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Error Inequalities for a Taylor-like Formula

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ABSTRACT

A Taylor-like formula is derived. Various error bounds for this formula are established.

RESUMEN

Se deduce una formula de tipo Taylor. Se establecen varias cotas de error para esta formula.

Key words and phrases: *Error Inequalities, Taylor formula.*

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1 Introduction

In recent years a number of authors have considered the Taylor and generalized Taylor formulas from an inequalities point of view. For example, this topic is considered in [1], [2], [3], [4], [5], [6] and [8]. In [5] we can find the following generalization of Taylor formula:

$$f(x) = f(a) + \sum_{k=1}^{n} (-1)^{k+1} \left[P_k(x) f^{(k)}(x) - P_k(a) f^{(k)}(a) \right] + R_n(f, a, x), \tag{1}$$

$$R_n(f, a, x) = (-1)^n \int_a^x P_n(t) f^{(n+1)}(t) dt,$$

where $\{P_k(t)\}_0^\infty$ is a harmonic (or Appell) sequence of polynomials, that is

$$P_k'(t) = P_{k-1}(t), \quad P_0(t) = 1.$$

If we substitute

$$P_k(t) = \frac{(t-x)^k}{k!}$$

in (1) then we get the classical Taylor formula:

$$f(x) = f(a) + \sum_{k=1}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + R_n^C(f, a, x),$$

$$R_n^C(f, a, x) = \frac{1}{n!} \int_a^x (x-t)^n f^{(n+1)}(t) dt.$$

In this paper we derive a Taylor-like formula. A way of obtaining this formula is similar to the way described in [5]. However, here we do not use an Appell sequence of polynomials. We use functions of the form

$$S_n(t) = \begin{cases} P_n(t), & t \in \left[a, \frac{a+x}{2}\right] \\ Q_n(t), & t \in \left(\frac{a+x}{2}, x\right] \end{cases},$$

where $P_n(t)$ and $Q_n(t)$ are Appell-like sequences of polynomials. We also establish various error bounds for this formula. Similar error inequalities are established in [7] for some quadrature rules.

Finally, we give an application of the mentioned Taylor-like formula to logarithmic function.

2 Main results

Theorem 1 *Let $f : [a, x] \to R$ be a function such that $f^{(n)}$ is absolutely continuous. Then*

$$f(x) = f(a) - \sum_{k=1}^{n} \frac{(-1)^k (x-a)^k}{4^k k!} (1+k) \left[f^{(k)}(x) - (-1)^k f^{(k)}(a) \right] \tag{2}$$

$$-\sum_{k=2}^{n}\frac{(-1)^k(x-a)^k}{4^k k!}(1-k)\left[1-(-1)^k\right]f^{(k)}(\frac{a+x}{2})+R(f),$$

where

$$R(f)=(-1)^n\int_a^x S_n(t)f^{(n+1)}(t)dt \tag{3}$$

and

$$S_n(t)=\begin{cases}\frac{(t-\frac{3a+x}{4})^{n-1}}{n!}\left[t+\frac{(n-3)a-(n+1)x}{4}\right], & t\in\left[a,\frac{a+x}{2}\right]\\ \frac{(t-\frac{a+3x}{4})^{n-1}}{n!}\left[t+\frac{(n-3)x-(n+1)a}{4}\right], & t\in\left(\frac{a+x}{2},x\right]\end{cases}. \tag{4}$$

Proof. We prove (2) by induction. We easily show that (2) holds for $n=1$. Now suppose that (2) holds for an arbitrary n. We have to prove that (2) holds for $n\to n+1$. To simplify the proof we introduce the notations

$$P_n(t)=\frac{(t-\frac{3a+x}{4})^{n-1}}{n!}\left[t+\frac{(n-3)a-(n+1)x}{4}\right] \tag{5}$$

$$Q_n(t)=\frac{(t-\frac{a+3x}{4})^{n-1}}{n!}\left[t+\frac{(n-3)x-(n+1)a}{4}\right]. \tag{6}$$

We see that P_n and Q_n form Appell sequences of polynomials, that is

$$P_n'(t)=P_{n-1}(t),\ \ Q_n'(t)=Q_{n-1}(t),\ \ P_0(t)=Q_0(t)=1.$$

We have

$$(-1)^{n+1}\int_a^x S_{n+1}(t)f^{(n+2)}(t)dt$$

$$=(-1)^{n+1}\int_a^{\frac{a+x}{2}}P_{n+1}(t)f^{(n+2)}(t)dt+(-1)^{n+1}\int_{\frac{a+x}{2}}^x Q_{n+1}(t)f^{(n+2)}(t)dt$$

$$=(-1)^{n+1}\left[P_{n+1}(\frac{a+x}{2})f^{(n+1)}(\frac{a+x}{2})-P_{n+1}(a)f^{(n+1)}(a)\right]$$

$$+(-1)^{n+1}\left[Q_{n+1}(x)f^{(n+1)}(x)-Q_{n+1}(\frac{a+x}{2})f^{(n+1)}(\frac{a+x}{2})\right]$$

$$+(-1)^n\int_a^{\frac{a+x}{2}}P_n(t)f^{(n+1)}(t)dt+(-1)^n\int_{\frac{a+x}{2}}^x Q_n(t)f^{(n+1)}(t)dt$$

$$=(-1)^n\int_a^x S_n(t)f^{(n+1)}(t)dt+(-1)^{n+1}\left[P_{n+1}(\frac{a+x}{2})-Q_{n+1}(\frac{a+x}{2})\right]f^{(n+1)}(\frac{a+x}{2})$$

$$-(-1)^{n+1}\left[P_{n+1}(a)f^{(n)}(a)-Q_{n+1}(x)f^{(n)}(x)\right]$$

$$=-\int_a^x f'(t)dt+\sum_{k=1}^{n}\frac{(-1)^k(x-a)^k}{4^k k!}\left[f^{(k)}(x)-(-1)^k f^{(k)}(a)\right]$$

$$+\sum_{k=2}^{n}\frac{(-1)^k(x-a)^k}{4^k k!}(1-k)\left[1-(-1)^k\right]f^{(k)}(\frac{a+x}{2})$$

$$+(-1)^{n+1}\left[P_{n+1}(\frac{a+x}{2})-Q_{n+1}(\frac{a+x}{2})\right]f^{(n+1)}(\frac{a+x}{2})$$

$$-(-1)^{n+1}\left[P_{n+1}(a)f^{(n)}(a)-Q_{n+1}(x)f^{(n)}(x)\right]$$

$$=-\int_a^x f'(t)dt+\sum_{k=1}^{n+1}\frac{(-1)^k(x-a)^k}{4^k k!}\left[f^{(k)}(x)-(-1)^k f^{(k)}(a)\right]$$

$$+\sum_{k=2}^{n+1}\frac{(-1)^k(x-a)^k}{4^k k!}(1-k)\left[1-(-1)^k\right]f^{(k)}(\frac{a+x}{2}),$$

since

$$(-1)^{n+1}\left[P_{n+1}(\frac{a+x}{2})-Q_{n+1}(\frac{a+x}{2})\right]f^{(n)}(\frac{a+x}{2})$$

$$-(-1)^{n+1}\left[P_{n+1}(a)f^{(n)}(a)-Q_{n+1}(x)f^{(n)}(x)\right]$$

$$=\frac{(-1)^{n+1}(x-a)^{n+1}}{4^{n+1}(n+1)!}(1-n-1)\left[1-(-1)^{n+1}\right]f^{(n+1)}(\frac{a+x}{2})$$

$$+\frac{(-1)^{n+1}(x-a)^{n+1}}{4^{n+1}(n+1)!}\left[f^{(n+1)}(x)-(-1)^{n+1}f^{(n+1)}(a)\right].$$

This completes the proof. ■

Lemma 2 *The functions $S_n(t)$ satisfy:*

$$\int_a^x S_n(t)dt=0, \quad \textit{if } n \textit{ is odd}, \tag{7}$$

$$\int_a^x |S_n(t)|\,dt=\frac{(4n+4)(x-a)^{n+1}}{4^{n+1}(n+1)!}, \tag{8}$$

$$\max_{t\in[a,x]}|S_n(t)|=\frac{(n+1)(x-a)^n}{4^n n!}. \tag{9}$$

Proof. A simple calculation gives

$$\int_a^x S_n(t)dt=\frac{(x-a)^{n+1}}{4^n(n+1)!}\left[1-(-1)^{n+1}\right].$$

From the above relation we see that (7) holds, since $1-(-1)^{n+1}=0$ if n is odd.

We now consider some properties of the Appell sequences of polynomials $P_n(t)$ and $Q_n(t)$, given by (5) and (6), respectively. Since

$$t+\frac{(n-3)a-(n+1)x}{4}\le 0,\ t\in\left[a,\frac{a+x}{2}\right]$$

and

$$t+\frac{(n-3)x-(n+1)a}{4}\ge 0,\ t\in\left(\frac{a+x}{2},x\right]$$

we easily show that the following facts are valid.

If n is odd then $P_n(t)\le 0$ and $Q_n(t)\ge 0$. Furthermore, $P_n(t)$ is an increasing function for $t\in\left[a,\frac{3a+x}{4}\right)$ and it is a decreasing function for $t\in\left(\frac{3a+x}{4},\frac{a+x}{2}\right]$. The function $Q_n(t)$ is decreasing for $t\in\left[\frac{a+x}{2},\frac{a+3x}{4}\right)$ and it is increasing for $t\in\left(\frac{3a+3x}{4},x\right]$.

If n is even then $P_n(t)$ is a decreasing function and $Q_n(t)$ is an increasing function. Furthermore, $P_n(t)>0$ for $t\in\left[a,\frac{3a+x}{4}\right)$ and $P_n(t)<0$ for $t\in\left(\frac{3a+x}{4},\frac{a+x}{2}\right]$, while $Q_n(t)<0$ for $t\in\left[\frac{a+x}{2},\frac{a+3x}{4}\right)$ and $Q_n(t)>0$ for $t\in\left(\frac{3a+3x}{4},x\right]$.

We use these properties to prove (8) and (9).

If n is odd then we have

$$\begin{aligned}\int_a^x |S_n(t)|\,dt &= \int_a^{\frac{a+x}{2}} |P_n(t)|\,dt+\int_{\frac{a+x}{2}}^x |Q_n(t)|\,dt\\ &= \left|\int_a^{\frac{a+x}{2}} P_n(t)dt\right|+\left|\int_{\frac{a+x}{2}}^x Q_n(t)dt\right|\\ &= \frac{(4n+4)(x-a)^{n+1}}{4^{n+1}(n+1)!}.\end{aligned}$$

If n is even then we find that the same result is valid. Thus (8) holds.

Finally, we have

$$\begin{aligned}\max_{t\in[a,x]}|S_n(t)| &= \max\left\{\max_{t\in\left[a,\frac{a+x}{2}\right]}|P_n(t)|,\ \max_{t\in\left[\frac{a+x}{2},x\right]}|Q_n(t)|\right\}\\ &= \max\left\{\left|P_n(\frac{a+x}{2})\right|,\left|Q_n(\frac{a+x}{2})\right|,|P_n(a)|,|Q_n(x)|\right\}\\ &= \frac{(n+1)(x-a)^n}{4^n n!}.\end{aligned}$$

■

We introduce the notation

$$\begin{aligned} F(x,a) &= f(a) - \sum_{k=1}^{n} \frac{(-1)^k (x-a)^k}{4^k k!} (1+k) \left[f^{(k)}(x) - (-1)^k f^{(k)}(a) \right] \\ &\quad - \sum_{k=2}^{n} \frac{(-1)^k (x-a)^k}{4^k k!} (1-k) \left[1 - (-1)^k \right] f^{(k)}\left(\frac{a+x}{2}\right). \end{aligned}$$

Theorem 3 *Let $f : [a,x] \to R$ be a function such that $f^{(n)}$ is absolutely continuous and there exist real numbers γ_n, Γ_n such that $\gamma_n \le f^{(n+1)}(t) \le \Gamma_n$, $t \in [a,x]$. Then*

$$|f(x) - F(x,a)| \le \frac{\Gamma_n - \gamma_n}{(n+1)!} \frac{(2n+2)(x-a)^{n+1}}{4^{n+1}} \quad \text{if } n \text{ is odd} \tag{10}$$

and

$$|f(x) - F(x,a)| \le \frac{(4n+4)(x-a)^{n+1}}{4^{n+1}(n+1)!} \left\| f^{(n+1)} \right\|_\infty \text{ if } n \text{ is even.} \tag{11}$$

Proof. Let n be odd. From (3) and (7) we get

$$R(f) = (-1)^n \int_a^x S_n(t) f^{(n+1)}(t) dt = (-1)^n \int_a^x S_n(t) \left[f^{(n+1)}(t) - \frac{\gamma_n + \Gamma_n}{2} \right] dt$$

such that we have

$$|R(f)| = |f(x) - F(x,a)| \le \max_{t \in [a,x]} \left| f^{(n+1)}(t) - \frac{\gamma_n + \Gamma_n}{2} \right| \int_a^x |S_n(t)|\, dt. \tag{12}$$

We also have

$$\max_{t \in [a,x]} \left| f^{(n+1)}(t) - \frac{\gamma_n + \Gamma_n}{2} \right| \le \frac{\Gamma_n - \gamma_n}{2}. \tag{13}$$

From (12), (13) and (8) we get

$$|f(x) - F(x,a)| \le \frac{\Gamma_n - \gamma_n}{(n+1)!} \frac{(2n+2)(x-a)^{n+1}}{4^{n+1}}.$$

Let n be even. Then we have

$$|R(f)| = |f(x) - F(x,a)| \le \int_a^x |S_n(t)|\, dt \left\| f^{(n+1)} \right\|_\infty = \frac{(4n+4)(x-a)^{n+1}}{4^{n+1}(n+1)!} \left\| f^{(n+1)} \right\|_\infty.$$

■

Theorem 4 *Let $f : [a,x] \to R$ be a function such that $f^{(n)}$ is absolutely continuous and let n be odd. If there exists a real number γ_n such that $\gamma_n \le f^{(n+1)}(t)$, $t \in [a,x]$ then*

$$|f(x) - F(x,a)| \le (T_n - \gamma_n) \frac{(n+1)(x-a)^{n+1}}{4^n n!}, \tag{14}$$

where

$$T_n = \frac{f^{(n)}(x) - f^{(n)}(a)}{x-a}.$$

If there exists a real number Γ_n *such that* $f^{(n+1)}(t) \le \Gamma_n$, $t \in [a,x]$ *then*

$$|f(x) - F(x,a)| \le (\Gamma_n - T_n)\,\frac{(n+1)(x-a)^{n+1}}{4^n n!}. \tag{15}$$

Proof. We have

$$|R(f)| = |f(x) - F(x,a)| = \left|\int_a^x (f^{(n+1)}(t) - \gamma_n) S_n(t)dt\right|,$$

since (7) holds. Then we have

$$\begin{aligned}
\left|\int_a^x (f^{(n+1)}(t) - \gamma_n) S_n(t)dt\right| &\le \max_{t\in[a,x]} |S_n(t)| \int_a^x (f^{(n+1)}(t) - \gamma_n)dt \\
&= \frac{(n+1)(x-a)^n}{4^n n!}\left[f^{(n)}(x) - f^{(n)}(a) - \gamma_n(x-a)\right] \\
&= \frac{(n+1)(x-a)^{n+1}}{4^n n!}\left(T_n - \gamma_n\right).
\end{aligned}$$

In a similar way we can prove that (15) holds. ∎

Remark 5 *Note that we can apply the estimations (10) and (11) only if* $f^{(n+1)}$ *is bounded. On the other hand, we can apply the estimation (14) if* $f^{(n+1)}$ *is unbounded above and we can apply the estimation (15) if* $f^{(n+1)}$ *is unbounded below.*

3 An application to logarithmic function

We now apply the formula (2) to logarithmic function. We have

$$f^{(j)}(t) = \frac{(-1)^j (j-1)!}{(1+t)^j} \text{ if } f(t) = \ln(1+t). \tag{16}$$

From (2), (16) and $a = 0$, $f(t) = \ln(1+t)$ we get

$$F(x) = -\sum_{k=1}^{n} \frac{(-1)^k x^k}{4^k k}\left[(1+k)\left(\frac{(-1)^{k+1}}{(1+x)^k} + 1\right) + \frac{(-1)^{k+1}(1-k)(1-(-1)^k)}{(1+\frac{x}{2})^k}\right] \tag{17}$$

$$\approx \ln(1+x), \;\; x \in \left(-\frac{4}{5}, 4\right).$$

The standard formula for this function is given by

$$S(x) = \sum_{k=1}^{m} \frac{(-1)^{k+1} x^k}{k} \approx \ln(1+x), \quad x \in (-1,1). \tag{18}$$

Many numerical examples show that n can be much less than m if we wish to obtain a prior given accuracy and if x is close to 1 ($x < 1$).

Let us choose $x = 0.99$ and give the accuracy of order $E - 14$. The "exact" value is $\ln(1 + 0.99) = 0.688134643528734$. If we use (17) with $n = 22$ then we get $F(0.99) \approx 0.688134643528725$. If we use (18) with $m = 5000$ then we get $S(0.99) \approx 0.688134643528737$. All calculations are done in double precision arithmetic. The first approximate result is obtained faster than the second one. Similar results are obtained when we chose $x = 0.9$, $x = 0.95$, etc.

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