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Arc-wise Essentially Tangentially Regular Set-valued Mappings and their Applications to Nonconvex Sweeping Process

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ABSTRACT

Recently, Borwein and Moors (1998) introduced a new class of tangentially regular sets in $I\!R^n$ (called *arc-wise essentially smooth sets*). They characterized the sets S of this class in terms of arc-wise essential smoothness of the distance function d_S. Very recently, the author (2002) gave an appropriate extension of this class to any Banach space X and he extended the above characterization to any Banach space X with a uniformly Gâteaux differentiable norm. In this paper we extend the concept of arc-wise essentially smooth sets to set-valued mappings $C : [0, T] \rightrightarrows X$ $(T > 0)$ and we will use this concept to establish an important application to nonconvex sweeping process.

RESUMEN

Ricientemente Borwein y Moors (1998) introducem una nueva clase de conjuntos tangencialmente regulares en $I\!R^n$ chamados conjuntos essencialmente suaves por arcos). Ellos caracterizan los conjuntos S de esta clase en terminos de la suavidad de la distancia por arco de la función d_S. Ricientemente, el autor (2002) dió una extensión apropriada de esta clase en cualquer espacio de Banach X y extiende la caractarización anterior a cualquer espacio de Banach X y con una norma de Gâteaux uniformemente diferenciable. En este artículo extendemos el concepto de conjunto essencialmente suave por arcos para el conjunto de aplicaciones $C : [0, T] \rightrightarrows X$ $(T > 0)$ y usaremos este concepto para establecer una importante aplicación a procesos no convexos generales.

Key words and phrases: *Nonconvex Sweeping Process, Tangentially Regular Sets.*

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1 Introduction

In [9] Borwein and Moors introduced, in $I\!R^n$, the concept of *arc-wise essential smoothness* for sets and for functions. They characterized the class of all sets S which are arc-wise essentially smooth in terms of arc-wise essential smoothness of the distance function d_S. Their definitions and results were strongly based on the finite dimensional structure. In [10] the author gave an appropriate extension of the arc-wise essentially smooth concept for sets and functions in any Banach space and he extended the above characterization of the class of arc-wise essentially smooth sets in any Banach space with a uniformly Gâteaux differentiable norm. In this paper we intend to extend the concept of arc-wise essentially smooth sets to set-valued mappings and to give some applications of this new concept of regularity of set-valued mappings. The paper is organized as follows. In section two we recall some notations and preliminaries that are used in the paper. Section three is devoted to introduce and to study the new concept of arc-wise essentially tangentially regular set-valued mappings. Many examples of this class of set-valued mappings are given in this section. We prove in this section various characterizations of arc-wise essentially tangentially regular set-valued mappings. The main characterization is given in Theorem 3.3 which establishes a relationship between arc-wise essentially tangential regularity of a set-valued mapping C and the arc-wise essentially smoothness of the distance function to the images of the set-valued mapping C. In the last section, we give an important application of this characterization to the nonconvex sweeping process.

2 Preliminaries

Throughout, X will be a real Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we will denote the canonical pairing between these spaces. Recall that a function f from X into $I\!R$ is Lipschitz around $x_0 \in X$ if there exist two real numbers $K > 0$ and $\delta > 0$ such that

$$|f(x') - f(x)| \leq K\|x' - x\| \quad \text{for all} \quad x', x \in x_0 + \delta I\!B,$$

where $\mathbb{B}$ denotes the closed united ball of X centered at the origin. We will say that f is locally Lipschitz over X if it is Lipschitz around any point of X.

Recall also that the usual directional derivative of f at x_0 in the direction v is,

$$f'(x_0; v) := \lim_{t\to 0} t^{-1}\big[f(x_0 + tv) - f(x_0)\big],$$

when this limit exists.

For a locally Lipschitz function $f : X \to \mathbb{R}$, we recall that the *Clarke generalized directional derivative* (resp. *the lower Dini directional derivative*) of f at $x_0 \in X$ in the direction v is given by,

$$f^0(x_0; v) := \limsup_{\substack{x\to x_0 \\ t\downarrow 0}} t^{-1}\big[f(x + tv) - f(x)\big],$$

(resp.

$$f^-(x_0; v) := \liminf_{t\downarrow 0} t^{-1}\big[f(x_0 + tv) - f(x_0)\big].)$$

One always has $f^-(x_0; v) \le f^0(x_0; v)$. The reverse inequality is not true in general (take for example $f(x) = -\|x\|$). The functions f satisfying the equality form in the last inequality are called *directionally regular* at x_0 in the direction v. Recall also that a locally Lipschitz function $f : X \to \mathbb{R}$ is *strictly differentiable* (in short s.d.) at x_0 in the direction v if

$$f^0(x_0; v) = -f^0(x_0; -v).$$

It is not difficult to check that, if f is s.d. at x_0 in the direction v, then one has $f^0(x_0; v) = f^-(x_0; v) = f'(x_0; v) = -f^0(x_0; -v)$ and so it is directionally regular at x_0 in the direction v.

Recall now, that the *Clarke subdifferential* (resp. *Fréchet subdifferential*) of f at $x_0 \in X$ is defined by

$$\partial^C f(x_0) = \{x^* \in X^* : \langle x^*, v\rangle \le f^0(x_0; v), \quad \text{for all } v \in X\},$$

(resp.

$$\partial^F f(x_0) = \{x^* \in X^* : \forall\epsilon > 0, \exists\delta > 0 : \langle x^*, x-x_0\rangle \le f(x)-f(x_0)+\epsilon\|x-x_0\|, \ \forall x \in x_0+\delta\mathbb{B}\}).$$

Let S be a nonempty subset of X. We will let $d(\cdot, S)$ (or $d_S(\cdot)$) stand for the usual distance function to S, i.e., $d(x, S) := \inf_{u\in S} \|x - u\|$. Recall (see [20]) that the *Clarke tangent cone* and the *contingent cone* of S at some point $x \in S$ are given respectively by

$$T_S(x) = \{v \in X : d_C^0(x; v) = 0\}, \tag{2.3}$$

and

$$K_S(x) = \{v \in X : d_S^-(x; v) = 0\}. \tag{2.4}$$

Note that one always has $T_S(x) \subset K_S(x)$. The sets S for which one has an equality in the last inclusion, will be called *tangentially regular* at x (see [20] for this definition). Let us recall (see for instance [12]) that the *Clarke normal cone* (resp. *Fréchet normal cone*) of S at $x \in S$ is defined by

$$N_S^C(x) = \{x^* \in X^* : \langle x^*, v\rangle \leq 0, \quad \text{for all } v \in T_S(x)\},$$

(resp.

$$N_S^F(x) = \{x^* \in X^* : \forall \epsilon > 0, \exists \delta > 0 : \langle x^*, x' - x\rangle \leq \epsilon \|x' - x\|, \ \forall x' \in x + \delta I\!B\}).$$

The following proposition is needed in the sequel. It was proved for the first time by Kruger [25] (see also Iofee [26].)

Proposition 2.1 **[12]** *Let S be a nonempty closed subset in X and let $x \in S$. Then*

$$\partial^F d_S(x) = N_S^F(x) \cap I\!B.$$

Let I be an interval and let Ω be an open subset of X. By absolutely continuous mapping one means a mapping $x : I \to \Omega$ such that $x(t) = x(a) + \int_a^t x'(s)ds$, for all $t \in I$, with $x' \in L_X^1(I)$ and $a \in I$. We will denote by $\mathcal{AC}(I, \Omega)$ the famility of all these mappings.

Remark 2.1 It is well known (see for instance [15]) that $F \circ x(\cdot)$, the composition of a locally Lipschitz mapping $F : \Omega \to Y$ with an absolutely continuous mapping $x : I \to \Omega$, is an absolutely continuous mapping, whenever the space Y is reflexive. For more details concerning absolutely continuous mappings we refer the reader to Brézis [15].

3 Arc-wise essentially tangentially regular set-valued mappings

We start with the following definition of *arc-wise essentially tangentially regular set-valued mappings*:

Definition 3.1 Let $I :=]0, 1[$ and let $C : I \rightrightarrows X$ be a set-valued mapping with nonempty closed values. We will say that C is *arc-wise essentially tangentially regular* and we will write $C \in \mathcal{AWETR}(I, X)$, if for each $x \in \mathcal{AC}(I, X)$, the set

$$\{t \in I : x(t) \in C(t) \ \text{ and } \ x'(t) \text{ or } -x'(t) \in K_{C(t)}(x(t)) \setminus T_{C(t)}(x(t))\}$$

has null measure.

In this paper we use the name *arc-wise essential tangential regularity* instead of *arc-wise essential smoothness* (used in [10] and [7, 8, 9]) because it seems for us that is more significant.

Remark 3.1 As one always has $K_S(x) = T_S(x) = X$, for each $x \in intS$ (the topological interior of S), we can take x only in $bd\, C(t)$ (the boundary of $C(t)$), in Definition 3.1, that is, C is arc-wise essentially tangentially regular if and only if for each $x \in \mathcal{AC}(I, X)$ one has

$$\mu\big(\{t \in \,]0,1[\, : x(t) \in bd\, C(t) \text{ and } x'(t) \text{ or } -x'(t) \in K_{C(t)}(x(t)) \setminus T_{C(t)}(x(t))\}\big) = 0.$$

Example 3.1

1. It is easy to see that all set-valued mappings $C : I \rightrightarrows X$ with closed tangentially regular values are arc-wise essentially tangentially regular.

2. Let S be a fixed set in X which is arc-wise essentially smooth in the sense of [9, 10]. Then using Proposition 4.1 in [10] we can check that the constant set-valued mapping $C : I \rightrightarrows X$ with $C(t) = S$ is arc-wise essentially tangentially regular.

3. Let S be a fixed set in X which is tangentially regular at each of its points except one point $x_0 \in S$. Define the set-valued mapping C as the translation of the set S in the direction $v(t)$, that is,

$$C(t) = S + v(t), \text{ with } v \in \mathcal{AC}(I, X). \tag{1}$$

   Assume now that v satisfies

$$\pm v'(t) \notin K_S(x_0) \setminus T_S(x_0), \text{ a.e. on } I.$$

   Then C is an arc-wise essentially tangentially regular set-valued mapping. Indeed, for any $x \in \mathcal{AC}(I, X)$ we can easily check that

$$\mu\left(\{t \in I : x(t) \in bdC(t) \text{ and } x'(t) \text{ or } -x'(t) \in K_{C(t)}(x(t)) \setminus T_{C(t)}(x(t))\}\right) =$$

$$\mu\left(\{t \in I : x(t) = v(t) + x_0 \text{ and } v'(t) \text{ or } -v'(t) \in K_S(x_0) \setminus T_S(x_0)\}\right) = 0.$$

   Take for example $X = \mathbb{R}^2$, S_1 is the epigraph of the absolute value function, and take S is the closure of the complement of S_1. Take $v(t) = (t, 2t)$, for all $t \in I \setminus N$ and $v(t) = (t, 1)$ for all $t \in N$, where N is a subset of I with null measure. Using what precedes we can easily check that the set-valued mapping C in (1) associated with the set S and v is arc-wise essentially tangentially regular.

4. More general and with the same manner we can prove that the set-valued mapping C in (1) is arc-wise essentially tangentially regular whenever the set S is tangentially regular at each of its points except on a countable set $\{x_n\}$ and with v satisfies

$$\pm v'(t) \notin K_S(x_n) \setminus T_S(x_n), \text{ for all } n \text{ and a.e. on } I. \tag{2}$$

5. The condition (2) on v cannot be removed in the last example. Take for example S is the closure of the complement of the epigraph of the absolute value function and take $v(t) = (t, 1)$, for all $t \in I$. The condition (2) is not satisfied and we can check that for some $x \in \mathcal{AC}(I, X)$ one has

$$\mu\left(\{t \in I : x(t) \in bdC(t) \text{ and } x'(t) \text{ or } -x'(t) \in K_{C(t)}(x(t)) \setminus T_{C(t)}(x(t))\}\right) = 1,$$

   and so the set-valued mapping C in this case is not arc-wise essentially tangentially regular. From this example we can conclude that a set-valued mapping with values $C(t)$ tangentially regular except on countable set is not necessarily arc-wise essentially tangentially regular.

6. Let C_0 be the Cantor ternary set with $0 \in C_0$. Let $C(t) = C_0 + t$. We claim that $C \notin \mathcal{AWETR}((0,1), I\!R)$. Let $x(t) = t$. Then

$$\{t \in (0,1) : x(t) \in C(t) \text{ with } -x'(t) \text{ or } x'(t) \in K_{C(t)}(x(t)) \setminus T_{C(t)}(x(t))\} =$$

$$\{t \in (0,1) : 0 \in C_0 \text{ with } x'(t) \neq 0\} = (0,1).$$

   and so

$$\mu(\{t \in (0,1) : x(t) \in C(t) \text{ with } -x'(t) \text{ or } x'(t) \in K_{C(t)}(x(t)) \setminus T_{C(t)}(x(t))\}) \neq 0,$$

   which ensures that C is not $\mathcal{AWETR}((0,1), I\!R)$.

A first question, which arises naturally, is to ask whether the epigraph set-valued mapping $t \rightrightarrows C(t) := epi\, f_t$ is arc-wise essentially tangentially regular, where

$$epi\, f_t := \{(x, r) \in X \times I\!R : \quad f(t, x) \leq r\}.$$

To give an answer to this question we introduce the following concepts. Let $f : I\!R \times X \to I\!R$ be a function from $I\!R \times X$ to $I\!R$. We define the following directional derivatives of f at $(t_0, x_0) \in I\!R \times X$ in a direction $v \in X$ by

$$f^0((t_0, x_0); v) := \limsup_{\substack{x \to x_0 \\ (\delta,t) \downarrow (0,t_0)}} \delta^{-1}\big[f(t, x + \delta v) - f(t, x)\big],$$

and

$$f^-((t_0, x_0); v) := \liminf_{(\delta,t)\downarrow(0,t_0)} \delta^{-1}\big[f(t, x_0 + \delta v) - f(t, x_0)\big].$$

It is clear that if the two above limits exist then the Clarke and the lower Dini directional derivatives of $f_{t_0}(\cdot) := f(t_0, \cdot)$ at x_0 in the direction v exist and equal respectively to $f^0((t_0, x_0); v)$ and $f^-((t_0, x_0); v)$. The converse is not true in general, take for example $f(t, x) = f_1(t) f_2(x)$ with f_1 is not right continuous.

Definition 3.2 We will say that f is *arc-wise essentially directionally regular* and we will write $f \in \mathcal{AWEDR}(I \times X)$, if for each $x \in \mathcal{AC}(I, X)$, the set

$$\{t \in I : f^-((t, x(t)); x'(t)) \neq f^0((t, x(t)); x'(t))\}$$

has null measure.

Example 3.2

1. Any mapping f defined as follows

$$f(t, x) = f_1(t) + f_2(x),$$

   is arc-wise essentially directionally regular whenever f_2 is directionally regular and without any assumptions on f_1.

2. Any mapping f defined as follows

$$f(t, x) = f_1(t) f_2(x),$$

   is arc-wise essentially directionally regular whenever f_2 is directionally regular and f_1 is continuous.

Recall that (see for instance [20]) for a function $f : I\!R \times X \to I\!R$ one has for all $t \in I\!R$

$$K_{epi\, f_t}((x, f_t(x)) = epi\, f_t^-(x; \cdot), \tag{3}$$

and

$$T_{epi\, f_t}((x, f_t(x)) = epi\, f_t^0(x; \cdot). \tag{4}$$

Now, we are ready to state the following result showing that the epigraph set-valued mapping $C(t) := epi\, f_t$ is arc-wise essentially tangentially regular whenever f is arc-wise essentially directionally regular.

Theorem 3.1 *Let I be an open interval and let $f : I \times X \to I\!R$ be a locally Lipschitz function from $I \times X$ to $I\!R$. Then the set-valued mapping $C : I \to X \times I\!R$ defined by $t \rightrightarrows epi\, f_t$ is arc-wise essentially tangentially regular whenever the function f is arc-wise essentially directionally regular.*

Proof. Put $C(t) = epi\, f_t$ and suppose that f is arc-wise essentially directionally regular. Let $(x, r) \in \mathcal{AC}(I, X \times I\!R)$ and put

$$J_1 := \{t \in I : \; f^-((t, x(t)); x'(t)) = f^0((t, x(t)); x'(t))\}, \text{ and}$$

$$J_2 := \{t \in I : \; (x(t), r(t)) \in bdC(t) \text{ and } (x'(t), r'(t)) \in K_{C(t)}(x(t), r(t)) \setminus T_{C(t)}(x(t), r(t))\}.$$

First, we have $\mu(J_1) = 1$ because f is arc-wise essentially directionally regular. Assume that there exists some $t_0 \in J_1 \cap J_2$. Then $f^0((t_0, x(t_0)); x'(t_0))$ and $f^-((t_0, x(t_0)); x'(t_0))$ exist and coincide and they equal to $f^0_{t_0}(x(t_0); x'(t_0)) = f^-_{t_0}(x(t_0); x'(t_0))$. We also have $x'(t_0)$ and $r'(t_0)$ exist and such that $(x(t_0), r(t_0)) \in bdC(t_0)$ $\big($that is, $r(t_0) = f_{t_0}(x(t_0))$, because the boundary of $C(t_0)$ is the graph of $f_{t_0}\big)$ and

$$(x'(t_0), r'(t_0)) \in K_{C(t)}(x(t_0), r(t_0)) \setminus T_{C(t)}(x(t_0), r(t_0)). \tag{5}$$

Further, (3) and (4) yield

$$r'(t_0) \geq f^-_{t_0}(x(t_0); x'(t_0)) = f^0_{t_0}(x(t_0); x'(t_0))$$

which means $(x'(t_0); r'(t_0)) \in T_{C(t_0)}(x(t_0), r(t_0))$ that is a contradiction with (5). Therefore, we obtain $J_1 \cap J_2 = \emptyset$ and hence as $\mu(J_1) = 1$ we get $\mu(J_2) = 0$. So, C is arc-wise essentially tangentially regular. $\square$

The following theorem states a necessary condition on f for the arc-wise essential tangential regularity of the epigraph set-valued mapping *epi* f_t. Its proof follows some ideas from [9].

Theorem 3.2 *Let I be an open interval and let $f : I \times X \to I\!R$ be a locally Lipschitz function from $I \times X$ to $I\!R$. If the set-valued mapping $C : I \to X \times I\!R$ defined by $t \rightrightarrows epi\, f_t$ is arc-wise essentially tangentially regular, then the function f is arc-wise essentially strictly differentiable function (i.e., $f \in \mathcal{AWESD}(X, I\!R)$), in the following sense: for any $x \in \mathcal{AC}(I, X)$, one has*

$$\mu\left(\{t \in I : f^0_t(x(t); -x'(t)) \neq -f^0_t(x(t); x'(t))\}\right) = 0. \tag{6}$$

Proof. Suppose that C is arc-wise essentially tangentially regular and fix any $x \in \mathcal{AC}(I, X)$. Since f is Lipschitz then by Remark 2.1 the function $\theta(t) := f(t, x(t)) \in \mathcal{AC}(I, X)$ and so $\theta'(t)$ exists a.e. on I. Fix now any $t \in I$ such that $x'(t)$ and $\theta'(t)$ exist. Then, by (4) one has

$$(x'(t), f^0(x(t); x'(t)) \in T_{C(t)}(x(t), f(x(t))).$$

By (3), one also has

$$(x(t), f_t(x(t))) \in C(t) \ \text{ and } \ (x'(t), f'_t(x(t); x'(t)) \in K_{C(t)}(x(t), f(x(t))).$$

Put $E := E_1 \cup E_2$ where

$$E_1 := \{s \in E_3 : \ (x'(s), \theta'(s)) \in K_{C(s)}(x(s), \theta(s)) \setminus T_{C(s)}(x(s), \theta(s))\},$$

$$E_2 := \{s \in E_3 : \ (-x'(s), -\theta'(s)) \in K_{C(s)}(x(s), \theta(s)) \setminus T_{C(s)}(x(s), \theta(s))\},$$

and

$$E_3 := \{s \in I : \ (x(s), \theta(s)) \in C\}.$$

As C is arc-wise essentially tangentially regular one gets by Definition 3.1 that $\mu(E) = 0$. If one assumes further that $t \notin E$, we obtain

$$(x'(t), \theta'(t)) \in T_{C(t)}(x(t), \theta(t)) = epi\, f_t^0(x(t); \cdot),$$

and

$$(-x'(t), -\theta'(t)) \in T_{C(t)}(x(t), \theta(t)) = epi\, f_t^0(x(t); \cdot).$$

This means respectively

$$f_t^0(x(t); x'(t)) \leq \theta'(t)$$

and

$$f_t^0(x(t); -x'(t)) \leq -\theta'(t),$$

which yields $f_t^0(x(t); x'(t)) \leq -f_t^0(x(t); -x'(t))$ and hence, because the reverse inequality always holds, one gets $f_t^0(x(t); x'(t)) = -f_t^0(x(t); -x'(t))$. Thus, the set

$$\tilde{E} := \{s \in I : \ f_t^0(x(s); x'(s)) \neq -f_t^0(x(s); -x'(s))\},$$

is included in E and so $\mu(\tilde{E}) = 0$. The proof then is complete. □

Remark 3.2 Combining Theorems 3.1-3.2 we get the following inclusion:

$$\mathcal{AWEDR}(X, I\!R) \subset \mathcal{AWESD}(X, I\!R). \tag{7}$$

This means that any arc-wise essentially directionally regular is arc-wise essentially strictly differentiable in the sense of (6). This inclusion is strict. Take for example the function f in Example 3.2 part (2) with f_1 is not continuous.

The following lemma will be used in the sequel.

Lemma 3.1 *Let f be a locally Lipschitz function defined from X into $I\!R$ and let $x_0, v \in X$. Then f is s.d. at x_0 in the direction v if and only if $\langle \partial^C f(x_0), v \rangle = \{f^0(x_0; v)\}$ iff $\langle \partial^C f(x_0), v \rangle$ is a singleton set. Here $\langle \partial^C f(x), v \rangle := \{\langle x^*, v \rangle : x^* \in \partial^C f(x)\}$.*

Proof. It is clear that it suffices to prove the following relation:

$$\langle \partial^C f(x_0), v \rangle = [-f^0(x_0; -v), f^0(x_0; v)].$$

By (b) in Proposition 2.1.2 in [20] one has $f^0(x_0; v) = \max \langle \partial^C f(x_0), v \rangle$ and hence $-f^0(x_0; -v) = \min \langle \partial^C f(x_0), v \rangle$ and as the set $\langle \partial^C f(x_0), v \rangle$ is convex, one obtains the desired equality. □

When the function f does not depend on t, that is, $f : X \to I\!R$, it is easy to see (by Proposition 3.1 in [10]) that the concept of arc-wise essential strict differentiability in the sense of Theorem 3.2, is equivalent to the concept of arc-wise essential smoothness in the sense of [10]. The next corollary summarizes further characterizations of arc-wise essentially smooth functions.

Corollary 3.1 *Let I be an open interval and let $f : X \to I\!R$ be a locally Lipschitz function. Then the following assertions are equivalent:*

1. *f is arc-wise essentially smooth in the sense of [10];*
2. *f is arc-wise essentially strictly differentiable;*
3. *epi f is arc-wise essentially tangentially regular;*
4. *f is arc-wise essentially directionally regular;*
5. *for each $x \in \mathcal{AC}(\,]0,1[\,,X)$*
$$\mu\big(\{t \in \,]0,1[\, : \big\langle \partial^C f(x(t); x'(t))\big\rangle = \{f^0(x(t); x'(t))\}\}\big) = 1.$$
6. *for each $x \in \mathcal{AC}(\,]0,1[\,,X)$*
$$\mu\big(\{t \in \,]0,1[\, : f^0(x(t); x'(t)) = (f \circ x)'(t)\}\big) = 1;$$
7. *for each $x \in \mathcal{AC}(\,]0,1[\,,X)$*
$$\mu\big(\{t \in \,]0,1[\, : f^0(x(t); x'(t)) = f'(x(t); x'(t))\}\big) = 1.$$

Proof. The following equivalences follow from Theorems 3.1-3.2, Lemma 3.1 and by what precedes the corollary:
$$(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5).$$

$(6) \Leftrightarrow (7)$:
Fix any $x \in \mathcal{AC}(\,]0,1[\,,X)$ and fix also any $t \in]0,1[$ where $x'(t)$ exists. If we put $\delta = \min\{t, 1-t\}$, the Lipschitz behavior of f ensures for all $s \in]-\delta, \delta[$
$$s^{-1}[(f \circ x)(t+s) - (f \circ x)(t)] = s^{-1}[f(x(t) + sx'(t)] + \epsilon(s) \tag{8}$$
with $\lim_{s \to 0} \epsilon(s) = 0$. Therefore, for any such t , $(f \circ x)'(t)$ exists if and only if $f'(x(t); x'(t))$ exists. The equivalence then holds.

$(4) \Leftrightarrow (6)$:
Fix any $x \in \mathcal{AC}(\,]0,1[\,,X)$ and $t \in]0,1[$ such that $x'(t)$ and $(f \circ x)'(t)$ exist. Note that the set of such points t has 1 as Lebesgue measure because x and $f \circ x$ are absolutely continuous, and note also that, by (8), for any such t $(f \circ x)'(t) = f^-(x(t); x'(t))$. So, the equivalence follows. □

Using Theorem 3.1, we get the following examples of arc-wise essentially tangentially regular set-valued mappings:

1. The translation of the epigraph of directionally regular functions in any direction of y-axis, i.e., $C(t) = epi\, f_2 + (0, f_1(t))$, with f_2 is a directionally regular function and f_1 is an arbitrary function.

2. The set-valued mapping $C : \mathbb{R} \rightrightarrows \mathbb{R}^2$

$$C(t) = \{(x, f_1(t)r) : \; f_2(t) \leq r\},$$

where $f_2 : \mathbb{R} \to \mathbb{R}$ is directionally regular and $f_1 : \mathbb{R} \to \mathbb{R}$ is continuous with $f_1 \not\equiv 0$.

Now we are going to establish a characterization of the class of arc-wise essentially tangentially regular set-valued mappings C, in terms of the distance function to the images of the set-valued mapping C. Its proof follows some ideas from [10]. It will be used to give an important application to nonconvex sweeping processes. In the proof of this theorem, we need the following characterization of the contingent cone $K_C(x)$. A vector $v \in K_C(x)$ if and only if there exist two sequences $\{t_n\}_{n\in\mathbb{N}}$ of positive real numbers converging to zero and $\{v_n\}_{n\in\mathbb{N}}$ in X converging to v such that

$$x + t_n v_n \in C, \text{ for each } n \in \mathbb{N}.$$

Theorem 3.3 *Let $C : I \rightrightarrows X$ be a set-valued mapping with nonempty closed values. Assume that $d_{C(\cdot)}(\cdot)$ is arc-wise essentially strictly differentiable. Then C is arc-wise essentially tangentially regular. If, in addition, X is a Banach space with uniformly Gâteaux differentiable norm, then C is arc-wise essentially tangentially regular if and only if $d_{C(\cdot)}(\cdot)$ is arc-wise essentially strictly differentiable.*

Proof.

1) Assume that $d_{C(\cdot)}(\cdot) \in \mathcal{AWESD}(X, \mathbb{R})$, i.e., for each $x \in \mathcal{AC}(]0,1[, X)$, the set

$$A := \{t \in \,]0,1[\, : d_{C(t)}(\cdot) \text{ is not s.d. at } x(t) \text{ in the direction } x'(t)\ \}$$

has null measure. We will show that C is arc-wise essentially tangentially regular, i.e., $\mu(B) = 0$ where $B := \{t \in \,]0,1[\, : x(t) \in C(t) \text{ and } x'(t) \text{ or } -x'(t) \in K_{C(t)}(x(t)) \setminus T_{C(t)}(x(t))\})$. It is enough to prove that $B \subset A$. Let $t_0 \notin A$. If $x(t_0) \notin C(t_0)$, then $t_0 \notin B$. So let us suppose that $x(t_0) \in C(t_0)$. If $x'(t)$ and $-x'(t) \notin K_{C(t)}(x(t)) \setminus T_{C(t)}(x(t))$, then $t_0 \notin B$. So, assume that $x'(t)$ or $-x'(t) \in K_{C(t)}(x(t)) \setminus T_{C(t)}(x(t))$. This ensures $d^-_{C(t_0)}(x(t_0); x'(t_0)) = 0$ or $d^-_{C(t_0)}(x(t_0); -x'(t_0)) = 0$. Since $d_{C(t_0)}$ is s.d. at $x(t_0) \in C(t_0)$ in the direction $x'(t_0)$, i.e., we have $d^0_{C(t_0)}(x(t_0); x'(t_0)) = -d^0_{C(t_0)}(x(t_0); -x'(t_0))$. On the other hand, the strict differentiability ensures the directional regularity, that is, $d^-_{C(t_0)}(x(t_0); x'(t_0)) = d^0_{C(t_0)}(x(t_0); x'(t_0))$ and $d^-_{C(t_0)}(x(t_0); -x'(t_0)) = d^0_{C(t_0)}(x(t_0); -x'(t_0))$ and hence

$$d^-_{C(t_0)}(x(t_0); x'(t_0)) = d^0_{C(t_0)}(x(t_0); x'(t_0)) = -d^0_{C(t_0)}(x(t_0); -x'(t_0)) = -d^-_{C(t_0)}(x(t_0); -x'(t_0)).$$

So in both cases $d^-_{C(t_0)}(x(t_0); x'(t_0)) = 0$ or $d^-_{C(t_0)}(x(t_0); -x'(t_0)) = 0$, we obtain $d^0_{C(t_0)}(x(t_0); x'(t_0)) = d^0_{C(t_0)}(x(t_0); -x'(t_0)) = 0$, that is, $x'(t_0) \in T_{C(t_0)}(x(t_0))$ and $-x'(t_0) \in T_{C(t_0)}(x(t_0))$. Thus, both directions $x'(t_0)$ and $-x'(t_0)$ lie in $T_{C(t_0)}(x(t_0))$ and hence $t_0 \notin B$. Consequently, each $t_0 \notin A$ does not lie in B. This completes the proof of the inclusion $B \subset A$.

2) Assume now that X is a Banach space with uniformly Gâteaux differentiable norm and assume that C is arc-wise essentially tangentially regular. Then, for each fixed x in $\mathcal{AC}(]0,1[, X)$ by Definition 3.1 we have

$$\mu(B_x) = 0,$$

where

$$B_x = B^1_x \cup B^2_x,$$

$$B^1_x := \{t \in]0,1[: x(t) \in C(t) \text{ and } x'(t) \in K_{C(t)}(x(t)) \setminus T_{C(t)}(x(t))\} \text{ and}$$

$$B^2_x := \{t \in]0,1[: x(t) \in C(t) \text{ and } -x'(t) \in K_{C(t)}(x(t)) \setminus T_{C(t)}(x(t))\}.$$

Put

$$A := \{t \in]0,1[: d_{C(t)} \text{ is not s.d. at } x(t) \text{ in the dir. } x'(t) \ \}.$$

It is not difficult to check that

$$A = \{t \in]0,1[: x(t) \in bd\, C(t),\ \ d_{C(t)} \text{ is not s.d. at } x(t) \text{ in the dir. } x'(t) \ \}.$$

Indeed, if $t \in]0,1[$ with $x(t) \in (X \setminus C(t)) \cup int\, C(t)$ and $d_{C(t)}$ is not s.d. at $x(t)$ in the direction $x'(t)$, then $(-d_{C(t)})$ is not s.d. at $x(t)$ in the direction $x'(t)$ and so $(-d_{C(t)})$ is not directionally regular at $x(t)$ in the direction $x'(t)$, which is impossible, because $x(t) \in (X \setminus C(t)) \cup int\, C(t)$, and Theorem 8 in [2]. Put now $D_{x'} := \{t \in]0,1[: x'(t) \text{ exists } \}$, hence

$$\mu(A \setminus D_{x'}) = 0. \tag{9}$$

Put also $I := I_r \cup I_l$ with I_r (resp. I_l) denotes the set of all isolated points in $A \cap D_{x'}$ relatively to the right topology (resp. the left topology). It is not difficult to check that I is countable and hence $\mu(I) = 0$. Fix $t_0 \in (A \cap D_{x'}) \setminus I$. Then there exist two sequences of real positive numbers $(\lambda_n)_n$ and $(\epsilon_n)_n$ converging to zero such that for n sufficiently large $t_0 + \lambda_n$ and $t_0 - \epsilon_n$ lie in $(A \cap D_{x'}) \setminus I$ and hence $x(t_0 + \lambda_n) \in bd\ C(t_0 + \lambda_n)$ and $x(t_0 - \epsilon_n) \in bd\ C(t_0 - \epsilon_n)$, for n sufficiently large. Put

$$v_n := \lambda_n^{-1}\big[x(t_0 + \lambda_n) - x(t_0)\big] \text{ and } w_n := \epsilon_n^{-1}\big[x(t_0 - \epsilon_n) - x(t_0)\big].$$

Clearly, $v_n \to x'(t_0)$ and $w_n \to -x'(t_0)$ and for n sufficiently large

$$x(t_0) + \lambda_n v_n \in bd\ C(t_0 + \lambda_n) \subset C(t_0) + R_C \lambda_n \mathbb{B} \subset cl\, C(t_0) = C(t_0)$$

and

$$x(t_0)+\epsilon_n w_n \in bd\, C(t_0-\epsilon_n) \subset C(t_0)+R_C\epsilon_n I\!B \subset cl\, C(t_0)=C(t_0).$$

It follows (by the characterization given above of the contingent cone) that $x'(t_0)$ and $-x'(t_0)$ lie in $K_{C(t_0)}(x(t_0))$. Now, we distinguish two cases. Firstly, if $x'(t_0) \in K_{C(t_0)}(x(t_0)) \setminus T_{C(t_0)}(x(t_0))$, then $t_0 \in B_x$. Secondly, if $x'(t_0) \in T_{C(t_0)}(x(t_0))$, then $-x'(t_0) \in K_{C(t_0)}(x(t_0)) \setminus T_{C(t_0)}(x(t_0))$ $\big($ because, if $-x'(t_0) \in T_{C(t_0)}(x(t_0))$, we would have $d^0_{C(t_0)}(x(t_0);x'(t_0)) = -d^0_{C(t_0)}(x(t_0);-x'(t_0))=0$, so $d_{C(t_0)}$ would be s.d. at $x(t_0)$ in the direction $x'(t_0)$, which would contradict that $t_0 \in A$ $\big)$. Hence $t_0 \in B_x$. Thus $(D_{x'} \cap A)\setminus I \subset B_x$ and hence

$$\mu((D_{x'}\cap A)\setminus I)=0. \tag{10}$$

Finally, according to (9) and (10), we obtain $\mu(A)=0$. This ensures that $d_{C(t_0)} \in \mathcal{AWESD}(X, I\!R)$ and hence the proof is finished. □

The following corollary follows from Theorem 3.3 and Lemma 3.1. It will be used in the next section.

Corollary 3.2 *Let H be a Hilbert space. A set-valued mapping $C : I \rightrightarrows cl(H)$ is arc-wise essentially tangentially regular if and only if for each $x \in \mathcal{AC}(I,H)$ one has*

$$\mu\left(\left\{t\in I : \left\langle \partial^C d_{C(t)}(x(t)), x'(t)\right\rangle \neq \{d^0_{C(t)}(x(t);x'(t))\}\right\}\right)=0. \tag{11}$$

4 Applications to nonconvex sweeping process

Throughout this section, we will let H (resp. $cl(H)$) denote a separable Hilbert space (resp. the collection of all nonempty closed sets in H).

Let $F : H \rightrightarrows H$ be a set-valued mapping from H to H. We will say that F is Hausdorff upper semicontinuous (for more details on Hausdorff upper semicontinuity see [23, 16]) if for any $y \in H$ one has

$$\limsup_{x\to\bar{x}} e(F(x),F(y)) \le e(F(\bar{x}),F(y)),$$

wheree

$$e(A,B) := \sup_{a\in A}\left[\inf_{b\in B}\|b-a\|\right] = \sup_{a\in A} d_B(a).$$

In all the sequel $T>0$, $I:=[0,T]$, and $C : I\!R \rightrightarrows cl(H)$ will denote a L'-Lipschitz set-valued mapping ($L'>0$) with nonempty closed values, i.e., for any $y\in H$ and any $t,s\in I$

$$|d(y,C(t))-d(y,C(s))| \le L'|t-s|.$$

We prove in the following theorem our main application of the concept of arc-wise essentially tangentially regular set-valued mappings. It proves a stability result for nonconvex sweeping processes with nonconvex noncontinous perturbation. Let us note that our assumption on F requiring the inclusion in the subdifferential of some function was introduced for the first time in the work by [14] and by many other authors (see for instance [1, 3, 4, 11, 28, 30]).

Theorem 4.1 *Assume that* $C : [0,T] \rightrightarrows H$ *is arc-wise essentially tangentially regular and it has ball compact values. Let* $F : H \rightrightarrows H$ *be Hausdorff u.s.c. on* H *contained in the subdifferential of a directionally regular locally Lipschitz function* $\psi : H \to I\!R$. *Let* $\{x_n(\cdot)\}_n$ *be a bounded sequence in* $\mathcal{AC}(I,H)$ *(that is,* $\|x_n(t)\| \le M$*, for some* $M > 0$*, for any* n *and any* $t \in I$*) such that*

$$(NSPP) \qquad \begin{cases} x_n'(t) \in -N^F_{C(t)}(x_n(t)) + f_n(t) + b_n(t) \ \text{a.e. on } [0,T]; \\ f_n(t) \in F(x_n(\theta_n(t))) \ \text{and } b_n(t) \in r_n(t) I\!B \ \text{a.e. on } [0,T] \\ x_n(t) \in C(t), \qquad \forall t \in [0,T]; \\ x_n(0) = x_0 \in C(0), \end{cases}$$

where $f_n, b_n \in L^2(I,H)$ *and* $r_n(t) \to 0^+$ *uniformly on* I*, and* $\theta_n(t) \to t$ *for all* $t \in [0,T]$*, and* $\|x_n'(t)\| \le L''$ *a.e. on* $[0,T]$*. Then there exist* $b \in]0,T]$ *and* $x \in \mathcal{AC}([0,b],H)$ *such that*

$$\begin{cases} x'(t) \in -N^C_{C(t)}(x(t)) + F(x(t)) \ \text{a.e. on } [0,T]; \\ x(t) \in C(t), \qquad \forall t \in [0,T]; \\ x(0) = x_0 \in C(0), \end{cases}$$

Proof. Let $\alpha > 0$ such that ψ is Lipschitz on $x_0 + \alpha I\!B$ with ratio $L > 0$. Put $b := \min\{\frac{\alpha}{L''}, T\}$ and $I := [0,b]$. Let $(f_n)_n$ and $(b_n)_n$ in $L^2(I,H)$ such that

$$f_n(t) \in F(x_n(\theta_n(t))) \text{ and } b_n(t) \in r_n(t) I\!B \text{ a.e. on } I.$$

So we have by $(NSPP)$

$$-x_n'(t) + f_n(t) + b_n(t) \in N^F_{C(t)}(x_n(t)) \text{ a.e. on } I$$

Since $\{x_n(\cdot)\}_n$ is bounded sequence in $\mathcal{AC}(I,H)$ and C has ball compact values we get the set $\{x_n(t) : \ n \ge 1\}$ is relatively strongly compact in H. Thus, as $\|x_n'(t)\| \le L''$, we get by Ascoli-Arzela's theorem

$$x_n \to^s x \text{ in } \mathcal{AC}(I,H),$$
$$x_n' \to^w x' \text{ in } L^2(I,H).$$

Since

$$\|x_n(t) - x_0\| \le \int_0^t \|x'{}_n(s)\| ds \le L'' b \le \alpha,$$

for all $t \in I$ we obtain

$$f_n(t) \in F(x_n(\theta_n(t))) \subset \partial\psi(x_n(\theta_n(t))) \subset L I\!B,$$

and so we get for n_0 large enough

$$\| - x_n'(t) + f_n(t) + b_n(t)\| \leq L'' + L + \frac{1}{n_0}$$

for all $n \geq n_0$. Thus, Proposition 2.1 ensures for $\sigma := L' + L + \frac{1}{n_0}$ and for a.e. $t \in I$

$$-x_n'(t) + f_n(t) + b_n(t) \in N^F_{C(t)}(x_n(t)) \cap \sigma I\!B = \sigma \partial^F d_{C(t)}(x_n(t)).$$

We can thus apply Castaing techniques (see for instance [17]). The convergence of the sequences $\{r_n\}_n$ and $\{x_n\}_n$ to 0 and x respectively and the weak convergence of the sequences $\{x_n'\}_n$ and $\{f_n\}_n$ to x' and f, and using Mazur's lemma yield

$$-x'(t) + f(t) \in \sigma \partial^C d_{C(t)}(x(t)) \text{ and } f(t) \in \partial^C \psi(x(t)). \tag{12}$$

Now, since the function ψ is directionally regular we obtain, by Corollary 3.1

$$(\psi \circ x)'(t) = \psi^0(x(t); x'(t)) = \langle f(t), x'(t) \rangle$$

and so

$$\int_0^b \psi^0(x(t); x'(t))dt = \int_0^b \langle f(t), x'(t) \rangle dt.$$

On one hand, as $f_n(t) \in \partial \psi(x_n(\theta_n(t)))$ one has

$$\langle f_n(t), x_n'(t) \rangle \leq \psi^0(x_n(\theta_n(t)); x_n'(t)),$$

because ψ is regular. On the other hand, since ψ is directionally regular we get $\psi^0(x_n(\theta_n(t)); x_n'(t)) = \psi'(x_n(\theta_n(t)); x_n'(t))$ and $\psi^0(x(t); x'(t)) = \psi'(x(t); x'(t))$ and so by Theorem 2.1 in [2] we obtain

$$\limsup_n \int_0^b \psi^0(x_n(\theta_n(t)); x_n'(t))dt = \limsup_n \int_0^b \psi'(x_n(\theta_n(t)); x_n'(t))dt$$

$$\leq \int_0^b \psi'(x(t); x'(t))dt = \int_0^b \psi^0(x(t); x'(t))dt.$$

Consequently, we get

$$\limsup_n \int_0^b \langle f_n(t), x_n'(t) \rangle dt \leq \int_0^b \langle f(t), x'(t) \rangle dt.$$

Coming back to (12) and using the fact C is arc-wise essentially tangentially regular and the fact that $x(t) \in C(t)$ for all $t \in I$, we get (by Corollary 3.2) for a.e. $t \in I$

$$\langle f(t) - x'(t), x'(t) \rangle = \sigma \langle \partial^C d_{C(t)}((x(t)), x'(t) \rangle = \sigma d^0_{C(t)}(x(t); x'(t)) = 0 \text{ and}$$

$$\langle b_n(t)+f_n(t)-x_n'(t),x_n'(t)\rangle=\sigma\langle\partial^C d_{C(t)}((x_n(t)),x_n'(t)\rangle=\sigma d^0_{C(t)}(x_n(t);x_n'(t))=0,$$

which gives

$$\|x'(t)\|^2=\langle f(t),x'(t)\rangle \text{ and } \|x_n'(t)\|^2=\langle b_n(t)+f_n(t),x_n'(t)\rangle.$$

Therefore,

$$\|x_n'\|^2_{L^2}=\int_0^b\langle b_n(t)+f_n(t),x_n'(t)\rangle dt \text{ and } \|x'\|^2_{L^2}=\int_0^b\langle f(t),x'(t)\rangle dt.$$

Finally, we have

$$\limsup_n \|x_n'\|^2_{L^2}\le\int_0^b\langle f(t),x'(t)\rangle dt=\|x'\|^2_{L^2}.$$

Since $x_n'\to^w x'$ in $L^2(I,H)$ and using the weak l.s.c. of the norm, together with the last inequality we get

$$\|x_n'\|_{L^2}\to\|x'\|_{L^2}.$$

Now, using the fact that $L^2(I,H)$ is a Hilbert space we conclude the strong convergence of x_n' to x' in $L^2(T,H)$.

Put now $\zeta_n(t):=-x_n'(t)+b_n(t)+f_n(t)$, a.e. on I. We have

$$\begin{aligned}d(\zeta_n(t),F(x(t))-x'(t))&=d(\zeta_n(t)+x'(t),F(x(t)))\\&\le\|b_n(t)\|+\|x_n'(t)-x'(t)\|+d\left(f_n(t),F(x(t))\right),\\&\le\|b_n(t)\|+\|x_n'(t)-x'(t)\|+e\left(F(x_n(\theta_n(t))),F(x(t))\right)\to 0 \text{ as } n\to+\infty,\end{aligned}$$

because of the Hausdorff u.s.c. of F and since $x_n(\theta_n(t))\to x(t)$ on I and $x_n'(t)\to x'(t)$ a.e. on I. So given $\epsilon>0$, we can find $n_0\ge 1$ such that for all $n\ge n_0$ we have

$$\zeta_n(t)+x'(t)\in F(x(t))+\epsilon\mathbb{B}.$$

Since $\epsilon>0$ was arbitrary and F has closed values we get

$$\Gamma(t):=\limsup\{\zeta_n(t)\}_{n\ge1}\subset F(x(t))-x'(t) \text{ a.e. on } I.$$

Let ζ be a measurable selection of Γ, i.e., $\zeta(t)\in\Gamma(t)$ a.e. on I. Then, we get for a.e. on I

$$\zeta(t)\in\Gamma(t)\subset\overline{co}^w\limsup\{\zeta_n(t)\}_{n\ge1}\subset\overline{co}^w\limsup\sigma\partial^F d_{C(t)}(x_n(t))\subset\sigma\partial^C d_{C(t)}(x_n(t)).$$

Therefore, we get for a.e. on I

$$\zeta(t)+x'(t)\in F(x(t)) \text{ and } \zeta(t)\in N^C_{C(t)}(x(t)),$$

which ensures

$$x'(t)\in -N^C_{C(t)}(x(t))+F(x(t)) \text{ a.e. on } I.$$

The proof then is complete. □

Using our stability result for sweeping processes, we prove a new existence result for nonconvex sweeping process with nonconvex and noncontinuous perturbation. First we recall the definition of r-prox-regularity (see [27]) (or equivalently r-proximal smoothness (see [21])) for subsets which is a generalization of convex subsets.

Definition 4.1 *Let S be a closed nonempty subset in H. We will say that S is r-prox-regular (or r-proximally smooth) if d_S is continuously Gâteaux differentiable on the tube $U(r) := \{u \in H : \ 0 < d_S(u) < r\}$.*

The following properties of uniformly prox-regular sets are necessary in the sequel.

Proposition 4.1 [13] *Let S be a r-prox-regular nonempty closed subset in H. Then following holds*

1. *S is tangentially regular at each point $x \in S$.*
2. *for any $x \in S$ and any $\xi \in \partial^F d_S(x)$ one has*
$$\langle \xi, x' - x\rangle \le \frac{2}{r}\|x' - x\|^2 + d_S(x') \text{ for all } x' \in H \text{ with } d_S(x') < r.$$
3. *The Clarke and the Fréchet subdifferentials of the distance function d_S coincide at each point $x \in S$, that is, $\partial^C d_S(x) = \partial^F d_S(x)$ for all $x \in S$. Therefore, in the sequel of all the paper we will denote $\partial d_S(x)$ for both subdifferentials for r-prox-regular sets.*
4. *The Clarke and the Fréchet normal cones coincide at each point $x \in S$, that is, $N_S^C(x) = N_S^F(x)$ for all $x \in S$. Hence, we will use the notation $N_S(x)$ for both normal cones for r-prox-regular sets.*

Note that the converse in the second assertion (even in the finite dimensional setting) is not true in general. For more details and examples, we refer the reader to [13].

Now, we are ready to state the following new existence result for prox-regular sweeping processes with nonconvex and noncontinuous perturbations.

Theorem 4.2 *Let $r : I \to]0, +\infty]$ such that $\int_0^T \frac{dt}{r(t)} < \infty$. Assume that $C : I \rightrightarrows cl(H)$ has $r(t)$-prox-regular and ball compact values for almost every t in I. Let $F : H \rightrightarrows H$ be Hausdorff u.s.c. on H contained in the subdifferential of a directionally regular locally Lipschitz function $\psi : H \to I\!R$. Then there exist $b \in]0, T]$ such that the following nonconvex sweeping process with nonconvex noncontinuous perturbation*

$$(NSPP) \qquad \begin{cases} x'(t) \in -N_{C(t)}(x(t)) + F(x(t)) \text{ a.e. on } [0, b]; \\ x(t) \in C(t), \qquad \forall t \in [0, b]; \\ x(0) = x_0 \in C(0), \end{cases}$$

has at least one solution.

To prove this theorem we need the following propositions:

Proposition 4.2 Let $r : I \to]0, +\infty]$ such that $\int_0^T \frac{dt}{r(t)} < \infty$. Assume that $C : I \rightrightarrows cl(H)$ has $r(t)$-prox-regular values and let $h \in L^2(I, H)$ with $\|h(t)\| \le m$ a.e. on I. Then the following sweeping process

$$(SP) \qquad \begin{cases} x'(t) \in -N_{C(t)}(x(t)) + h(t) \text{ a.e. on } I; \\ x(t) \in C(t), \qquad \forall t \in I; \\ x(0) = x_0 \in C(0), \end{cases}$$

has one and only one solution satisfying $\|x'(t)\| \le L' + 2m$ a.e. on I.

Proof. Put $u(t) := x(t) + \int_0^t h(s)ds$, $K(t) := C(t) - \int_0^t h(s)ds$. Then (SP) is equivalent to

$$(SP') \qquad \begin{cases} u'(t) \in -N_{K(t)}(u(t)) \text{ a.e. on } I; \\ u(t) \in K(t), \qquad \forall t \in I; \\ u(0) = x_0 \in K(0). \end{cases}$$

By Theorem 4.1 in [12] (SP') has one and only one solution u satisfying $\|u(t)\| \le L' + m$ a.e. on I. This completes the proof. □

Note that Theorem 4.1 in [12] is given for set-valued mappings C with r-prox-regular values with r does not depend on t but an inspection of the proof of Theorem 4.1 in [12] shows that it is also true if we take $C(t)$ is $r(t)$-prox-regular for almost every t in I and with r satisfies $\int_0^T \frac{dt}{r(t)} < \infty$.

Proposition 4.3 Let $r : I \to]0, +\infty]$ such that $\int_0^T \frac{dt}{r(t)} < \infty$. Assume that $C : I \rightrightarrows cl(H)$ has $r(t)$-prox-regular values for almost every t in I. Let $x_0, y_0 \in C(0)$, and $f, g \in L^2(I, H)$, and let x and y be two solutions of the two following problems, respectively

$$(SP_f) \qquad \begin{cases} x'(t) \in -N_{C(t)}(x(t)) + f(t) \text{ a.e. on } I; \\ x(t) \in C(t), \qquad \forall t \in I; \\ x(0) = x_0 \in C(0), \end{cases}$$

and

$$(SP_g) \qquad \begin{cases} y'(t) \in -N_{C(t)}(y(t)) + g(t) \text{ a.e. on } I; \\ y(t) \in C(t), \qquad \forall t \in I; \\ y(0) = y_0 \in C(0), \end{cases}$$

and satisfying

$$\|x'(t)\| \le \delta_f \text{ and } \|y'(t)\| \le \delta_g, \text{ for a.e. on } I,$$

with $\delta_f, \delta_g > 0$. Then for $p(t) = \int_0^t \frac{2}{r(\tau)} \max\{\delta_f + L_f, \delta_g + L_g\} d\tau$ one has

$$\|x(t) - y(t)\| \leq \|x_0 - y_0\| e^{p(t)} + \int_0^t \|f(\tau) - g(\tau)\| e^{p(t)-p(\tau)} d\tau \text{ for all } t \in I,$$

where L_f and L_g are constants depending on f and g respectively.

Proof. By (SP_f) and (SP_g) we have for a.e. $t \in I$

$$-x'(t) + f(t) \in N_{C(t)}(x(t)), \text{ with } x(0) = x_0$$

and

$$-y'(t) + g(t) \in N_{C(t)}(y(t)), \text{ with } y(0) = y_0$$

and $\|f(t) - x'(t)\| \leq \delta_f + L_f$ and $\|g(t) - y'(t)\| \leq \delta_g + L_g$. So, by part (1) in Proposition 2.1 we get

$$-x'(t) + f(t) \in \delta \partial d_{C(t)}(x(t)) \text{ and } -y'(t) + g(t) \in \delta \partial d_{C(t)}(y(t)),$$

where $\delta := \max\{\delta_f + L_f, \delta_g + L_g\}$. Now, by using the property of the uniform prox-regularity of the values of C recalled in part (3) in Proposition 2.1, we obtain

$$\big\langle -x'(t) + f(t) + y'(t) - g(t), x(t) - y(t) \big\rangle \geq \frac{-2\delta}{r(t)} \|x(t) - y(t)\|^2.$$

Hence

$$\big\langle x'(t) - y'(t), x(t) - y(t) \big\rangle \leq \big\langle f(t) - g(t), x(t) - y(t) \big\rangle + \frac{2\delta}{r(t)} \|x(t) - y(t)\|^2,$$

and hence

$$\frac{\big\langle x'(t) - y'(t), x(t) - y(t) \big\rangle}{\|x(t) - y(t)\|} \leq \|f(t) - g(t)\| + \frac{2\delta}{r(t)} \|x(t) - y(t)\|, \tag{3.1}$$

whenever $x(t) \neq y(t)$. Put $s(t) := \|x(t) - y(t)\|$, a function which is Lipschitz continuous on I, as the composition of two Lipschitz mappings. Let t be in the set of full measure in which $s'(t)$, $x'(t)$, and $y'(t)$ exist and for which $C(t)$ is $r(t)$-prox-regular. Then

$$s'(t) = \begin{cases} \dfrac{\big\langle x'(t) - y'(t), x(t) - y(t) \big\rangle}{\|x(t) - y(t)\|}, & \text{if } x(t) \neq y(t) \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the relation (3.1) ensures for a. e. $t \in I$

$$s'(t) \leq \|f(t) - g(t)\| + \frac{2\delta}{r(t)} s(t).$$

We rewrite this inequality in the form

$$\left(s'(t) - \frac{2\delta}{r(t)}s(t)\right)e^{-p(t)} \leq \|f(t) - g(t)\|e^{-p(t)},$$

where $p(t) = \displaystyle\int_0^t \frac{2\delta}{r(\tau)}d\tau$. As the left side is the derivative of the function $t \mapsto s(t)e^{-p(t)}$, we can write

$$s(t)e^{-p(t)} - s(0) \leq \int_0^t \|f(\tau) - g(\tau)\|e^{-p(\tau)}d\tau$$

and then

$$\|x(t) - y(t)\| = s(t) \leq \|x(0) - y(0)\|e^{p(t)} + \int_0^t \|f(\tau) - g(\tau)\|e^{p(t)-p(\tau)}d\tau.$$

This completes the proof. $\square$

Now, we are ready to prove Theoerem 4.2.

Proof of Theoerem 4.2. Let $\alpha > 0$ such that ψ is Lipschitz on $x_0 + \alpha I\!B$ with ratio $L > 0$. Put $\gamma(t) = \int_0^t \frac{2(L'+L)}{r(\tau)}d\tau$ and $b = \min\{\frac{\alpha}{2(e^{\gamma(T)}L+L')}, T\}$. Now, we consider a sequence of mappings defined on $I := [0, b]$ and prove that a subsequence converges to a solution of (NSPP).

For very $n \in I\!N$ put $I_k^n := [0, t_k^n]$, $t_k^n := \frac{kb}{n}$, $k \in \{1, \ldots, n\}$ and we are going to construct $f_n, x_n : I \to H$. Pick $y_0^n \in F(x_0)$ and define f_n on $I_1^n = [0, \frac{b}{n}]$ by $f_n(t) = y_0^n$ for all $t \in I_1^n$. Then consider the problem

$$(SPP_{n,0}) \qquad \begin{cases} x'(t) \in -N_{C(t)}(x(t)) + f_n(t) \text{ a.e. on } I_1^n; \\ x(t) \in C(t), \qquad \forall t \in I_1^n; \\ x(0) = x_0. \end{cases}$$

By Proposition 4.1, problem (SP_n) has a unique solution $x_n \in AC(I_1^n, H)$ with $\|x_n(t)\| \leq L' + 2L$. Let $y_0 \in AC(I_1^n, H)$ be the unique solution of

$$(SP_{n,0}) \qquad \begin{cases} x'(t) \in -N_{C(t)}(x(t)) \text{ a.e. on } I_1^n; \\ x(t) \in C(t), \qquad \forall t \in I_1^n; \\ x(0) = x_0. \end{cases}$$

Then Proposition 4.2 ensures

$$\|x_n(t) - y_0(t)\| \leq e^{\gamma(t)} \int_0^t \|f_n(\tau)\|d\tau \text{ for all } t \in I_1^n.$$

Also

$$\|y_0(t) - x_0\| \leq \int_0^t \|y_0'(\tau)\|d\tau \leq L't \text{ for all } t \in I_1^n.$$

Therefore, we get

$$\|x_n(t)-x_0\| \le \|x_n(t)-y_0(t)\| + \|y_0(t)-x_0\|$$
$$\le e^{\gamma(T)}\int_0^t \|f_n(\tau)\|d\tau + L't \le (e^{\gamma(T)}L+L')t \le \frac{\alpha}{2n},$$

which ensures that $\|x_n(t)-x_0\| < \frac{\alpha}{n}$ on I_1^n.

Assume now that f_n and x_n have defined on the interval I_k^n and we will extend these mappings to the interval I_{k+1}^n, for all $k \in \{1,\dots,n\}$. Taking $y_k^n \in F(x_n(t_k^n))$, we define f_n on $(t_k^n, t_{k+1}^n]$ by $f_n(t) = y_k^n$. Again let $x_n \in AC(I_{k+1}^n, H)$ be the unique solution of

$$(SPP_{n,k}) \qquad \begin{cases} x'(t) \in -N_{C(t)}(x(t)) + f_n(t) \text{ a.e. on } I_{k+1}^n; \\ x(t) \in C(t), \qquad \forall t \in I_{k+1}^n; \\ x(0) = x_n(t_k^n). \end{cases}$$

Then as above we have $\|x_n(t)-x_0\| < \frac{\alpha(k+1)}{n}$ on I_{k+1}^n. Indeed, let $y_{n,k} \in AC(I_{k+1}^n, H)$ be the unique solution of

$$(SP_{n,k}). \qquad \begin{cases} x'(t) \in -N_{C(t)}(x(t)) \text{ a.e. on } I_{k+1}^n; \\ x(t) \in C(t), \qquad \forall t \in I_{k+1}^n; \\ x(0) = x_n(t_k^n). \end{cases}$$

By Proposition 4.2 we have

$$\|x_n(t)-y_{n,k}(t)\| \le e^{\gamma(t)}\int_0^t \|f_n(\tau)\|d\tau \le e^{\gamma(T)}L't \text{ for all } t \in I_1^n.$$

Also, we have for all $t \in I_{k+1}^n$

$$\|y_{n,k}(t)-x_0\| \le \|y_{n,k}(t)-y_{n,k}(0)\| + \|x_n(t_k^n)-x_0\| < \int_0^t \|y'_{n,k}(\tau)\|d\tau + \frac{k\alpha}{n} < L't + \frac{k\alpha}{n}.$$

Therefore, we get for all $t \in I_{k+1}^n$

$$\|x_n(t)-x_0\| < (e^{\gamma(T)}L'+L')t + \frac{k\alpha}{n} < \frac{(k+1)\alpha}{2n} + \frac{k\alpha}{n} < \frac{(k+1)\alpha}{n}.$$

So we have obtained two sequences of mappings $(f_n)_n$ and $(x_n)_n$, defined on I. Let $\theta_n : I \to I$ be defined by

$$\theta_n(t) = t_k^n, \text{ if } t \in (t_k^n, t_{k+1}^n] \text{ and } \theta_n(t_0^n) = 0.$$

Then by our construction we have $x'_n(t) \in -N_{C(t)}(x_n(t)) + F(x_n(\theta_n(t)))$ and $\|x'_n(t)\| \le L' + 2L$ a.e. on I and $\theta_n(t) \to t$ for all $t \in I$. Furthermore we have $\|x_n(t)-x_0\| < \alpha$, for all $t \in I$. Thus, Theorem 4.1 completes the proof. □

We close the paper with a direct and important corollary of Theorem 4.2. It establishes an existence result for the following differential inclusion:

$$(*) \qquad \begin{cases} x'(t) \in -N_C(x(t)) + F(x(t)) \text{ a.e. on } [0,b]; \\ x(t) \in C, \quad \forall t \in [0,b]; \quad x(0) = x_0, \end{cases}$$

First, we recall that this type of differential inclusion has been introduced by Henry [24] for studying some economic problems. In the case when F is an u.s.c set-valued mapping, he proved an existence result of $(*)$ under the convexity assumption on the set C and on the images of the set-valued mapping F. This result has been extended by Cornet [22] by assuming the tangential regularity assumption on the set C and the convexity on the images of F with the u.s.c of F. Thibault in [29], proved an existence result of $(*)$ for any closed subset C (without any assumption on C), which also required the convexity of the images of F and the u.s.c. of F. Recently, the author proved in [11], without any assumption of convexity on the images of F, the existence of solutions of $(*)$, but a heavy price was payed for the absence of the convexity. The price is the continuity of F and a standard tangential condition. Noting that all the results mentioned above in [11, 24, 22, 29] are given in the finite dimensional setting. The question arises whether we can drop the assumption of convexity of the images of F, without assuming any tangential condition and without the continuity of F, and if possible in the infinite dimensional setting. Our next corollary establishes a positive answer to this question.

Corollary 4.1 *Let $r \in]0,+\infty]$ and C be a uniformly prox-regular set in H which is ball compact. Let $F : H \rightrightarrows H$ be Hausdorff u.s.c. on H contained in the subdifferential of a directionally regular locally Lipschitz function $\psi : H \to I\!R$. Then for any $x_0 \in C$ there exists $b \in]0,T]$ such that the nonconvex sweeping process with nonconvex noncontinuous perturbation $(*)$ has at least one solution.*

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