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On the Index of Clifford Algebras of Quadratic Forms

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ABSTRACT

In this paper, we determine the index of the Clifford algebras of 6-dimensional quadratic forms over a field whose characteristic is unequal to 2. In the case that the characteristic is equal to 2, we compute the Clifford algebras of the Scharlau's transfer of 4-dimensional quadratic forms with trivial Arf invariant, and then investigate how the index of the Clifford algebra of q depends on orthogonal decompositions of q when q is a low dimensional quadratic form.

RESUMEN

En este artículo determinamos el índice de la algebra de Clifford de formas quadraticas 6-dimensionales cuja característica es distinta de dos. En el caso de caracteristica dos cálculamos la algebra de Clifford de la traslación de Scharlau de formas quadraticas 4-dimensionales con Art invariante trivial e se investiga como el indice de la algebra de Clifford de q depende de la descomposición ortogonal de q quando q es una forma quadrática de dimensión baja.

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1 Introduction

In his book [5], Knus classified the Clifford algebras $C(q)$, the even Clifford algebras $C_0(q)$ and the discriminant algebras $Z(q)$ of low dimensional quadratic forms q over a field F. In the case of dimension 6, Knus showed the following classification Table 1 [5, Appendix A].

Table 1

$q : \dim_F q = 6$	$Z(q)$	$C_0(q)$	$C(q)$
$\nu = 0$	$F \times F$	$D_4 \times D_4$	$M_2(D_4)$
$\nu = 0, \nu_L = 0$	L	$L \otimes D_4$	?
$\nu = 0, \nu_L = 1$	L	$M_2(L \otimes D_2)$	?
$\nu = 0, \nu_L = 3$	L	$M_4(L)$	?
$\nu = 1$	$F \times F$	$M_2(D_2) \times M_2(D_2)$	$M_4(D_2)$
$\nu = 1, \nu_Q = 1$	L	$M_2(L \otimes D_2)$	$M_2(D_4)$
$\nu = 1, \nu_Q = 2$	L	$M_2(L \otimes D_2)$	$M_4(D_2)$
$\nu = 2, 1 \in q(H(F^2)^\perp)$	L	$M_4(L)$	$M_8(F)$
$\nu = 2, 1 \notin q(H(F^2)^\perp)$	L	$M_4(L)$	$M_4(D_2)$
$\nu = 3$	$F \times F$	$M_4(F) \times M_4(F)$	$M_8(F)$

Here Q is the 8-dimensional quadratic form defined by $n_Z \perp -q$, where n_Z denotes the reduced norm form on $Z(q)$, and L is a separable quadratic extension over F. In the first column of the Table 1, ν, ν_Q and ν_L denote the Witt index of q, Q and q_L, respectively. In the third and fourth columns of the Table 1, D_n denotes a central division F-algebra of dimension n^2.

In this paper we study the question marks of the Table 1. For the 8-dimensional quadratic form $Q = n_Z \perp -q$, it is known that $C(Q) \simeq M_2(C(q))$ and $\mathrm{ind}C(Q) = \mathrm{ind}C(q)$. Hence $\mathrm{ind}C(q)$ is determined by Q. The solutions of second and third question marks of the Table 1 are given as in the Table 2 by considering how the form Q is decomposed into the orthogonal sum of subform of 2 or 4 dimensions.

In the case of $\mathrm{ch}(F) \neq 2$, Izhboldin and Karpenko [8, Theorem 16.10] proved that an 8-dimensional quadratic form ϕ has the trivial Arf invariant and satisfies $\mathrm{ind}C(\phi) \leq 4$ if and only if ϕ is isometric either to (1) an orthogonal sum of two quadratic forms which are each similar to 2-fold Pfister forms or (2) a Scharlau's transfer of a 4-dimensional quadratic form which is similar to a 2-fold Pfister form over a quadratic extension of F. The solution of first question mark of the Table 1 is given as in the Table 3 by applying this result to the form Q.

In the case of $\mathrm{ch}(F) = 2$, we will prove that the if part of Izhboldin and Karpenko's theorem also holds. Whether the only if part of Izhboldin and Karpenko's theorem holds or not for $\mathrm{ch}(F) = 2$ is not known, but we will give some sufficient conditions for Q to decomposed into an orthogonal sum of 2-fold Pfister forms.

We summarize in the following Tables 2, 3 and 4 all the results we proved in this paper on $\mathrm{ind}C(q)$ of a 6-dimensional anisotropic quadratic form q with non-trivial Arf invariant. The Table 2 gives a classification of $C(q)$ in the case that $\nu_L \geq 1$ and the characteristic of F is arbitrary. The Table 3 (resp. the Table 4) gives a classification of $C(q)$ in the case that $\nu_L = 0$ and $\mathrm{ch}(F) \neq 2$ (resp. $\mathrm{ch}(F) = 2$). Any positive condition of q such that $\mathrm{ind}C(q) = 8$ is not known.

We use the following notations.

$GP_r(F)$: a set of similar forms of r-fold Pfister forms over F.

$GP_2(F)_n := \{\perp_{i=1}^n \pi_i \mid \pi_i \in GP_2(F)\}$ $(GP_2(F)_1 = GP_2(F).)$

$\mathfrak{E}$: a set of separable quadratic extensions of F.

$s^*_{E/F}(GP_2(E))$: image of $GP_2(E)$ by Scharlau's transfer $s^*_{E/F}$.

$\mathfrak{S} = \cup_{E\in\mathfrak{E}} s^*_{E/F} GP_2(E) \cup GP_2(F)_2$.

Table 2

$\mathrm{ch}(F) \geq 0, q : \dim_F q = 6, \nu = 0, Z(q) = L$	$C_0(q)$	$C(q)$
$\nu_L = 1, \nu_Q = 0, Q$ is of type E_7	$M_2(L \otimes D_2)$	$M_4(D_2)$
$\nu_L = 1, \nu_Q = 0, Q$ is not of type E_7 $(Q \in GP_2(F)_2)$	$M_2(L \otimes D_2)$	$M_2(D_4)$
$\nu_L = 1, \nu_Q = 1$	$M_2(L \otimes D_2)$	$M_2(D_4)$
$\nu_L = 1, \nu_Q = 2$	$M_2(L \otimes D_2)$	$M_4(D_2)$
$\nu_L = 3, \nu_Q = 0, Q$ is of type E_7	$M_4(L)$	$M_4(D_2)$
$\nu_L = 3, \nu_Q = 0, Q$ is not of type E_7 $(Q \in GP_3(F))$	$M_4(L)$	$M_8(F)$
$\nu_L = 3, \nu_Q = 2$	$M_4(L)$	$M_4(D_2)$

Table 3

$\mathrm{ch}(F) \neq 2, q : \dim_F q = 6, \nu = 0, Z(q) = L$	$C_0(q)$	$C(q)$
$\nu_L = 0, \nu_Q = 0, Q \notin \mathfrak{S}$	$L \otimes D_4$	D_8
$\nu_L = 0, \nu_Q = 0, Q \in \mathfrak{S}$	$L \otimes D_4$	$M_2(D_4)$
$\nu_L = 0, \nu_Q = 1$	$L \otimes D_4$	$M_2(D_4)$

Table 4

$\mathrm{ch}(F) = 2, q : \dim_F q = 6, \nu = 0, Z(q) = L$	$C_0(q)$	$C(q)$
$\nu_L = 0, \nu_Q = 0, Q \notin \mathfrak{S}$	$L \otimes D_4$	$? \in \{D_8, M_2(D_4)\}$
$\nu_L = 0, \nu_Q = 0, Q \in \mathfrak{S}$	$L \otimes D_4$	$M_2(D_4)$
$\nu_L = 0, \nu_Q = 1$	$L \otimes D_4$	$M_2(D_4)$

By these results, we can make the following Table 5 on the 8-dimensional quadratic forms with trivial Arf invariant if $\mathrm{ch}(F) \neq 2$.

Table 5

$\mathrm{ch}(F) \neq 2, q : \dim_F q = 8, Z(q) = F \times F$	$C_0(q)$	$C(q)$
$\nu = 0, q \notin \mathfrak{S}$	$D_8 \times D_8$	$M_2(D_8)$
$\nu = 0, q \in \mathfrak{S}$, q does not have a norm splitting	$M_2(D_4) \times M_2(D_4)$	$M_4(D_4)$
$\nu = 0, q$ is of type E_7	$M_4(D_2) \times M_4(D_2)$	$M_8(D_2)$
$\nu = 0, q \in GP_3(F)$	$M_8(F) \times M_8(F)$	$M_{16}(F)$
$\nu = 1$	$M_2(D_4) \times M_2(D_4)$	$M_4(D_4)$
$\nu = 2$	$M_4(D_2) \times M_4(D_2)$	$M_8(D_2)$
$\nu = 4$	$M_8(F) \times M_8(F)$	$M_{16}(F)$

As an application of Tables 2, 3 and 4, we will show a Minkowski-Hasse type theorem in Theorem 4.5.

2 Notation and Definition

In this section we recall the basic notations on the quadratic forms.

Let F be a field of arbitrary characteristic. A quadratic space (V, q) over F is a pair of a finite dimensional F-vector space V and a quadratic form $q : V \longrightarrow F$ such that q satisfies:

1. $q(\lambda v) = \lambda^2 q(v)$ for $\lambda \in F, v \in V$;
2. $b_q : V \times V \longrightarrow F$ defined by $b_q(v, w) = q(v + w) - q(v) - q(w)$ is an F-bilinear form.

A quadratic form q is called regular if b_q is nonsingular. We assume that all the quadratic forms are regular throughout this paper.

A morphism of quadratic spaces $\phi : (V, q) \longrightarrow (V', q')$ is an F-linear map $\phi : V \longrightarrow V'$ such that $q(x) = q'(\phi(x))$ for all $x \in V$. If ϕ is an F-isomorphism, then it is called isometry.

A quadratic form which represents 0 for some nonzero element in V is called isotropic, otherwise it is called anisotropic. A 2-dimensional isotropic quadratic space defined by

$q_H(x) = x_1x_2$ for $x = (x_1, x_2) \in F^2$ is called hyperbolic space and denoted by $H(F) = (F^2, q_H)$. A quadratic form q is decomposed to an orthogonal sum of n-hyperbolic forms and an anisotropic form q_0, i.e.,$q \simeq q_H^n \perp q_0$. Then n is uniquely determined by q and is called the Witt index of q and denoted by $\nu(q)$.

If $\mathrm{ch}(F) \neq 2$, then n-dimensional quadratic form is isometric to a diagonal form $q(x) = \sum_{i=1}^n a_i x_i^2, (x = (x_1, \cdots, x_n) \in F^n)$ The q is denoted by $< a_1, \cdots, a_n >$.

In characteristic 2, the dimension of a regular quadratic form is always even and the diagonal quadratic forms are not regular. We can decompose $2m$-dimensional quadratic form into $q(x) = \sum_{i=1}^m (a_i x_{2i-1}^2 + x_{2i-1}x_{2i} + b_i x_{2i}^2)$. This q is denoted by $[a_1, b_1] \perp \cdots \perp [a_m, b_m]$.

In general the signed discriminant $\delta(q)$ of $2m$-dimensional quadratic form q is defined to be $\delta(q) = (-1)^m \det b_q$ as an element of $F^\bullet/F^{\bullet 2}$. If $\mathrm{ch}(F) = 2$, then the signed discriminant of q is trivial. In this case, for a quadratic form $q = [a_1, b_1] \perp \cdots \perp [a_m, b_m]$, we define the classical Arf invariant $\alpha(q)$ of q by $\alpha(q) = a_1b_1 + \cdots + a_mb_m$ as an element of $F/\wp(F)$, where $\wp(F) = \{x + x^2, x \in F\}$. We have $\delta(q \perp q') = \delta(q)\delta(q')$ and $\alpha(q \perp q') = \alpha(q) + \alpha(q')$.

A form $\ll a_1, \cdots, a_n \gg = < 1, a_1 > \otimes \cdots \otimes < 1, a_n >$ if $\mathrm{ch}(F) \neq 2$, and a form $[[b, a_1, \cdots, a_{n-1} \gg = [1, b] \otimes \ll a_1, \cdots, a_{n-1} \gg$ if $\mathrm{ch}(F) = 2$ are called an n-fold Pfister form. We denote by $GP_n(F)$ the set of all similar forms to n-fold Pfister forms

Let $F \subset E$ be a field extension. We can extend a quadratic form $q : V \longrightarrow F$ to a form $q_E : E \otimes V \longrightarrow E$ by putting

$$q_E\left(\sum_i \lambda_i \otimes v_i\right) = \sum_i \lambda_i^2 q(v_i) + \sum_{i<j} \lambda_i\lambda_j b_q(v_i, v_j).$$

We denote $(E \otimes V, q_E)$ by $E \otimes (V, q)$.

The Clifford algebra of a quadratic space (V, q) is defined as $C(V, q) = C(q) = T(V)/I(V)$, where $T(V)$ is a tensor algebra of V and $I(V)$ is a two-sided ideal of $T(V)$ generated by all elements of the form $v \otimes v - q(v)$, $(v \in V)$. The even Clifford algebra $C_0(V, q) = C_0(q)$ is the subalgebra of $C(q)$ generated by $uv, (u, v \in V)$.

Let $\dim q$ be even. Then $C(q)$ is a central simple F-algebra, and by Wedderburn's Theorem, $C(q) \simeq M_t(D)$ for some central division F-algebra D. We denote by $[C(q)] = [D]$ the Brauer equivalent class of $C(q)$. If $q \perp q'$ denotes the orthogonal sum of q and q', then $C(q \perp q')$ is isomorphic to $C(q) \otimes C(\delta(q)q')$. The center of $C_0(q)$ is a separable quadratic F-algebra. It is called the discriminant algebra of q and denoted by $Z(V, q) = Z(q)$. The isomorphism class of $Z(q)$ is called the Arf invariant of q. We say that the Arf invariant is trivial if $Z(q) \simeq F \times F$. Two quadratic forms q and q' have the same Arf invariant if and only if they have the same signed discriminant (the same classical Arf invariant if $\mathrm{ch}(F) = 2$)(cf. Knus [5, section 5]).

Let $\mathfrak{M}(F)$ be the set of all regular quadratic forms over F. If $F \subset L$ is a field extension and $s : L \longrightarrow F$ is a nonzero F-linear map, then Scharlau's transfer $s^*_{L/F}$ is a map from $\mathfrak{M}(L)$ to $\mathfrak{M}(F)$ defined by $s^*_{L/F}(q) = s \cdot q$. It is known that $\mathrm{Im}(s^*_{L/F})$ is independent of s and that $\dim_F s^*_{L/F}(q) = [L : F] \dim_L q$.

3 Basic properties

The notion of a norm splitting of a quadratic space was first introduced by Tits and Weiss (cf. Medts [10]). We say that a $2m$-dimensional quadratic space (V, q) over F has a norm splitting if there exists a separable quadratic extension $F \subset E$ with reduced norm n_E and some elements $a_1, \cdots, a_m \in F^\bullet$ such that $(V, q) \simeq (E, a_1 n_E) \perp \cdots \perp (E, a_m n_E)$. The following Theorems were proved in [5], [8], or [10].

Theorem 3.1 ([10, Theorem 3.9]) *Let $F \subset E$ be a separable quadratic extension and $a_1, \cdots, a_m \in F^\bullet$. Then $[C(\perp_{i=1}^m (E, a_i n_E))] = [C(E, (-1)^{[m/2]}(\prod_{i=1}^m a_i) n_E)]$.*

The index of Clifford algebra of 2-dimensional quadratic space depends only on the elements which the space represents. If the quadratic space represents 1, then the index is equal to 1, otherwise it is equal to 2. Hence the index of Clifford algebra of quadratic space which has norm splitting $(V, q) \simeq (E, a_1 n_E) \perp \cdots \perp (E, a_m n_E)$ is equal to 1 or 2 according as (E, n_E) represents $(-1)^{[m/2]} \prod_{i=1}^m a_i$ or not.

We recall that an 8-dimensional anisotropic quadratic space (V, q) is said to be of type E_7 if (V, q) has a norm splitting $(E, a_1 n_E) \perp \cdots \perp (E, a_4 n_E)$ such that $\prod_{i=1}^4 a_i \notin n_E(E^\bullet)$.

Theorem 3.2 ([8, Theorem 16,10]) *We assume that $\mathrm{ch}(F) \neq 2$. Let q be an 8-dimensional quadratic form over F. Then the following two conditions are equivalent each other.*

(1) *The Arf invariant of q is trivial and $\mathrm{ind} C(q) \leq 4$.*

(2) *At least one of the following conditions hold:*

  (a) *There exist $\pi_1, \pi_2 \in GP_2(F)$ such that $q = \pi_1 \perp \pi_2$.*

  (b) *There exist a field extension $F \subset L$ of degree 2 and a quadratic form $\tau \in GP_2(L)$ such that $q = s^*_{L/F}(\tau)$.*

Theorem 3.3 ([5, Ch.11]) *Let (V, q) be a 6-dimensional quadratic space with trivial Arf invariant. We assume that (V, q) represents $\lambda \in F^\bullet$. Then there exist a 16-dimensional central simple algebra A and an even symplectic involution σ over A such that $(V, q) \simeq (\mathrm{Alt}^\sigma(A), \lambda pf)$, where pf is a pfaffian. Moreover we have $C(q) \simeq M_2(A)$ and*

(1) *if* $\nu(q) = 0$, *then* $\mathrm{ind}C(q) = 4$,

(2) *if* $\nu(q) = 1$, *then* $\mathrm{ind}C(q) = 2$,

(3) *if* $\nu(q) = 3$, *then* $\mathrm{ind}C(q) = 1$.

4 Main Theorem

Let (V, q) be a 6-dimensional quadratic space and Z be a discriminant algebra of q with reduced norm n_Z. If $Q = n_Z \perp -q$ is isotropic, then we have an orthogonal decomposition $Q = q_H \perp Q'$ by some 6-dimensional quadratic form Q' with trivial Arf invariant, and hence $M_2(C(q)) \simeq C(Q) \simeq M_2(C(Q'))$. By Theorem 3.3, $\mathrm{ind}C(q)$ is determined by the Witt index of Q. Therefore we treat the 6-dimensional quadratic spaces (V, q) with non-trivial Arf invariant such that $Q = n_Z \perp -q$ are anisotropic.

First we consider the quadratic forms which satisfy $\nu(q_Z) = 3$.

Lemma 4.1 *Let (V, q) be an anisotropic quadratic space over F and $F \subset E$ be a separable quadratic extension with reduced norm n_E. If $E \otimes (V, q)$ is isotropic, then (V, q) is decomposed into $(V, q) \simeq (E, \lambda n_E) \perp (V', q')$ for some $\lambda \in F^\bullet$ and quadratic space (V', q') over F.*

Proof. See [10, Lemma 4.1]. □

By Lemma 4.1, a 6-dimensional quadratic form q with discriminant algebra $Z \not\simeq F \times F$ and $\nu(q_Z) = 3$ is decomposed into $q \simeq \lambda_1 n_Z \perp \lambda_2 n_Z \perp \lambda_3 n_Z$ for some $\lambda_i \in F^\bullet$. Therefore q has a norm splitting and the index of the Clifford algebra of q is equal to 1 or 2 according as $n_Z \perp -q$ is of type E_7 or not. In a similar fashion, if q is a quadratic form with discriminant algebra $Z \not\simeq F \times F$ and $\nu(q_Z) = 1$, then q is decomposed into $q \simeq \lambda n_Z \perp q'$, where q' is a 4-dimensional quadratic form with trivial Arf invariant, hence $q' \in GP_2(F)$. Therefore $n_Z \perp -q \in GP_2(F)_2$. On the other hand, we have $\mathrm{ind}C(q) = 2$ or 4 since $\mathrm{ind}_Z C(q_Z) = 2$ by Theorem 3.3. If $\mathrm{ind}C(q) = 2$, then both $C(\lambda n_Z)$ and $C(q')$ have the common splitting quadratic field $F \subset E$. Since both $n_Z \perp -\lambda n_Z$ and q' are hyperbolic over E, we have $n_Z \perp -q$ has a norm splitting by E.

Therefore $n_Z \perp -q$ is in $GP_2(F)_2$ and the index of the Clifford algebra of q is equal to 2 or 4 according as $n_Z \perp -q$ has a norm splitting or not.

If q is a quadratic form with discriminant algebra $Z \not\simeq F \times F$ and $\nu(q_Z) = 0$, then $\mathrm{ind}_Z C(q_Z) = 4$ by Theorem 3.3, hence we have $\mathrm{ind}_F C(q) = 4$ or 8. The condition that the quadratic forms satisfy $\mathrm{ind}_F C(q) = 4$ is given by applying Theorem 3.2 to $n_Z \perp -q$ if $\mathrm{ch}(F) \neq 2$.

By these consideration and Theorem 3.2, we can determine the index of Clifford algebra of 6-dimensional quadratic form if $\mathrm{ch}(F) \neq 2$. In the case of $\mathrm{ch}(F) = 2$, we need a counterpart to Theorem 3.2. The implication (2) $\Rightarrow$ (1) of Theorem 3.2 is true even if $\mathrm{ch}(F) = 2$. We have the followings.

Theorem 4.1 *We assume that* $\mathrm{ch}(F) = 2$. *Let* q *be an 8-dimensional quadratic form over* F. *Then the Arf invariant of* q *is trivial and* $\mathrm{ind}C(q) \leq 4$, *if at least one of the following conditions hold:*

(*a*) *there exist* $\pi_1, \pi_2 \in GP_2(F)$ *such that* $q \simeq \pi_1 \perp \pi_2$.

(*b*) *there exist a field extension* $F \subset L$ *of degree* 2 *and a quadratic form* $\tau \in GP_2(L)$ *such that* $q \simeq s^*_{L/F}(\tau)$.

Proof. If q satisfies (a), then the Theorem is trivial since both π_1 and $\pi_2 \in GP_2(F)$ have trivial classical Arf invariants and $\mathrm{ind}C(\pi_1), \mathrm{ind}C(\pi_2) \leq 2$. Therefore we assume that q satisfies (b). Let $L = F(z)$ be a separable field extension with $z^2 = z + r, r \in F$ and n_L a reduced norm of L. We take the F-linear map $L \ni x_1 + x_2 z \longrightarrow x_1 \in F$ as a map s. If $\tau = q_H^2$, then $s^*_{L/F}(\tau) = q_H^4$ and the Theorem is trivial. Hence we assume that τ is anisotropic. A quadratic form $\tau \in GP_2(L)$ is generally given for some $\lambda = \lambda_1 + \lambda_2 z, a = a_1 + a_2 z, b = b_1 + b_2 z \in L^\bullet$ $(\lambda_i, a_i, b_i \in F)$, by

$$\begin{array}{rcl} \tau & = & \lambda[[a, b \gg \\ & \simeq & [\lambda_1 + \lambda_2 z, \frac{(\lambda_1+\lambda_2)a_1+\lambda_2 r a_2}{n_L(\lambda)} + \frac{\lambda_2 a_1 + \lambda_1 a_2}{n_L(\lambda)} z] \\ & \perp & [\lambda_1 b_1 + \lambda_2 r b_2 + \{\lambda_2 b_1 + (\lambda_1 + \lambda_2) b_2\} z, \frac{(\lambda_1+\lambda_2)A_1+\lambda_2 r A_2}{n_L(\lambda)} + \frac{\lambda_2 A_1 + \lambda_1 A_2}{n_L(\lambda)} z] \end{array}$$

where $A_i \in F$ is given by $\frac{a_1+a_2 z}{b_1+b_2 z} = A_1 + A_2 z$.

Hence we have

$$\begin{array}{rcl} s^*_{L/F}(\tau) & = & [\lambda_1, \frac{(\lambda_1+\lambda_2)a_1+\lambda_2 r a_2}{n_L(\lambda)}] \perp [\lambda_1 b_1 + \lambda_2 r b_2, \frac{(\lambda_1+\lambda_2)A_1+\lambda_2 r A_2}{n_L(\lambda)}] \\ & \perp & [(\lambda_1 + \lambda_2) r, \frac{\lambda_1 a_1 + (\lambda_1+\lambda_2 r) a_2}{n_L(\lambda) r}] \\ & \perp & [\{(\lambda_1 + \lambda_2) b_1 + (\lambda_1 + \lambda_2 + \lambda_2 r) b_2\} r, \frac{\lambda_1 A_1 + (\lambda_1+\lambda_2 r) A_2}{n_L(\lambda) r}]. \end{array}$$

The Arf invariant of $s^*_{L/F}(\tau)$ is trivial since

$$\begin{array}{rcl} \alpha(s^*_{L/F}(\tau)) & = & \lambda_1 \frac{(\lambda_1+\lambda_2)a_1+\lambda_2 r a_2}{n_L(\lambda)} + (\lambda_1 b_1 + \lambda_2 r b_2) \frac{(\lambda_1+\lambda_2)A_1+\lambda_2 r A_2}{n_L(\lambda)} \\ & + & (\lambda_1 + \lambda_2) \frac{\lambda_1 a_1 + (\lambda_1+\lambda_2 r) a_2}{n_L(\lambda)} \\ & + & \{(\lambda_1 + \lambda_2) b_1 + (\lambda_1 + \lambda_2 + \lambda_2 r) b_2\} \frac{\lambda_1 A_1 + (\lambda_1+\lambda_2 r) A_2}{n_L(\lambda)} \\ & = & a_2 + b_1 A_2 + b_2 A_1 + b_2 A_2 \\ & = & \frac{a_2(b_1^2+b_1 b_2+b_2^2 r)+b_1(a_1 b_2+a_2 b_1)+b_2(a_1 b_1+a_1 b_2+a_2 b_2 r)+b_2(a_1 b_2+a_2 b_1)}{n_L(b)} \\ & = & 0. \end{array}$$

In the followings, we consider the Clifford algebra of $s^*_{L/F}(\tau)$. We denote the Brauer equivalent class of $C[x,y]$ by $[[x,y]]$. Since the signed discriminant of 2-dimensional quadratic form is trivial if $\mathrm{ch}(F)=2$, we have

$$\begin{aligned}[C(s^*_{L/F}(\tau))] &= [[\lambda_1, \tfrac{(\lambda_1+\lambda_2)a_1+\lambda_2 ra_2}{n_L(\lambda)}]] \otimes [[\lambda_1 b_1+\lambda_2 rb_2, \tfrac{(\lambda_1+\lambda_2)A_1+\lambda_2 rA_2}{n_L(\lambda)}]]\\ &\otimes [[(\lambda_1+\lambda_2)r, \tfrac{\lambda_1 a_1+(\lambda_1+\lambda_2 r)a_2}{n_L(\lambda)r}]]\\ &\otimes [[\{(\lambda_1+\lambda_2)b_1+(\lambda_1+\lambda_2+\lambda_2 r)b_2\}r, \tfrac{\lambda_1 A_1+(\lambda_1+\lambda_2 r)A_2}{n_L(\lambda)r}]].\end{aligned}$$

By two relations

$[[w,x]]\otimes[[w,y]]=[[w,x+y]]$ and

$[[w,xy]]\otimes[[x,yw]]\otimes[[y,wx]]=1 (w,x,y\in F)$, we have

$$\begin{aligned}[C(s^*_{L/F}(\tau))] &= [[\lambda_1, \tfrac{(\lambda_1+\lambda_2)a_1+\lambda_2 ra_2}{n_L(\lambda)}]] \otimes [[\lambda_1 b_1+\lambda_2 rb_2, \tfrac{(\lambda_1+\lambda_2)A_1+\lambda_2 rA_2}{n_L(\lambda)}]]\\ &\otimes [[\lambda_1+\lambda_2, \tfrac{\lambda_1 a_1+(\lambda_1+\lambda_2 r)a_2}{n_L(\lambda)}]] \otimes [[r, \tfrac{(\lambda_1+\lambda_2)\{\lambda_1 a_1+(\lambda_1+\lambda_2 r)a_2\}}{n_L(\lambda)r}]]\\ &\otimes [[(\lambda_1+\lambda_2)b_1+(\lambda_1+\lambda_2+\lambda_2 r)b_2, \tfrac{\lambda_1 A_1+(\lambda_1+\lambda_2 r)A_2}{n_L(\lambda)}]]\\ &\otimes [[r, \tfrac{\{(\lambda_1+\lambda_2)b_1+(\lambda_1+\lambda_2+\lambda_2 r)b_2\}\{\lambda_1 A_1+(\lambda_1+\lambda_2 r)A_2\}}{n_L(\lambda)r}]]\\ &= [[\lambda_1, \tfrac{\lambda_2 a_1+\lambda_1 a_2}{n_L(\lambda)}]] \otimes [[\lambda_1 b_1+\lambda_2 rb_2, \tfrac{\lambda_2 A_1+\lambda_1 A_2}{n_L(\lambda)}]]\\ &\otimes [[\lambda_2, \tfrac{\lambda_1(a_1+a_2)+\lambda_2 ra_2}{n_L(\lambda)}]] \otimes [[r, \tfrac{(\lambda_1+\lambda_2)\{\lambda_1(a_1+a_2)+\lambda_2 ra_2\}}{n_L(\lambda)r}]]\\ &\otimes [[\lambda_1 b_2+\lambda_2(b_1+b_2), \tfrac{\lambda_1(A_1+A_2)+\lambda_2 rA_2}{n_L(\lambda)}]]\\ &\otimes [[r, \tfrac{\{\lambda_1(b_1+b_2)+\lambda_2(b_1+b_2+rb_2)\}\{\lambda_1(A_1+A_2)+\lambda_2 rA_2\}}{n_L(\lambda)r}]]\\ &= [[\lambda_1, \tfrac{\lambda_1 a_2+\lambda_2 a_1}{n_L(\lambda)}]] \otimes [[\lambda_1, \tfrac{\lambda_1 b_1 A_2+\lambda_2 b_1 A_1}{n_L(\lambda)}]] \otimes [[b_1, \tfrac{\lambda_1^2 A_2+\lambda_1\lambda_2 A_1}{n_L(\lambda)}]]\\ &\otimes [[\lambda_2, \tfrac{\lambda_1 rb_2 A_2+\lambda_2 rb_2 A_1}{n_L(\lambda)}]] \otimes [[r, \tfrac{\lambda_1\lambda_2 rb_2 A_2+\lambda_2^2 rb_2 A_1}{n_L(\lambda)r}]] \otimes [[b_2, \tfrac{\lambda_1\lambda_2 rA_2+\lambda_2^2 rA_1}{n_L(\lambda)}]]\\ &\otimes [[\lambda_2, \tfrac{\lambda_1(a_1+a_2)+\lambda_2 ra_2}{n_L(\lambda)}]] \otimes [[r, \tfrac{(\lambda_1+\lambda_2)\{\lambda_1(a_1+a_2)+\lambda_2 ra_2\}}{n_L(\lambda)r}]]\\ &\otimes [[\lambda_1, \tfrac{\lambda_1 b_2(A_1+A_2)+\lambda_2 rb_2 A_2}{n_L(\lambda)}]] \otimes [[b_2, \tfrac{\lambda_1^2(A_1+A_2)+\lambda_1\lambda_2 rA_2}{n_L(\lambda)}]]\\ &\otimes [[\lambda_2, \tfrac{\lambda_1(b_1+b_2)(A_1+A_2)+\lambda_2 r(b_1+b_2)A_2}{n_L(\lambda)}]] \otimes [[b_1+b_2, \tfrac{\lambda_1\lambda_2(A_1+A_2)+\lambda_2^2 rA_2}{n_L(\lambda)}]]\\ &\otimes [[r, \tfrac{\{\lambda_1(b_1+b_2)+\lambda_2(b_1+b_2+rb_2)\}\{\lambda_1(A_1+A_2)+\lambda_2 rA_2\}}{n_L(\lambda)r}]].\end{aligned}$$

Since $a_2+b_1A_2=b_2(A_1+A_2)$, $a_1+b_1A_1=rb_2A_2$, we have

$$\begin{aligned}[C(s^*_{L/F}(\tau))] &= [[b_1,A_2]]\otimes[[r,b_2A_2]]\otimes[[b_2,A_1+A_2]]\\ &= [[b_1,A_2]]\otimes[[rb_2,A_2]]\otimes[[b_2,rA_2]]\otimes[[b_2,A_1+A_2]]\\ &= [[b_1+rb_2,A_2]]\otimes[[b_2,A_1+A_2+rA_2]].\end{aligned}$$

Hence we have $\mathrm{ind}C(s^*_{L/F}(\tau))\le 4$.

If $L=F(z)$ is inseparable, then we can set $z^2=r$. Then Scharlau's transfer of $\tau=\lambda[[a,b\gg$ is given by

$$\begin{aligned}s^*_{L/F}(\tau) &= [\lambda_1, \tfrac{\lambda_1 a_1+\lambda_2 a_2 r}{\lambda^2}] \perp [\lambda_1 b_1+\lambda_2 b_2 r, \tfrac{\lambda_1(a_1b_1+a_2b_2r)+\lambda_2(a_1b_2+a_2b_1)r}{\lambda^2 b^2}]\\ &\perp r[\lambda_1, \tfrac{\lambda_1 a_1+\lambda_2 a_2 r}{\lambda^2}] \perp r[\lambda_1 b_1+\lambda_2 b_2 r, \tfrac{\lambda_1(a_1b_1+a_2b_2r)+\lambda_2(a_1b_2+a_2b_1)r}{\lambda^2 b^2}].\end{aligned}$$

Hence the index of the Clifford algebra of $s^*_{L/F}(\tau)$ is less than 2 since we have

$$\begin{array}{rcl}[C(s^*_{L/F}(\tau))] & = & [[r, \frac{\lambda_1^2 a_1+\lambda_1\lambda_2 a_2 r}{\lambda^2 r}]] \otimes [[r, \frac{(\lambda_1 b_1+\lambda_2 b_2 r)\{\lambda_1(a_1b_1+a_2b_2r)+\lambda_2(a_1b_2+a_2b_1)r\}}{\lambda^2 b^2 r}]] \\ & = & [[r, \frac{a_1b_2^2+a_2b_1b_2}{b^2}]].\end{array}$$

□

The implication (1) ⇒ (2) of Theorem 3.2 in the case of $\mathrm{ch}(F) = 2$ is easily proved as follows if q is isotropic or q satisfies $\mathrm{ind}C(q) < 4$.

Lemma 4.2 *Let q be an isotropic quadratic form of dimension 8 with trivial Arf invariant. Then there exist $\pi_1, \pi_2 \in GP_2(F)$ such that $q \simeq \pi_1 \perp \pi_2$.*

Proof. Let $q = q_H \perp [a_1, b_1] \perp [b_2, b_3] \perp [a_2, \frac{a_1b_1+b_2b_3}{a_2}]$ for $a_i \in F^\bullet, b_i \in F$. Then we have

$$q \simeq [a_1, \frac{b_2b_3}{a_1}] \perp [a_1, \frac{a_1b_1+b_2b_3}{a_1}] \perp [b_2, b_3] \perp [a_2, \frac{a_1b_1+b_2b_3}{a_2}].$$

Let $\pi_1 = [a_1, \frac{b_2b_3}{a_1}] \perp [b_2, b_3]$ and $\pi_2 = [a_1, \frac{a_1b_1+b_2b_3}{a_1}] \perp [a_2, \frac{a_1b_1+b_2b_3}{a_2}]$. Then we have $q \simeq \pi_1 \perp \pi_2$. □

Theorem 4.2 *Let q be an 8-dimensional quadratic form over F. If the Arf invariant of q is trivial and $\mathrm{ind}C(q) < 4$, then there exist $\pi_1, \pi_2 \in GP_2(F)$ such that $q \simeq \pi_1 \perp \pi_2$.*

Proof. For any decomposition $q = q_1 \perp q_2$ with $\dim q_1 = 2$, the index of the Clifford algebra of q_2 over $Z(q_1)$ is less than 2 by Theorem 3.3. Therefore $q_2 = \lambda q_1 \perp \pi$ for some $\lambda \in F^\bullet$ and $\pi \in GP_2(F)$ by Lemma 4.1. Hence $q \in GP_2(F)_2$. □

We consider the implication (1) ⇒ (2) of Theorem 3.2 under the condition of $\mathrm{ind}C(q) = 4$ in the case of $\mathrm{ch}(F) = 2$. For an 8-dimensional quadratic form q with trivial Arf invariant, we have $C(q) \simeq C(\lambda q)$ for any $\lambda \in F^\bullet$. Therefore we may assume that q represents 1. Let $q = [1, *] \perp q'$. Then q' is a 6-dimensional quadratic form such that $[C(q)] = [C(q')]$. We consider a relation between a subform of q' and the index of $C(q)$.

Lemma 4.3 *Let q be a quadratic form of dimension 8 with trivial Arf invariant and $\mathrm{ind}C(q) = 4$. We denote by D a division algebra such that $[D] = [C(q)]$. If there exists a 2-dimensional quadratic subform $\phi \subset q$ such that $L = Z(\phi) \subset D$, then $q \simeq \pi_1 \perp \pi_2$ for some $\pi_i \in GP_2(F)$.*

Proof. Let $q = \phi \perp q'$. Then ϕ and q' have the same Arf invariant. Since $q_L = q_H \perp q'_L$ and $L \subset D$, we have $\mathrm{ind}_L C(q'_L) = \mathrm{ind}_L C(q_L) = \mathrm{ind}_L L \otimes C(q) = 2$. Obviously q'_L has the trivial

Arf invariant. This show $q' \simeq \lambda\phi \perp \pi_1$ for some $\lambda \in F^\bullet, \pi_1 \in GP_2(F)$. We set $\pi_2 = \phi \perp \lambda\phi$, then we have $q \simeq \pi_1 \perp \pi_2$. □

Theorem 4.3 *Let q be a 6-dimensional quadratic form over F with $\mathrm{ind}C(q) = 4$ and Z a discriminant algebra of q with reduced norm n_Z. If there exists some decomposition $q = q_1 \perp q_2$ with $\dim_F q_1 = 4$ and $\dim_F q_2 = 2$ such that both $C(q_1)$ and $C(q_2)$ contain a common separable extension field $F \subset L$ of degree 2, then $n_Z \perp q \simeq \pi_1 \perp \pi_2$ for some $\pi_1, \pi_2 \in GP_2(F)$.*

Proof. Since

$$\begin{aligned} C(L \otimes (n_Z \perp q_2)) &\simeq L \otimes C(n_Z \perp q_2) \\ &\simeq L \otimes M_2(C(q_2)) \\ &\simeq M_2(L \otimes C(q_2)) \\ &\simeq M_4(L), \end{aligned}$$

$(n_Z \perp q_2)_L$ is isotropic and $n_Z \perp q_2 \simeq \alpha n_L \perp \phi$ for some $\alpha \in F^\bullet$ and a 2-dimensional quadratic form ϕ. Let $\psi = \phi \perp q_1$. Then ψ is a 6-dimensional quadratic form with discriminant algebra L and so ψ_L has the trivial Arf invariant. On the one hand, we have

$$\begin{aligned} \mathrm{ind}C((n_Z \perp q)_L) &= \mathrm{ind}(L \otimes C(n_Z \perp q)) \\ &= \mathrm{ind}(L \otimes M_2(C(q))) \\ &= \mathrm{ind}(L \otimes C(q_1) \otimes C(q_2)) \\ &\leq 2. \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathrm{ind}C((n_Z \perp q)_L) &= \mathrm{ind}C((\alpha n_L \perp \psi)_L) \\ &= \mathrm{ind}C(q_H \perp \psi_L) \\ &= \mathrm{ind}C(\psi_L). \end{aligned}$$

Hence $\mathrm{ind}C(\psi_L) \leq 2$. This show that ψ_L is isotropic and we have $\psi \simeq \beta n_L \perp \pi_1$ for some $\beta \in F^\bullet$ and $\pi_1 \in GP_2(F)$. Therefore if we set $\pi_2 = \alpha n_L \perp \beta n_L \in GP_2(F)$, then we have

$$\begin{aligned} n_Z \perp q &\simeq n_Z \perp q_2 \perp q_1 \\ &\simeq \alpha n_L \perp \phi \perp q_1 \\ &\simeq \alpha n_L \perp \beta n_L \perp \pi_1 \\ &\simeq \pi_2 \perp \pi_1. \end{aligned}$$

□

Corollary 4.1 *Let q be a quadratic form of dimension 8 with trivial Arf invariant and $\mathrm{ind}C(q) = 4$. If there exists a decomposition $q = q_1 \perp q_2$, where q_i are forms of dimension 4 with $\mathrm{ind}C(q_i) = 2$, then $q \simeq \pi_1 \perp \pi_2$ for some $\pi_i \in GP_2(F)$.*

Proof. We may assume that q is anisotropic by Lemma 4.2. Let Z be a discriminant algebra of q_1 (so it is also of q_2). Since $\mathrm{ind}C(q_1) = 2$, $n_Z \perp q_1$ is isotropic, and so q_1 represents some non-zero element $\lambda \in \mathrm{Im}(n_Z)$. Let $q_1 = \phi_1 \perp \phi_2$ such that ϕ_1 represents λ and $\dim \phi_1 = 2$. We set $\psi = \lambda\phi_2 \perp \lambda q_2$. Since $C(\lambda q_2) \simeq C(q_2)$ and $C(\lambda\phi_2) \simeq C(q_1)$, we have $\mathrm{ind}C(\lambda q_2) = 2$ and ψ satisfies the condition of Theorem 4.3. Then we have

$$\begin{array}{rcl} n_{Z(\psi)} \perp -\psi & \simeq & n_{Z(-\lambda\phi_1)} \perp \lambda\phi_2 \perp -\lambda q_2 \\ & \simeq & n_{Z(\phi_1)} \perp \lambda\phi_2 \perp -\lambda q_2 \\ & \simeq & \lambda\phi_1 \perp \lambda\phi_2 \perp -\lambda q_2 \\ & \simeq & \lambda q. \end{array}$$

Therefore $q \simeq \pi_1 \perp \pi_2$ for some $\pi_i \in GP_2(F)$ by Theorem 4.3. □

The converse of Theorem 4.3 is also hold.

Theorem 4.4 *Let q be a 6-dimensional quadratic form over F with $\mathrm{ind}C(q) = 4$ and Z a discriminant algebra of q with reduced norm n_Z. If $n_Z \perp q \simeq \pi_1 \perp \pi_2$ for some $\pi_1, \pi_2 \in GP_2(F)$, then there exists a decomposition $q = q_1 \perp q_2$ with $\dim_F q_1 = 4$ and $\dim_F q_2 = 2$ such that both $C(q_1)$ and $C(q_2)$ contain a same separable quadratic field L over F.*

Proof. We set $Z = F \cdot 1 + F \cdot z, z^2 = z + r, r \in F$ and $\pi_i = \lambda_i n_{A_i} (i = 1, 2)$, where n_{A_i} are the reduced norm of quaternion F-algebras A_i and $\lambda_i \in F^\bullet$. Since $n_Z \perp q \simeq \pi_1 \perp \pi_2$, we have that 1 and $z \in Z$ each correspond to $v_1 + v_2$ and $w_1 + w_2$ for some $v_i, w_i \in A_i$.

Since $b_n(1, z) = b_{\pi_1+\pi_2}(v_1 + v_2, w_1 + w_2) = b_{\pi_1}(v_1, w_1) + b_{\pi_2}(v_2, w_2) = 1$, we have $b_{\pi_1}(v_1, w_1) \neq 0$ or $b_{\pi_2}(v_2, w_2) \neq 0$. We may assume $b_{\pi_1}(v_1, w_1) \neq 0$. Then we have $v_1, w_1 \neq 0$. Let $V = F \cdot v_1 + F \cdot w_1$ and $\phi = \pi_1|_V$. Since $b_{\pi_1}(v_1, w_1) \neq 0$, ϕ is nonsingular and $\psi_1 = \phi^\perp$ in π_1 is a 2-dimensional quadratic form. We have $\pi_2 \perp \phi \simeq n_Z \perp \psi_2$ for some 4-dimensional quadratic form ψ_2. Hence we have $q \simeq \psi_1 \perp \psi_2$ and set $q_1 = \psi_2$ and $q_2 = \psi_1$. Let $L = Z(q_2)$. We show $L \subset C(q_1)$. Since $\pi_{1L} \simeq q_H^2$ and $q_{2L} \simeq q_H$, we have

$$\begin{array}{rcl} \mathrm{ind}(C((n_Z \perp q)_L)) & = & \mathrm{ind}(C((\pi_1 \perp \pi_2)_L)) \\ & = & \mathrm{ind}(C(\pi_{2L})) \\ & = & \mathrm{ind}(L \otimes A_2) \\ & \leq & 2 \end{array}$$

and on the other hand

$$\begin{array}{rcl} \mathrm{ind}(C((n_Z \perp q)_L)) & = & \mathrm{ind}(L \otimes C(q)) \\ & = & \mathrm{ind}(L \otimes C(q_1) \otimes C(q_2)) \\ & = & \mathrm{ind}(L \otimes C(q_1)). \end{array}$$

This show that $\mathrm{ind}(L \otimes C(q_1)) \leq 2$ and $L \subset C(q_1)$. □

As an application of Tables 2, 3 and 4, we show a Minkowski–Hasse type theorem. We denote by $Br_2(F)$ the subgroup $\{[A] \in Br(F) \mid [A]^2 = 1\}$ of the Brauer group $Br(F)$ and by $Br_2(F)'$ the subset of $Br_2(F)$ consisting of Brauer classes of quaternion F-algebras. By Merkurjev's theorem and the theory of simple p-algebras in the case of $\mathrm{ch}(F) = p = 2$, it is known that $Br_2(F)$ is generated by $Br_2(F)'$.

Theorem 4.5 *Assume that F satisfies the following two conditions*

1) $Br_2(F) = Br_2(F)'$.

2) For any quaternion F-algebra A, its reduced norm n_A is surjective.

Then the dimension of any anisotropic quadratic form over F is less than or equal to 4.

Proof. Let (V, q) be an anisotropic quadratic form over F, $L = Z(q)$ the discriminant algebra of q and ν_L the Witt index of $L \otimes (V, q)$. We first suppose that $\dim_F V = 6$. By the assumption 1), the index of $C(q)$ must be less than or equal to 2. Then, it follows from the classification Tables 2, 3 and 4 that $\nu_L \geq 1$, and hence there exists a nonsingular 2-dimensional subspace U of V such that $(V, q) = (U, q|_U) \perp (U^\perp, q|_{U^\perp})$ and $Z(q|_U) = L$. Since the Arf invariant of $(U^\perp, q|_{U^\perp})$ is trivial, $(U^\perp, q|_{U^\perp})$ is similar to the reduced norm form of some quaternion F-algebra. By the assumption 2), $(U^\perp, q|_{U^\perp})$ is universal, i.e., $q|_{U^\perp}$ represents an arbitrary element of F. This contradicts to $\nu(V, q) = 0$. Therefore, $\dim_F V$ must be less than or equal to 5. If $\mathrm{ch}(F) = 2$, we have $\dim_F V \leq 4$ as q is nonsingular. Thus, we next suppose that $\mathrm{ch}(F) \neq 2$ and $\dim_F V = 5$. Let $\delta(q)$ be the signed discriminant of (V, q). It was proved in [5, Ch.12, Proposition 5] that the index of $C_0(q)$ is 4 if and only if q does not represent $\delta(q)$. This is not the case by the assumption 1). Hence there exists a $v \in V$ such that $q(v)F^{\bullet 2} = \delta(q)$. Then the 4-dimensional quadratic form $(\{v\}^\perp, q|_{\{v\}^\perp})$ is similar to the reduced norm form of some quaternion F-algebra since the Arf invariant of $(\{v\}^\perp, q|_{\{v\}^\perp})$ is trivial. By the assumption 2) and the same argument as above, this leads us to a contradiction. $\square$

The above proof of Theorem 4.5 was given by Watanabe in the Appendix of [4]. It is well known that non-Archimedean local fields, totally imaginary algebraic number fields and function fields of one variable over a finite field satisfy the conditions 1) and 2) (cf. [4, Ch. X – XIII]). Theorem 4.5 gives a uniform and characteristic free proof of Minkowski–Hasse theorem of such local and global fields.

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