CUBO *A Mathematical Journal*
Vol. 7, Nº 3, (27 - 37). December 2005.

Sufficiency of the maximum principle for time optimality

H. O. Fattorini
Department of Mathematics, University of California
Los Angeles, California 90095-1555
hof@math.ucla.edu

ABSTRACT

For infinite dimensional linear systems, Pontryagin's maximum principle is shown to be sufficient for time optimality with conditions on the initial condition and on the target. These conditions cannot be given up and are shown to be best possible by means of counterexamples.

RESUMEN

Para sistemas lineales en dimensión infinita, el principio del máximo de Pontryagin es suficiente para alcanzar optimalidad en el tiempo con condiciones en el valor inicial y el final. Estas condiciones no se pueden relajar y se muestra que son las mejores posibles, por medio de contraejemplos.

Key words and phrases: *linear control systems in Banach spaces, time optimal problem.*
Math. Subj. Class.: *93E20, 93E25.*

1 Introduction.

Consider the *time optimal* problem of driving the solution $y(t)$ of

$$y'(t) = Ay(t) + u(t)\,, \quad y(0) = \zeta \tag{1.1}$$

from the initial point ζ to a point target,

$$y(T) = \bar{y} \tag{1.2}$$

with maximum-norm bound

$$\|u(t)\| \leq 1 \quad \text{a. e. in } 0 \leq t \leq T \tag{1.3}$$

in minimum time T; A is the infinitesimal generator of a strongly continuous semigroup $S(t)$ in a Banach space E and the controls $u(t)$ are strongly measurable (so that, in view of (1.3), belong to the unit ball of $L^\infty(0,T;E)$). Solutions or *trajectories*

$$y(t) = S(t)\zeta + \int_0^t S(t-\sigma)u(\sigma)d\sigma$$

of (1.1) are named $y(t) = y(t,\zeta,u)$ and controls satisfying (1.3) are called *admissible.* Let $\mathcal{Z}$ be an arbitrary linear space with $E^* \subseteq \mathcal{Z}$. We say that $\mathcal{Z}$ is a *multiplier space* if (i) $S(t)^*$ is defined in $\mathcal{Z}$,(ii) $S(t)^*\mathcal{Z} \subseteq E^*$ for $t > 0$. A control $\bar{u}(t)$ in the interval $0 \leq t \leq T$ satisfies the *weak maximum principle* if there exists z in a multiplier space $\mathcal{Z}$ such that $S(t)^*z$ is not identically zero in $0 < t \leq T$ and

$$\langle S(T-t)^*z, \bar{u}(t)\rangle = \max_{\|u\|\leq 1} \langle S(T-t)^*z, u\rangle \quad \text{a. e. in } 0 \leq t \leq T\,, \tag{1.4}$$

where $\langle \cdot, \cdot \rangle$ is the duality of the space E and the dual E^*. The control satisfies the *strong maximum principle* if (1.4) holds and

$$\int_0^T \|S(t)^*z\|_{E^*} dt < \infty\,. \tag{1.5}$$

The space $Z(T)$ consists of all multipliers that satisfy [1] (1.5). In Hilbert space, (1.4) is equivalent to

$$\bar{u}(t) = \frac{S(T-t)^*z}{\|S(T-t)^*z\|} \tag{1.6}$$

whenever the denominator is not zero.It is known [3] that if the control $\bar{u}(t)$ drives $\zeta \in E$ to $\bar{y} = y(T,\zeta,\bar{u}) \in D(A)$ then the strong maximum principle (1.4)-(1.5) is a necessary condition for time optimality.

It is also known [4] that(1.4)-(1.5) is a sufficient condition if $\zeta = 0$ or $\bar{y} = y(T,\zeta,\bar{u}) = 0$; then $\bar{u}(t)$ drives ζ to $\bar{y}$ time optimally.[2] We prove below (Theorem 1.2) that these conditions on the initial and final point of the trajectory can be relaxed to one of the two assumptions

$$\zeta \in D(A),\ \|A\zeta\| \leq 1 \quad \text{or} \quad \bar{y} \in D(A),\ \|A\bar{y}\| \leq 1\,, \tag{1.7}$$

[1]The semigroup $S(t)^*$ may not be strongly continuous, but in all cases the norm $\|S(t)^*\|$ is lower semicontinuous, thus theintegral (1.5) makes sense. The spaces $Z(T)$ are the same for all $T > 0$; condition (1.5) only bears on the behavior of $\|S(t)^*\|$ near zero.

[2]The weak maximum principle(1.4) is not a sufficient condition for time optimality; for a counterexample, see[6]

(the first with an additional condition on the adjoint semigroup). That restrictions on the initial condition ζ or the target $\bar{y}$ cannot be completely given up is illustrated with several examples, two of which show that conditions (1.7) are the best possible of their kind. We also see (in Example 4.2) that restrictions on $\|\zeta\|, \|\bar{y}\|$ (rather than on $\|A\zeta\|, \|A\bar{y}\|$) do not guarantee sufficiency of the maximum principle for time optimality.

Remark 1.1. If $S(t)$ is a group or, more generally, if $S(T)E = E$ $(t > 0)$ then the condition $\bar{y} \in D(A)$ is not required to show that the maximum principle is a necessary condition for time optimality; moreover, $Z(T) = E^*$. Sufficiency of the maximum principle, however, requires the same conditions as those in the general case.

2 Sufficiency of the maximum principle.

Let $R^\infty(T) \subseteq E$ be the space of all elements of the form

$$y = y(T, 0, u) = \int_0^T S(T-\sigma)u(\sigma)d\sigma\,, \quad u(\cdot) \in L^\infty(0, T; E)\,. \tag{2.1}$$

The norm $\|y\|_{R^\infty(T)}$ is the infimum of $\|u(\cdot)\|_{L^\infty(0,T;E)}$ for all $u(\cdot)$ that satisfy (2.1); in other words, $R^\infty(T)$ is the quotient of $L^\infty(0, T; E)$ by the closed subspace characterized by $y(T, 0, u) = 0$. An element $z \in Z(T)$ defines a bounded linear functional ξ_z in $R^\infty(T)$ through the formula

$$\langle\langle \xi_z, y\rangle\rangle = \int_0^T \langle S(T-\sigma)^* z, u(\sigma)\rangle d\sigma \tag{2.2}$$

where y and $u(\cdot)$ are related by (2.1) and $\langle\langle \cdot, \cdot\rangle\rangle$ indicates the duality of the space $R^\infty(T)$ and its dual $R^\infty(T)^\star$; the norm of ξ_z satisfies

$$\|\xi_z\|_{R^\infty(T)^\star} = \int_0^T \|S(t)^* z\|_{E^*} dt\,. \tag{2.3}$$

Theorem 2.1. *Assume that $\bar{u}(t)$ satisfies* (1.4) - (1.5) *and that either*

$$\begin{aligned} &(a) \quad \bar{y} = y(T, \zeta, \bar{u}) \in D(A), \quad \|A\bar{y}\| < 1\,, \ \text{or} \\ &(b) \qquad\qquad \zeta \in D(A), \quad \|A\zeta\| < 1\,, \quad S(t)^* z \neq 0 \ \text{in} \ 0 \le t < T\,. \end{aligned} \tag{2.4}$$

Then $\bar{u}(t)$ drives ζ to $\bar{y}$ time optimally in $0 \le t \le T$.

Proof of case (a). Assume $\bar{u}(\cdot)$ does not drive ζ to $\bar{y}$ time optimally. Then there exists $\delta > 0$ and a control $\tilde{u}(\cdot) \in L^\infty(0, T-\delta; E)$, $\|\tilde{u}(\cdot)\|_{L^\infty(0,T-\delta;E)} \le 1$, that drives ζ to $\bar{y}$ in time $T - \delta$. The control

$$v(\sigma) = \begin{cases} \tilde{u}(\sigma) & (0 \le \sigma < T - \delta) \\ -A\bar{y} & (T - \delta \le \sigma \le T) \end{cases} \tag{2.5}$$

satisfies $\|v(\cdot)\|_{L^\infty(0,T;E)} \le 1$. We have

$$\begin{aligned} \bar{y} - S(t - (T-\delta))\bar{y} &= -\int_0^{t-(T-\delta)} S(\sigma)A\bar{y}\, d\sigma \\ &= -\int_0^{t-(T-\delta)} S(t - (T-\delta) - \sigma)A\bar{y}\, d\sigma \\ &= \int_{T-\delta}^{t} S(t-\sigma)(-A\bar{y})\, d\sigma \quad (T-\delta \le t \le T) \end{aligned} \tag{2.6}$$

hence the trajectory $y(t,\zeta,v)$ starts at ζ, reaches $\bar{y}$ at time $T-\delta$ and stays at $\bar{y}$ for $T - \delta \le t \le T$;

$$y(t,\zeta,v) = \bar{y} \quad (T-\delta \le t \le T)\,. \tag{2.7}$$

This can be also be seen noting that if $y(t) = \bar{y}$ then we have $y'(t) - Ay(t) = -A\bar{y}$ in $T - \delta \le t \le T$.

The maximum principle (1.4) is equivalent to

$$\begin{gathered} \int_0^T \langle S(T-\sigma)^* z, u(\sigma)\rangle d\sigma \le \int_0^T \langle S(T-\sigma)^* z, \bar{u}(\sigma)\rangle d\sigma \\ (u(\cdot) \in L^\infty(0,T;E), \quad \|u(\cdot)\|_{L^\infty(0,T;E)} \le 1)\,. \end{gathered} \tag{2.8}$$

In terms of the linear functional ξ_z in (2.2), this is

$$\begin{gathered} \left\langle\!\!\left\langle \xi_z, \int_0^T S(T-\sigma)u(\sigma)d\sigma \right\rangle\!\!\right\rangle \le \left\langle\!\!\left\langle \xi_z, \int_0^T S(T-\sigma)\bar{u}(\sigma)d\sigma \right\rangle\!\!\right\rangle \\ (u(\cdot) \in L^\infty(0,T;E), \quad \|u(\cdot)\|_{L^\infty(0,T;E)} \le 1)\,. \end{gathered} \tag{2.9}$$

We have $y(T,\zeta,v) = y(T,\zeta,\bar{u})$, thus

$$\int_0^T S(T-\sigma)v(\sigma)d\sigma = \int_0^T S(T-\sigma)\bar{u}(\sigma)d\sigma\,, \tag{2.10}$$

and it follows from (2.9) that

$$\begin{gathered} \left\langle\!\!\left\langle \xi_z, \int_0^T S(T-\sigma)u(\sigma)d\sigma \right\rangle\!\!\right\rangle \le \left\langle\!\!\left\langle \xi_z, \int_0^T S(T-\sigma)v(\sigma)d\sigma \right\rangle\!\!\right\rangle \\ (u(\cdot) \in L^\infty(0,T;E), \quad \|u(\cdot)\|_{L^\infty(0,T;E)} \le 1)\,, \end{gathered} \tag{2.11}$$

which, being equivalent to (1.4), gives

$$\langle S(T-t)^* z, v(t)\rangle = \max_{\|u\| \le 1} \langle S(T-t)^* z, u\rangle \quad \text{a. e. in } 0 \le t \le T\,. \tag{2.12}$$

We have $S(T-\sigma)^*z \neq 0$ near[3] T, hence (2.12) implies $\|v(\sigma)\| = 1$ near T. This is a contradiction, since by hypothesis $\|v(\sigma)\| = \|A\bar{y}\| < 1$. ■

Proof of case (b). This time we define

$$v(\sigma) = \begin{matrix} -A\zeta & (0 \leq \sigma \leq \delta) \\ \tilde{u}(\sigma-\delta) & (\delta \leq \sigma \leq T)\,. \end{matrix} \tag{2.13}$$

As in (2.6) we have

$$\zeta - S(t)\zeta = \int_0^t S(t-\sigma)(-A\zeta)d\sigma\,,$$

hence the trajectory $y(t,\zeta,v)$ stays at ζ for $0 \leq t \leq \delta$,

$$y(t,\zeta,v) = \bar{y} \qquad (0 \leq t \leq \delta)\,,$$

and then starts for the target $\bar{y}$, which hits at time T. The proof ends in the same way as that of (a) noting that $S(T-\sigma)^*z \neq 0$ in $0 \leq \sigma \leq T$ (in particular, in $0 \leq \sigma \leq \delta$) hence (2.12) implies $\|v(\sigma)\| = 1$ in $0 \leq \sigma \leq \delta$, in contradiction to the fact that $\|v(\sigma)\| = \|A\zeta\| < 1$ in $0 \leq \sigma \leq \delta$. ■

3 Counterexamples, I.

To see that (2.4) cannot be relaxed, we have

Example 3.1. Consider the one dimensional system

$$y'(t) = -ay(t) + u(t), \quad y(0) = \zeta \tag{3.1}$$

with $a > 0$. We have $S(t) = e^{-at} = S(t)^*$, thus controls satisfying (1.4) with $z \neq 0$ are of one of the two forms

$$\bar{u}(t) = \begin{cases} 1 & \text{if } z > 0\,, \\ -1 & \text{if } z < 0\,. \end{cases} \tag{3.2}$$

For the initial condition and target $\zeta = 1/a$, $\bar{y} = 1/a$ we have

$$\int_0^T S(T-\sigma)\cdot 1\,d\sigma = \int_0^T e^{-a(T-\sigma)}d\sigma = \frac{1-e^{-aT}}{a} = \bar{y} - S(T)\zeta$$

so that the first control in (3.2) drives ζ to $\bar{y}$ in any time $T \geq 0$; in other words, $y(T,\zeta,\bar{u}) = \bar{y}$ for all $T \geq 0$. None of these drives is time optimal except forthe one where $T = 0$.

[3]The semigroup equation for the adjoint semigroup $S(t)^*$ implies: if $S(T-t)^*z = 0$, then $S(T-\sigma)^*z = S(t-\sigma)^*S(T-t)^*z = 0$ for $\sigma \leq t$. Accordingly, unless $S(T-t)^*z \neq 0$ in an interval (ρ, T), $\rho < T$,$S(T-t)^*z$ will be identically zero in $0 < t \leq T$.

Example 3.2. For another counterexample (or, rather, family of counterexamples) we use an arbitrary unitary group $S(t)$ in Hilbert space. Here we have $Z(T) = E$ (see Remark 1.1), $S(t)^* = S(-t) = S(t)^{-1}$, $\|S(t)y\| = \|y\|$. Controls satisfying the maximum principle are given by (1.6) (the denominator satisfies $\|S(T-t)^*z\| = \|z\|$). Assuming (as we may) that $\|z\| = 1$ we have

$$\int_0^T S(T-\sigma)\frac{S(T-\sigma)^*z}{\|S(T-\sigma)^*z\|}d\sigma = \int_0^T S(T-\sigma)S(T-\sigma)^*z\,d\sigma = Tz \tag{3.3}$$

so that the control (1.6) drives ζ to $\bar{y}$ in time T if and only if T is a solution of the equation

$$Tz = \bar{y} - S(T)\zeta\,. \tag{3.4}$$

This equation implies the scalar equation

$$T = \|S(T)\zeta - \bar{y}\| \tag{3.5}$$

and, conversely, if $T > 0$ is a solution of (3.5) it is clear that (3.4) will holdwith

$$z = \frac{\bar{y} - S(T)\zeta}{\|\bar{y} - S(T)\zeta\|}\,. \tag{3.6}$$

Theorem 3.3. *Assume* (3.5) *has only one nonnegative solution* T. *Then the control* (1.6) *drives* ζ *to* $\bar{y}$ *in optimal time* T. *If* (3.5) *has multiple solutions, only the control* (1.6) *corresponding to the smallest* T *drives* ζ *to* $\bar{y}$ *time optimally.*

Proof. Let $T \geq 0$ be the smallest solution of (3.5). If $T = 0$ we don't need to drive at all so that T is the optimal time. If $T > 0$ there exists an admissible control driving from ζ to $\bar{y}$, hence the standard existence theorem [2, Theorem 1.2] provides a control $\bar{u}(t)$ driving from ζ to $\bar{y}$ in optimal time $\mathcal{T}$. Since $S(t)$ is a group (Remark 1.1) this control must satisfy the maximum principle (1.4) with a nonzero multiplier $z \in Z(T) = E^* = E$. We are in a Hilbert space, which means this control must of the form (1.6) (with $\|z\| = 1$),

$$\bar{u}(t) = \frac{S(\mathcal{T}-t)^*z}{\|S(\mathcal{T}-t)^*z\|} = S(\mathcal{T}-t)^*z$$

(the denominator cannot be zero since $\|S(T-t)^*z\| = \|z\|$). As in (3.3) we then have

$$\int_0^{\mathcal{T}} S(\mathcal{T}-\sigma)\bar{u}(\sigma)d\sigma = \mathcal{T}z = \bar{y} - S(\mathcal{T})\zeta$$

hence $\mathcal{T}$ is a solution of (3.5) and, as $\mathcal{T}$ is the optimal time, we must have $\mathcal{T} = T$. ∎

Corollary 3.4. *Assume that, either* (*a*) $\zeta \in D(A), \|A\zeta\| > 1$, *or* (*b*) $\zeta \notin D(A)$. *Then there exists a control of theform* (1.6) *that drives* ζ *to* ζ *in time* $T > 0$, *thus is not time optimal.*

Proof. We write (3.5) for $\zeta = \bar{y}$ as

$$\frac{\|S(t)\zeta - \zeta\|}{t} = 1\,. \tag{3.7}$$

In case (a) we have

$$\lim_{t\to 0+}\frac{\|S(t)\zeta-\zeta\|}{t}=\lim_{t\to 0+}\left\|\frac{S(t)\zeta-\zeta}{t}\right\|\to\|A\zeta\|>1\,,$$

and we deduce that (3.7) has a positive solution, since the left side tends to 0 as $t\to\infty$.

In case (b),

$$\liminf_{t\to 0+}\frac{\|S(t)\zeta-\zeta\|}{t}=\liminf_{t\to 0+}\left\|\frac{S(t)\zeta-\zeta}{t}\right\|=\infty$$

since a finite lim inf implies that $\zeta\in D(A)$ ([1, Theorem 2.1.2. (c), p.88]). ∎

Remark 3.5. Corollary 3.4 has an interesting application. The equivalence

$$S(T)\zeta+\int_0^T S(T-\sigma)u(\sigma)d\sigma=\bar y \quad\Longleftrightarrow\quad \int_0^T S(T-\sigma)u(\sigma)d\sigma=\bar y-S(T)\zeta$$

says that $u(t)$ drives ζ to $\bar y$ in time $T \Longleftrightarrow u(t)$ drives 0 to $\bar y-S(T)\zeta$ in time T. If "drives" is changed to "drives optimally", the implication $\Longrightarrow$ remains. In fact, if $\bar u(\cdot)$ does not drive 0 to $\bar y-S(T)\zeta$ time optimally then there exists $\delta>0$ and acontrol $u(\cdot)$ with $\|u(\cdot)\|_{L^\infty(0,T-\delta;E)}\le 1$ and

$$\int_0^{T-\delta} S(T-\delta-\sigma)u(\sigma)d\sigma=\bar y-S(T)\zeta\,.$$

Then, if we define

$$v(\sigma)=\begin{cases}0 & (0\le\sigma<\delta)\\ u(\sigma-\delta) & (\delta\le\sigma\le T)\end{cases}$$

we have

$$\int_0^T S(T-\sigma)v(\sigma)d\sigma=\int_0^{T-\delta} S(T-\delta-\sigma)u(\sigma)d\sigma=\bar y-S(T)\zeta\,,$$

thus $v(t)$ drives from ζ to $\bar y$ in time T. If this drive were time optimal,the "bang-bang" Theorem 2.2 in [2] would say that $\|v(\sigma)\|=1$ a. e., which is not the case since $v(\sigma)=0$ in $0\le\sigma\le\delta$. Accordingly, the optimal driving time from ζ to $\bar y$ is $<T$.

The implication $\Longleftarrow$ is not true; in the setting of unitary semigroups in Hilbert spaces it suffices to take $\bar y=\zeta\in D(A)$ with $\|A\zeta\|>1$, and, applying Corollary 3.4 construct a control $\bar u(\cdot)$ satisfying (1.6) and driving ζ to ζ in time $T>0$. The same control drives 0 to $\zeta-S(T)\zeta$, but this drive is optimal since the initial condition satisfies (b) in Theorem 2.1.

4 Counterexamples, II.

The next example belongs to the family in Example 3.2.

Example 4.1. Consider $E = I\!R^2$,

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{4.1}$$

The semigroup generated by A,

$$S(t) = e^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \tag{4.2}$$

is unitary. In polar coordinates, $\zeta = (r\cos\theta, r\sin\theta)$,$\bar{y} = (s\cos\varphi, s\sin\varphi)$, and

$$\begin{aligned} \|S(t)\zeta - \bar{y}\|^2 = & \left\| S(t)\begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} - \begin{bmatrix} s\cos\varphi \\ s\sin\varphi \end{bmatrix} \right\|^2 \\ & = (r\cos t\cos\theta + r\sin t\sin\theta - s\cos\varphi)^2 \\ & \quad +(-r\cos\theta\sin t + r\cos t\sin\theta - s\sin\varphi)^2 \\ & = r^2 + s^2 - 2rs\cos(t - \theta + \varphi)\,. \end{aligned}$$

We have $\|Ay\| = \|y\|$. For $\zeta = \bar{y} = (1.1, 0)$ (so that $\|A\zeta\| = \|A\bar{y}\| = 1.1$) we have $r = s = 1.1$,$\theta = \varphi = 0$. Equation (3.5) (Figure 1) has a positive solution

$$T = 1.49797 \tag{4.3}$$

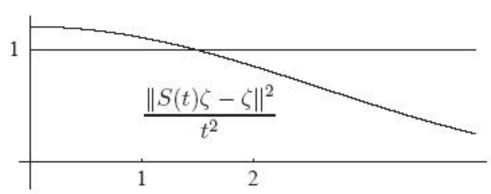


Figure 1

thus we can drive from $\zeta = (1.1, 0)$ back to ζ in time T with a control satisfying (1.6),

$$\bar{u}(\sigma) = S(T - \sigma)z\,, \tag{4.4}$$

with z given by (3.6),

$$z = (0.68090, 0.73238)\,.$$

Figure 2 below shows the drive (moving clockwise) whichis obviously not time optimal.

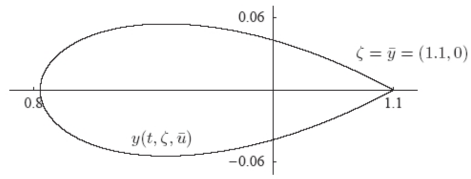
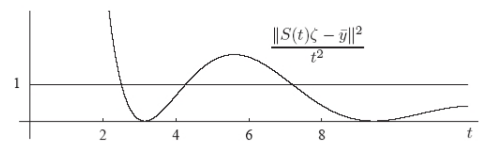


Figure 2

For $\zeta = (4, 0)$, $\bar{y} = (-4, 0)$ we have $r = s = 4$, $\theta = 0$, $\varphi = \pi$. Equation (3.5) (Figure 3) has three solutions,

$$T_0 = 2.50471\,, \quad T_1 = 4.26666\,, \quad T_2 = 7.19061\,. \tag{4.5}$$



thus we can drive from $(4, 0)$ to $(-4, 0)$ with three different controls that satisfy (1.6),

$$\bar{u}_j(\sigma) = S(T_j - \sigma)^* z_j \qquad j = 0\,, 1\,, 2\,, \tag{4.6}$$

where the z_j are given by (3.6) for each T_j,

$$z_0 = (-0.31308,\ 0.94972)\,,\ z_1 = (-0.53333,\ -0.84590)\,,\ z_2 = (-0.89882,\ 0.43830)\,.$$

Figure 2 shows the three trajectories, each plotted for $0 \le t \le T_j$; only the first (thicker curve) is time optimal.

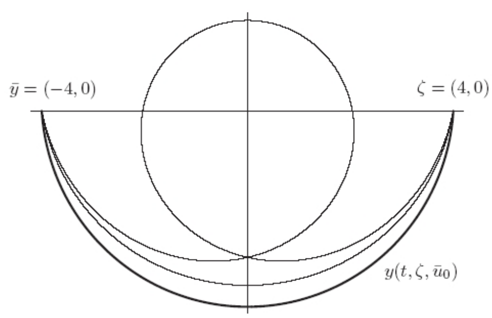


Figure 4

Remark 4.2. The strong maximum principle (1.4)-(1.5) is a sufficient condition for norm optimality [4] with no conditions on ζ or $\bar{y}$ so that each of the controls $\bar{u}_j(t)$,

$j = 0, 1, 2$ in (4.6) is norm optimal in its own interval; this means, if $\zeta = (4, 0)$ can be drivento $\bar{y} = (-4, 0)$ in the interval $0 \le t \le T_j$ by means of a control $u(t)$ then

$$\|u(\cdot)\|_{L^\infty(0,T_j;E)} \ge 1 = \|\bar{u}_j(\cdot)\|_{L^\infty(0,T_j;E)} .$$

The same observation applies to the control (4.4); it drives ζ back to ζ norm optimally in the interval $0 \le t \le T$.

Example 4.3. Example 3.1 can be manipulated into evidence that restrictions on $\|\zeta\|$ or $\|\bar{y}\|$ such as $\|\zeta\| \le \epsilon$ or $\bar{y} \le \epsilon$ don't guarantee sufficiency of the maximum principle for time optimality. To this end we consider thespace $E = \ell^2$ of all sequences $y = \{y_1, y_2, \dots\}$ such that $\|y\|^2 = \sum |y_k|^2 < \infty$, equipped with the norm $\| \cdot \|$. The operator is

$$Ay = A\{y_k\} = \{-ny_k\}$$

with maximal domain. It generates the semigroup

$$S(t)\{y_k\} = \{e^{-kt}y_k\} = S(t)^* .$$

Let

$$\zeta_n = \frac{1}{n}\{\delta_{nk}\} = \bar{y}_n , \quad z_n = \{\delta_{nk}\}$$

(δ_{nk} the Kronecker delta). We have

$$\|\zeta_n\| = \|\bar{y}_n\| = \frac{1}{n} , \quad \|A\zeta_n\| = \|A\bar{y}_n\| = 1 .$$

If $\bar{u}_n(\cdot)$ satisfies (1.6) in an interval $0 \le t \le T$ with $z = z_n$ then $\bar{u}_n(\sigma) = \{\delta_{nk}\}$ and

$$\int_0^T S(T-\sigma)\bar{u}(\sigma)d\sigma = \frac{1 - e^{-nT}}{n}\delta_{nk} = \bar{y}_n - S(T)\zeta_n$$

for any $T > 0$. Accordingly, $\bar{u}_n(\sigma)$ drives ζ_n to y_n in an arbitrary interval $0 \le t \le T$. The driveis not optimal unless $T = 0$.

Received: April 2004. Revised: May 2004.

References

[1] P. L. Butzer, H. Berens, *Semi-Groups of Operators and Approximation*, Springer, Berlin 1967.

[2] H. O. Fattorini, *Time-optimal control of solutions of operational differential equations*, SIAM J. Control **2** (1964) 54-59.

[3] H. O. Fattorini, *The maximum principle in infinite dimension*,Discrete & Continuous Dynamical Systems **6** (2000) 557-574.

[4] H. O. Fattorini, *Existence of singular extremals and singular functionals in reachable spaces*, Jour. Evolution Equations **1** (2001)325-347.

[5] H. O. Fattorini, *A survey of the time optimal problem and the norm optimal problem in infinite dimension*, Cubo Mat. Educacional **3** (2001) 147-169.

[6] H. O. Fattorini, *Time optimality and the maximum principle in infinite dimension*, Optimization **50** (2001) 361-385.