchas aplicaciones, las que varían desde imagenología médica, hasta diversos métodos de detección [BB], [Che-Is]. En este artículo, discutimos varias conjeturas en problemas inversos de valores en la frontera para ecuaciones elípticas quasi-lineales y sus progresos recientes. Estos problemas dicen relación con ecuaciones elípticas quasilineales anisotrópicas en conexión con materiales nolineales y sistemas de elasticidad no lineal.

Key words and phrases: *Inverse boundary value problem. Dirichlet to Neumann map*
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1 Anisotropic Quasilinear Conductivity Equations

Consider the quasilinear elliptic equation

$$L_A u = \sum_{i,j=1}^{n} (a_{ij}(x,u)u_{x_i})_{x_j} = 0, u|_\Gamma = f \in C^{2,\alpha}(\Gamma) \tag{1}$$

on a bounded domain $\Omega \subset R^n$, $n \geq 2$, with smooth boundary Γ. Here $A(x,t) = (a_{ij}(x,t))_{n\times n}$ is the quasilinear coefficient matrix which is assumed to be in the $C^{1,\alpha}$ class with $0 < \alpha < 1$. The nonlinear Dirichlet to Neumann map

$$\Lambda_A : f \to \nu \cdot A(x,f)\nabla u|_\Gamma$$

is an operator from $C^{2,\alpha}(\Gamma)$ to $C^{1,\alpha}(\Gamma)$, which carries essentially all information about the solution u which can be measured on the boundary. Here we denote ν to be the unit outer normal of Ω. The inverse problem under discussion is to recover information about the quasilinear coefficient matrix A from the knowledge of Λ_A.

This problem was raised by R. Kohn and M. Vogelius [KV] in mid 80's as a nonlinear analogue of the well known inverse conductivity problem posed by A.P. Calderon [C]. Physically, the problem is connected to Electrical Impedance Tomography in nonlinear media.

It has been shown in [Su1] that, in the isotropic case of the problem, i.e., when A is a scalar matrix, the Dirichlet to Neumann map Λ_A gives full information about A. In other words, Λ_A determines A uniquely as a function on $\Omega \times R$. This generalizes to the quasilinear case the well known uniqueness theorems of the linear case (i.e., when A is scalar and is indenpendent on t)[SU1,2][SuU2] [N].

In the anisotropic case, however, one only expects to recover A module the group

$$G = \{\text{all } C^{3,\alpha} \text{ diffeomorphisms } \Phi\text{: } \bar{\Omega} \to \bar{\Omega} \text{ with } \Phi\big|_{\partial\Omega} = \text{identity}\}.$$

In fact, Λ_A is invariant under G: For any A and $\Phi \in G$, $\Lambda_A = \Lambda_{H_\Phi A}$. Here $H_\Phi A$ is the pull back of A under Φ:

$$H_\Phi A(x,t) = (|detD\Phi|^{-1}(D\Phi)^T A(x,t)(D\Phi)) \circ \Phi^{-1} \tag{2}$$

where $D\Phi$ is the Jacobian matrix of Φ. One should observe that (2) holds only when Φ is independent on t. Thus, the following conjecture is natural:

Conjecture 1: Assume that $\Lambda_{A_1} = \Lambda_{A_2}$. Then there exists a unique diffeomorphism $\Phi \in G$ so that $A_2 = H_\Phi A_1$.

In [SuU1] we have verified this conjecture in the $C^{2,\alpha}$ category for dimension $n = 2$ and in the real analytic category for dimension $n \geq 3$. These results extend all known results regarding this conjecture in the case of linear coefficient matrices (i.e. when A is independent of t), obtained earlier in the works of Sylvester [S], Nachman [N] and Lee-Uhlmann [LU]. We mention that in the two dimensional case the unique diffeomorphism Φ in the result belongs to the $C^{3,\alpha}$ class, which is one order smoother than A_1 and A_2 and in the case $n \geq 3$, Φ is in the real analytic category. Assuming Holder smoothness for the coefficient seems quite essential to assure that Φ is one order smoother than the coefficient matrices. As explained in [SuU1], this is closely related to the elliptic regularity theory.

The proof is based on a well known linearization technique introduced in [I1] and further developed in [I2][IS][IN][Su1,3] which reduces the nonlinear problem to a linear one. Let $t \in R$ and $g \in C^{2,\alpha}(\Gamma)$. From Λ_A one determines two linear operators:

$$K^{(1)}_{A,t} : g \to \ d/ds\Lambda_A(t + sg)|_{s=0}$$

$$K^{(2)}_{A,t} : g \to \ d^2/ds^2(s^{-1}\Lambda_A(t + sg))|_{s=0} \tag{3}$$

One observes that $K^{(1)}_{A,t} = \Lambda_{A^t}$, the Dirichlet to Neumann map corresponding to the linear coefficient matrix $A^t(x) = A(x,t)$ for a fixed t. So, if $\Lambda_{A_1} = \Lambda_{A_2}$ for two quasilinear coefficient matrices A_1 and A_2, then $\Lambda_{A_1^t} = \Lambda_{A_2^t}$, $\forall t \in R$, and since the conjecture is true in the linear case, one obtains a family of diffeomorphisms $\Phi^t \in G$, depending on the parameter t, so that

$$H_{\Phi^t} A_1^t = A_2^t, \forall t \in R. \tag{4}$$

The mathematical difficulty is to show that Φ^t is actually independent on t, which would imply the result. It has been verified in [SuU1] that Φ^t is smooth in t. For dimension $n \geq 3$, this was achieved by studying a related geometrical problem in which Φ^t becomes a family of isometries between two families of Riemannian metrics $|A_i^t|^{1/(n-2)}(A_i^t)^{-1}$ on $\bar{\Omega}, i = 1,2$. For $n = 2$, One can transform it to a similar problem where Φ^t becomes a family of conformal diffeomorphisms between Riemannian metrics $(A_i^t)^{-1}$, $i = 1,2$. In the latter case, the smoothness is verified via the standard theory of the Beltrami equation [AB].

So, the task is to show that $\dot{\Phi}|_{t=0}$, where dot means differentiation in t variable. We only give a very brief description of the proof. One only needs to show

$$\dot{\Phi}^0 = \dot{\Phi}^t|_{t=0} = 0 \tag{5}$$

since the same argument works for $t \neq 0$. By a transformation one may assume that $\Phi^0 =$ identity map. The proof of (5) is then based on the information obtained from (3):

$$K^{(2)}_{A_1,t} = K^{(2)}_{A_2,t}. \tag{6}$$

A crucial step of the proof is to show that one can recover from $K^{(2)}_{A,t}$ information about $\partial A/\partial t(x,0)$. So (6) implies

$$\frac{\partial}{\partial t} A_1(x,0) = \frac{\partial}{\partial t} A_2(x,0), \forall x \in \Omega. \tag{7}$$

One views (7) as a certain control over the flows A_1^t and A_2^t at $t = 0$. Actually, the assumption $\Phi^0 =$Id. together with (7) give $A_1^0 = A_2^0$ and $\dot{A}_1^0 = \dot{A}_2^0$. Consider now the solution flows $u_{i,f}^t$ for the linear equations $L_{A_i^t}(u_{i,f}^t) = 0$ with $u_{i,f}^t|_\Gamma = f$, $i = 1, 2$. One observes that the control over the flows of coefficient matrices translates to a control over the solution flows. In fact, for every f, $u_{1,f}^0 = u_{2,f}^0$ and $\dot{u}_{1,f}^0 = \dot{u}_{2,f}^0$. Since the transformation in (4) links $u_{1,f}^t$ to $u_{2,f}^t$ via the relation $\dot{u}_{1,f}^t = \dot{u}_{2,f}^t \circ \Phi^t$, one differentiates it in t at $t = 0$ to get $\dot{\Phi}^0 \cdot \nabla u_{1,f}^0 = 0$ for all boundary value f, from which (5) follows by an argument based on Runge approximation. See [SuU1] for details.

The above result obtained in [SuU1] covers the two dimensional case and the real analytic case in dimension three or higher. However, the remaining case in dimension $n \geq 3$ is essentially open even when the equation (1) is linear. An interesting problem for further study in this direction is whether one can reduce the conjecture in the quasilinear case directly to the conjecture in the linear case. In other words, one would like to verify Conjecture 1 under the assumption that Conjecture 1 holds in the linear case. Such a full reduction has already been obtained in the scalar case (where A is a scalar matrix) [Su1]. It is possible that the same reduction also hold in the anisotropic case. One possible approach to attack this problem is to further study the relation between (6) and (7) in the general case, which is the heart of proof in [SuU1]. The main issue is how to avoid the use of the property of completeness of products of solutions which is currently available only in the two dimensional case and the case of real analytic coefficient matrices.

2 Quasilinear Equations in Connection with Nonlinear Elastic Materials

Consider the quasilinear elliptic equation

$$\nabla \cdot A(x, \nabla u) = 0, u|_\Gamma = f \in C^{3,\alpha}(\Gamma), \tag{8}$$

on a bounded domain $\Omega \subset R^n$, $n \geq 2$, with smooth boundary Γ. Here $A(x,p) = (a_1(x,p), a_2(x,p), ..., a_n(x,p))$ is the quasilinear coefficient vector. We assume that A and A_p (which is assumed to be symmetric) are both in $C^{2,\alpha}(\bar{\Omega} \times R)$ with $0 < \alpha < 1$, $A(x,0) = 0$ and the structure conditions which guarantee the unique solvability in the $C^{3,\alpha}$ class [HSu].

The nonlinear Dirichlet to Neumann map

$$\Lambda_A : f \to \nu \cdot A(x, \nabla u)|_\Gamma, \tag{9}$$

is an operator from $C^{3,\alpha}(\Gamma)$ to $C^{2,\alpha}(\Gamma)$, which carries essentially all information about the solution u observable on the boundary. One verifies that Λ_A is invariant under the group G: $\Lambda_A = \Lambda_{H_\Phi A}$ for all $\Phi \in G$. Here the transformation H_Φ is defined as

$$H_\Phi A(x, p) = (|det D\Phi|^{-1}(D\Phi)^T A(x, (D\Phi)p)) \circ \Phi^{-1}.$$

The main problem is whether the converse is true.

Conjecture 2: Assume that $\Lambda_{A_1} = \Lambda_{A_2}$. Then there exists a unique diffeomorphism $\Phi \in G$ so that $A_2 = H_\Phi A_1$.

The equation (8) can be considered as a simple scalar model of the nonlinear elasticity system, which takes the form

$$\nabla\{\sigma(x, E) + (\nabla u)\sigma(x, E)\} = 0, \tag{10}$$

where u is the displacement vector function resulting from a deformation of an elastic body and the matrix function σ is the constitutive relation with the strain tensor

$$E = \frac{1}{2}(\nabla u^T + \nabla u + \nabla u^T \nabla u).$$

In [HSu], we developed a mathematical framework towards proving this conjecture in the case of two dimensions. In the discussion below, we assume $\Lambda_{A_1} = \Lambda_{A_2}$ for two quasilinear coefficient vectors A_1 and A_2 in dimension two. By linearizing (9) one obtains, as in the case of Conjecture 1, a family of diffeomorphisms $\{\Phi_f\} \subset G$ which transforms $A_{1,p}(x, \nabla u_{1,f})$ to $A_{2,p}(x, \nabla u_{2,f})$:

$$A_{2,p}(x, \nabla u_{2,f}) = H_{\Phi_f} A_{1,p}(x, \nabla u_{1,f}),$$

and the main problem is to show that Φ_f is independent on f. Here we denote by $u_{i,f}$ solution of (11) with A replaced by A_i, $i = 1, 2$.

One notices that $\{\Phi_f,\ f \in C^{2,\alpha}(\Gamma)\}$ is an infinite dimensional family rather than an one dimensional family in the case of Conjecture 1. Also, contrary to (3), any further linearization on (9) would not provide any new information about Φ_f. So, technically, the task in this case is much harder to accomplish.

For a $f \in C^{3,\alpha}(\Gamma)$, let $g_{i,f}$ be the Riemannian metric (on $\bar{\Omega}$) generated by the metrix $A^{-1}_{i,p}(x, \nabla u_{i,f})$, $i = 1, 2$. One verifies that Φ_f is a family of conformal diffeomorphisms sending $(\bar{\Omega},\ g_{1,f})$ to $(\bar{\Omega},\ g_{2,f})$. If one uses $\Phi_f^* g$ to denote the pullback of a tensor g under Φ_f, then (15) can be rewritten as

$$\Phi_f^* g_{2,f} = |D\Phi_f| g_{1,f}.$$

Given $f,\ h \in C^{3,\alpha}(\Gamma)$, Let's denote by $\dot{g}_{i,f,h}$ the Frechet derivative of $g_{i,f}$ at the reference point f in the direction h, $i = 1, 2$. Once again, one can show that Φ_f

is smooth in f (parallel to those in Conjecture 1) and we denote by $X = \dot{\Phi}_{f,h}$ the corresponding derivative of Φ_f in the direction h (viewed as a vector field). For a fixed f, we may once again assume that $\Phi_f =$ identity and set $g_{1,f} = g_{2,f} =: g_f$ and $u_{1,f} = u_{2,f} =: u_f$.

In order to prove the conjecture by showing

$$X = \dot{\Phi}_{f,h} = 0, \forall h \in C^{3,\alpha}(\Gamma), \tag{11}$$

Let us take a deep look at the relation $\Phi_f^* g_{2,f} =\mid D\Phi_f \mid g_{1,f}$ by differentiating it in f with a direction $h \in C^{3,\alpha}(\Gamma)$. We get

$$\dot{g}_{1,f,h} - \dot{g}_{2,f,h} = L_X g_f - (e^{\sigma} \nabla_{g_f} \cdot (e^{-\sigma} X)) g_f. \tag{12}$$

where $L_X g_f$ stands for Lie derivative of g_f under the vector field X and $\sigma = log\sqrt{det(g)}$. Equation (12) implies that X is connected to the inhomogeneous conformal Killing field equation (with respect to the metric g_f) with the boundary condition $X \mid_{\Gamma} = 0$. However, this equation has no real consequence if one just considers one direction. The main observation made in [HSu] is that if one considers a pair of directions, then one can use the theory of conformal Killing field to obtain useful consequences leading to (11). Indeed, when one is given a pair of directions h_1, $h_2 \in C^{2,\alpha}(\Gamma)$, one can show that the following symmetric relation

$$\dot{g}_{f,h_1} \, l_{f,h_2} = \dot{g}_{f,h_2} \, l_{f,h_1}$$

holds for $\dot{g}_{f,h_1} = \dot{g}_{1,f,h_1}$ or $\dot{g}_{2,f,h_1}$ and $l_{f,h} = \nabla_{g_f} \dot{u}_{f,h} = g_f^{-1} \nabla \dot{u}_{f,h}$. This is proven in [HSu] using the special structure of the linearized coefficient matrix. Combining this symmetric relation together with (12) one gets

$$l_{f,h_2} \rfloor (L_{X_1} g_f - (e^{\sigma} \nabla_{g_f} \cdot (e^{-\sigma} X_1)) g_f) = l_{f,h_1} \rfloor (L_{X_2} g_f - (e^{\sigma} \nabla_{g_f} \cdot (e^{-\sigma} X_2)) g_f), \tag{13}$$

where $X_i = \dot{\Phi}_{f,h_i}$, $i = 1,2$. Equation (13) implies that both X_i, $i = 1,2$, satisfy the inhomogeneous conformal Killing field equation of the type

$$l \rfloor (L_X(g) - (e^{\sigma} \nabla \cdot (e^{-\sigma} X)) g) = F \tag{14}$$

with the same inhomogeneous term F, which is a 1-form. The equation (14) is the crucial equation for the proof. We have proven that if X and l satisfy the equation (14) with $X \mid_{\Gamma} = 0$, then both inner products $\langle l, X \rangle_g$ and $\langle l^{\perp}, X \rangle_g$ are uniquely determined by F, where $l^{\perp}$ stands for the unique vector perpendicular to l with $\| l^{\perp} \| = \| l \|$ in the counterclockwise direction under the metric g [Su2], Base on this result, one concludes from (13) that the vector fields X_i and l_{f,h_i} must satisfy the following system of equations:

$$\begin{cases} \langle X_1, l_{f,h_2} \rangle_{g_f} = \langle X_2, l_{f,h_1} \rangle_{g_f} \\ \langle X_1, l_{f,h_2}^{\perp} \rangle_{g_f} = \langle X_2, l_{f,h_1}^{\perp} \rangle_{g_f} \end{cases}, \tag{15}$$

To understand (15) better, consider now a two-parameter family of conformal diffeomorphisms $\Phi_{f+\eta_1 h_1+\eta_2 h_2} \subset G$ with parameters η_1 and η_2 in R. For a fixed point $x \in \Omega$, define

$$\omega(\eta_1,\eta_2) = \Phi_{f+\eta_1 h_1+\eta_2 h_2}(x): \ R^2 \to \bar{\Omega}$$

as a function from (η_1, η_2) to the image of x under $\Phi_{f+\eta_1 h_1+\eta_2 h_2}$. One checks that

$$\omega_{\eta_1} = \dot{\Phi}_{f+\eta_1 h_1+\eta_2 h_2, h_1}(x), \ \ \omega_{\eta_2} = \dot{\Phi}_{f+\eta_1 h_1+\eta_2 h_2, h_2}(x).$$

By Replacing f by $f + \eta_1 h_1 + \eta_2 h_2$ one can shows from (15) that the function ω satisfies the following first order system:

$$\begin{cases} \left\langle \omega_{\eta_1}, l_2 \right\rangle_g = \left\langle \omega_{\eta_2}, l_1 \right\rangle_g \\ \\ \left\langle \omega_{\eta_1}, l_2^{\perp} \right\rangle_g = \left\langle \omega_{\eta_2}, l_1^{\perp} \right\rangle_g \end{cases}, \tag{16}$$

where

$$l_j = l_{f+\eta_1 h_1+\eta_2 h_2, h_j} \circ \Phi_{f+\eta_1 h_1+\eta_2 h_2}, \ j = 1, 2.$$

Here the additional term $\Phi_{f+\eta_1 h_1+\eta_2 h_2}$ is needed once one removes the assumption $\Phi_f = identity$.

System (16) can be viewed as a generalized Cauchy-Riemann system under the vector fields l_1 and l_2. The proof of (11) with $h = h_1$ and h_2 is now reduced to showing that System (16) admits no bounded nonconstant solution ω. Note that ω is always bounded. In order to do that, one way is to apply Liouville's type theorems to the system (16). However, one must choose the directions h_1 and h_2 in a way that the gradients of the solution l_1 and l_2 are uniformly independent. Once (11) is proven with two independent directions, one can show that (11) holds for all directions. This is proven in [HSu] using the geometric argument developed in [Su2].

In [HSu] the above framework has been successfully to two important special cases: The case in which $A(x,p)$ is independent of x and the case in which $A_p(x,p)$ is independent of p. In both cases one is allowed to construct the needed independent directions h_1 and h_2. See [HSu] for details.

To verify the conjecture completely, the main difficulty is the construction of special directions. The construction of special directions in the known cases has been completed by using techniques of exponentially growing solutions, which is not available in the general case. One possible way to overcome this difficulty is to replace the two-parameter family of conformal diffeomorphisms $\Phi_{f+\eta_1 h_1+\eta_2 h_2}$ by $\Phi_{F(\eta_1,\eta_2)}$, where $F(\eta_1, \eta_2)$ is a two dimensional nonlinear variety in $C^{3,\alpha}(\Gamma)$ passing through f. The nonlinearity of $F(\eta_1, \eta_2)$ should correspond to the quasilinear nature of $A(x,p)$. Once one identifies the correct form of $F(\eta_1, \eta_2)$, the rest of the argument can be modified to cover the general case.

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