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The Hilbert Transform on a Smooth Closed Hypersurface

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ABSTRACT

In this paper a condensed account is given of results connected to the Hilbert transform on the smooth boundary of a bounded domain in Euclidean space and some of its related concepts, such as Hardy spaces and the Cauchy integral, in a Clifford analysis context. Clifford analysis, also known as the theory of monogenic functions, is a multidimensional function theory, which is at the same time a generalization of the theory of holomorphic functions in the complex plane and a refinement of classical harmonic analysis. It offers a framework which is particularly suited for the integrated treatment of higher dimensional phenomena, without having to rely on tensorial approaches.

RESUMEN

En este artículo damos un relato condensado de los resultados conectados con la transformada de Hilbert sobre dominios acotados con frontera suave en espacios euclideanos y también damos conceptos relacionados, tales como espacios de Hardy y la integral de Cauchy en el contexto del análisis de Clifford. El análisis de Clifford, también conocido como la teoria de funciones monogénicas, es una teoría de funciones multidimensionales, la cual es al mismo tiempo una generalización de la teoría de funciones holomorfas en el plano complejo y un refinamiento del análisis armónico clásico. El artículo ofrece un referencial que es particularmente conveniente para el tratamiento integrado de fenómenos en dimensiones altas, sin tener que recurrir a un abordaje tensorial.

Key words and phrases: *Hilbert transform, Hardy space, Cauchy integral.*

Math. Subj. Class.: *30G35, 44A15*

1 Introduction

In one–dimensional signal processing the Hilbert transform is an indispensable tool for both global and local descriptions of a signal, yielding information on various independent signal properties. The instantaneous amplitude, phase and frequency are estimated by means of so–called quadrature filters, which allow for distinguishing different frequency components and in this way locally refine the structure analysis. These filters are essentially based on the notion of *analytic signal*, which consists of the linear combination of a bandpass filter, selecting a small part of the spectral information, and its Hilbert transform, the latter basically being the result of a phase shift by $\frac{\pi}{2}$ on the original filter (see e.g. [18]). Mathematically, if $f(x) \in L_2(\mathbb{R})$ is a real valued signal of finite energy, and $\mathcal{H}[f]$ denotes its Hilbert transform given by the Cauchy Principal Value

$$\mathcal{H}[f](x) = \frac{1}{\pi} \text{ Pv} \int_{-\infty}^{+\infty} \frac{f(y)}{x-y} \, dy$$

then the corresponding analytic signal is the function $\frac{1}{2}f + \frac{i}{2}\mathcal{H}[f]$, which belongs to the Hardy space $H^2(\mathbb{R})$ and arises as the L_2 non–tangential boundary value (NTBV) for $y \to 0+$ of the holomorphic Cauchy integral of f in the upper half of the complex plane. Though initiated by Hilbert, the concept of a *conjugated pair* $(f, \mathcal{H}[f])$, nowadays called a Hilbert pair, was developed mainly by Titchmarch and Hardy.

The multidimensional approach to the Hilbert transform usually is a tensorial one, considering the so–called Riesz transforms in each of the Cartesian variables separately. As opposed to these tensorial approaches Clifford analysis is particularly suited for a treatment of multidimensional phenomena encompasssing all dimensions at the same time as an intrinsic feature. During the last fifty years Clifford analysis has gradually developed to a comprehensive theory offering a direct, elegant and powerful generalization to higher dimension of the theory of holomorphic functions in the complex plane. In its most simple but still useful setting, flat m–dimensional Euclidean space, Clifford analysis focusses on so–called *monogenic functions*, i.e. null solutions of the Clifford–vector valued Dirac operator $\underline{\partial} = \sum_{j=1}^{m} e_j \partial_{x_j}$ where $(e_1, \ldots, e_m)$ forms an orthogonal basis for the quadratic space $\mathbb{R}^m$ underlying the construction of the Clifford algebra $\mathbb{R}_{0,m}$. Monogenic functions have a special relationship with harmonic functions of several variables in that they are refining their properties. The reason is that, as does the Cauchy–Riemann operator in the complex plane, the rotation–invariant Dirac operator factorizes the m–dimensional Laplace operator. This has, a.o., allowed for a nice study of Hardy spaces of monogenic functions, see [7, 24, 8, 9, 1, 11]. In this context the Hilbert transform, as well as more general singular integral operators have been studied in higher dimensional Euclidean space (see [14, 24, 31, 19, 10, 12]), in particular on Lipschitz hypersurfaces (see [25, 21, 20, 22]) and also on smooth closed hypersurfaces, in particular the unit sphere (see [11, 3, 6]).

The subject of this paper is the Hilbert transform, within the Clifford analysis context, on the smooth boundary of a bounded domain in Euclidean space of dimension at least three. For the two–dimensional case we refer to the inspiring book [2]. We have gathered the relevant results spread over the literature and have moulded them together with some new results and insights into a comprehensive text.

2 Clifford analysis: the basics

In this section we briefly present the basic definitions and some results of Clifford analysis which are necessary for our purpose. For an in–depth study of this higher dimensional function theory and its applications we refer to e.g. [4, 13, 14, 15, 16, 17, 26, 27, 28, 29, 30].

Let $\mathbb{R}^{0,m}$ be the real vector space $\mathbb{R}^m$, endowed with a non–degenerate quadratic form of signature $(0, m)$, let $(e_1, \ldots, e_m)$ be an orthonormal basis for $\mathbb{R}^{0,m}$, and let $\mathbb{R}_{0,m}$ be the universal Clifford algebra constructed over $\mathbb{R}^{0,m}$.

The non–commutative multiplication in $\mathbb{R}_{0,m}$ is governed by the rules

$$e_i e_j + e_j e_i = -2\delta_{i,j}, \qquad i, j \in \{1, \ldots, m\}.$$

For a set $A = \{i_1, \ldots, i_h\} \subset \{1, \ldots, m\}$ with $1 \leq i_1 < i_2 < \ldots < i_h \leq m$, let $e_A = e_{i_1} e_{i_2} \ldots e_{i_h}$. Moreover, we put $e_\emptyset = 1$, the latter being the identity element. Then $(e_A : A \subset \{1, \ldots, m\})$ is a basis for the Clifford algebra $\mathbb{R}_{0,m}$. Any $a \in \mathbb{R}_{0,m}$ may thus be written as $a = \sum_A a_A\, e_A$ with $a_A \in \mathbb{R}$ or still as $a = \sum_{k=0}^m [a]_k$ where $[a]_k = \sum_{|A|=k} a_A\, e_A$ is the so–called k–vector part of a $(k = 0, 1, \ldots, m)$. If we denote the space of k–vectors by $\mathbb{R}^k_{0,m}$, then the Clifford algebra $\mathbb{R}_{0,m}$ decomposes as $\bigoplus_{k=0}^m \mathbb{R}^k_{0,m}$. We will identify an element $\underline{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$ with the one–vector (or vector for short) $\underline{x} = \sum_{j=1}^m x_j\, e_j$. The multiplication of any two vectors $\underline{x}$ and $\underline{y}$ is given by

$$\underline{x}\,\underline{y} = \underline{x} \circ \underline{y} + \underline{x} \wedge \underline{y}$$

with

$$\begin{aligned} \underline{x} \circ \underline{y} &= -\sum_{j=1}^{m} x_j y_j = \frac{1}{2}(\underline{x}\,\underline{y} + \underline{y}\underline{x}) = -\langle \underline{x}, \underline{y} \rangle \\ \underline{x} \wedge \underline{y} &= \sum_{i<j} e_{ij}(x_i y_j - x_j y_i) = \frac{1}{2}(\underline{x}\,\underline{y} - \underline{y}\underline{x}) \end{aligned}$$

being a scalar and a 2–vector (also called bivector), respectively. In particular one has that $\underline{x}^2 = -\langle \underline{x}, \underline{x} \rangle = -|\underline{x}|^2 = -\sum_{j=1}^m x_j^2$. Conjugation in $\mathbb{R}_{0,m}$ is defined as the anti–involution for which $\overline{e}_j = -e_j$, $j = 1, \ldots, m$. In particular for a vector $\underline{x}$ we have $\overline{\underline{x}} = -\underline{x}$.

The Dirac operator in $\mathbb{R}^m$ is the first order vector valued differential operator

$$\underline{\partial} = \sum_{j=1}^{m} e_j \partial_{x_j}$$

its fundamental solution being given by

$$E(\underline{x}) = \frac{1}{a_m} \, \frac{\overline{\underline{x}}}{|\underline{x}|^m}.$$

We consider functions f defined in $\mathbb{R}^m$ and taking values in $\mathbb{R}_{0,m}$. Such a function may be written as $f(\underline{x}) = \sum_A \; f_A(\underline{x}) \; e_A$ and each time we assign a property such as continuity, differentiability, etc. to f it is meant that all components f_A share this property. We say that the function f is left monogenic in the open region Ω of $\mathbb{R}^m$ iff f is continuously differentiable in Ω and satisfies in Ω the equation $\underline{\partial}\, f = 0$. As $\overline{\underline{\partial}\, f} = \overline{f}\;\overline{\underline{\partial}} = -\overline{f}\underline{\partial}$, a function f is left monogenic in Ω if and only if $\overline{f}$ is right monogenic in Ω. As moreover the Dirac operator factorizes the Laplace operator Δ, $-\underline{\partial}^2 = \underline{\partial}\,\overline{\underline{\partial}} = \overline{\underline{\partial}}\,\underline{\partial} = \Delta$, a monogenic function in Ω is harmonic (and hence C_∞) in Ω, and so are its components.

3 The Hilbert transform

Let Ω be a bounded domain in $\mathbb{R}^m$ with a C_∞–smooth boundary $\partial\Omega$ splitting $\mathbb{R}^m$ into the interior of the domain Ω^+ and its exterior Ω^-. In what follows the Clifford algebra valued $L_2(\partial\Omega)$ inner product

$$\langle f, g \rangle = \int_{\partial\Omega} \overline{f(\underline{\zeta})}\, g(\underline{\zeta})\, dS(\underline{\zeta})$$

will be used, as well as its associated norm $||f|| = \sqrt{[\langle f, f \rangle]_0}$, where the notation $[\cdot]_0$ stands for taking the scalar part of the expression.

Let u be a C_∞–smooth function on $\partial\Omega$, then its *Cauchy integral* in $\mathbb{R}^m \setminus \partial\Omega = \Omega^+ \cup \Omega^-$ is defined by

$$\mathcal{C}[u](\underline{x}) = \int_{\partial\Omega} E(\underline{\zeta} - \underline{x}) \; d\sigma_{\underline{\zeta}} \; u(\underline{\zeta}) = \frac{1}{a_m} \int_{\partial\Omega} \frac{\underline{x} - \underline{\zeta}}{|\underline{x} - \underline{\zeta}|^m} \; \nu(\underline{\zeta}) \; u(\underline{\zeta}) \; dS(\underline{\zeta})$$

where the Clifford–vector valued oriented surface element $d\sigma_{\underline{\zeta}}$ has been rewritten as $\nu(\underline{\zeta})\,dS(\underline{\zeta})$, with $\nu(\underline{\zeta})$ denoting the outward pointing unit normal vector at $\underline{\zeta} \in \partial\Omega$. Defining the *Cauchy kernel* by

$$C(\underline{\zeta}, \underline{x}) = \frac{1}{a_m}\, \overline{\nu}(\underline{\zeta})\, \frac{\overline{\underline{x}} - \overline{\underline{\zeta}}}{|\underline{x} - \underline{\zeta}|^m}, \quad \underline{\zeta} \in \partial\Omega, \quad \underline{x} \in \Omega^+ \cup \Omega^-$$

the Cauchy integral may be rewritten in terms of the $L_2(\partial\Omega)$ inner product as

$$\mathcal{C}[u](\underline{x}) = \ \langle C(\underline{\zeta}, \underline{x}), u(\underline{\zeta})\rangle \ = \int_{\partial\Omega} \overline{C(\underline{\zeta}, \underline{x})}\ u(\underline{\zeta})\ dS(\underline{\zeta}).$$

The fundamental properties of the Cauchy kernel and the Cauchy integral are:

(i) the Cauchy kernel $C(\underline{\zeta}, \underline{x})$ is right–monogenic in $\underline{x} \in \Omega^+ \cup \Omega^-$;

(ii) the Cauchy integral $\mathcal{C}[u](\underline{x})$ is left–monogenic in Ω^+ and in Ω^-;

(iii) $\lim_{\underline{x} \to \infty} \mathcal{C}[u](\underline{x}) = 0$.

The operator $\mathcal{C}$ is sometimes called the Cauchy–Bitsadze operator. A simple yet important example is furnished by the constant function, say $u = 1$, on $\partial\Omega$, for which

$$\mathcal{C}[1](\underline{x}) = \frac{1}{a_m} \int_{\partial\Omega} \frac{\underline{x} - \underline{\zeta}}{|\underline{x} - \underline{\zeta}|^m}\ d\sigma_{\underline{\zeta}} = \begin{cases} 1, & \underline{x} \in \Omega^+, \\ 0, & \underline{x} \in \Omega^-. \end{cases}$$

Now we investigate the non–tangential boundary behaviour of the Cauchy integral $\mathcal{C}[u](\underline{x}), u \in C_\infty(\partial\Omega)$, for $\underline{x} \to \underline{\xi} \in \partial\Omega$. First assume that $\underline{x} \in \Omega^+$. As

$$\mathcal{C}[u](\underline{x}) - u(\underline{\xi}) = \mathcal{C}[u(\underline{\zeta}) - u(\underline{\xi})](\underline{x}) = \int_{\partial\Omega} E(\underline{\zeta} - \underline{x})\ d\sigma_{\underline{\zeta}}\ (u(\underline{\zeta}) - u(\underline{\xi}))$$

we have

$$\lim_{\underline{x} \to \underline{\xi}}\ \mathcal{C}[u](\underline{x}) = u(\underline{\xi}) + \lim_{\underline{x} \to \underline{\xi}} \int_{\partial\Omega} E(\underline{\zeta} - \underline{x})\ d\sigma_{\underline{\zeta}}\ (u(\underline{\zeta}) - u(\underline{\xi}))$$

– where the last integral is no longer singular – or still

$$\lim_{\underline{x} \to \underline{\xi}}\ \mathcal{C}[u](\underline{x}) = u(\underline{\xi}) + \lim_{\varepsilon \to 0+} \int_{\partial\Omega_\varepsilon} E(\underline{\zeta} - \underline{x})\ d\sigma_{\underline{\zeta}}\ u(\underline{\zeta}) - \lim_{\varepsilon \to 0+} \int_{\partial\Omega_\varepsilon} E(\underline{\zeta} - \underline{x})\ d\sigma_{\underline{\zeta}}\ u(\underline{\xi})$$

where we have introduced $\partial\Omega_\varepsilon = \left\{\underline{\zeta} \in \partial\Omega : d(\underline{\zeta}, \underline{\xi}) > \varepsilon\right\}$. By a classical argument involving Cauchy's Theorem it is found that

$$\lim_{\varepsilon \to 0+} \int_{\partial\Omega_\varepsilon} E(\underline{\zeta} - \underline{\xi})\ d\sigma_{\underline{\zeta}} = \frac{1}{2}$$

leading to

$$\lim_{\underline{x} \to \underline{\xi}}\ \mathcal{C}[u](\underline{x}) = \frac{1}{2} u(\underline{\xi}) + \lim_{\varepsilon \to 0+} \int_{\partial\Omega_\varepsilon} E(\underline{\zeta} - \underline{\xi})\ d\sigma_{\underline{\zeta}}\ u(\underline{\zeta})$$

or finally

$$\lim_{\Omega^+ \ni \underline{x} \to \underline{\xi}} \mathcal{C}[u](\underline{x}) = \frac{1}{2} u(\underline{\xi}) + \frac{1}{2} H[u](\underline{\xi}), \quad \underline{\xi} \in \partial\Omega \tag{3.1}$$

where we have put for $\underline{\xi} \in \partial\Omega$:

$$\begin{aligned} H[u](\underline{\xi}) &= 2 \lim_{\varepsilon \to 0+} \int_{\partial\Omega_\varepsilon} E(\underline{\zeta} - \underline{\xi})\, d\sigma_{\underline{\zeta}}\, u(\underline{\zeta}) \\ &= \frac{2}{a_m} \mathrm{Pv} \int_{\partial\Omega} \frac{\overline{\underline{\zeta}} - \overline{\underline{\xi}}}{|\underline{\zeta} - \underline{\xi}|^m}\, d\sigma_{\underline{\zeta}}\, u(\underline{\zeta}) \;=\; \frac{2}{a_m} \mathrm{Pv} \int_{\partial\Omega} \frac{\underline{\xi} - \underline{\zeta}}{|\underline{\xi} - \underline{\zeta}|^m}\, \nu(\underline{\zeta})\, u(\underline{\zeta})\, dS(\underline{\zeta}) \end{aligned}$$

This integral transform H is mostly called the Hilbert transform; it is sometimes denoted by $S_{\partial\Omega}$, a notation we will not use any further in this paper. Note that, in view of the example given above

$$H[1](\underline{\xi}) = \frac{2}{a_m} Pv \int_{\partial\Omega} \frac{\underline{\xi} - \underline{\zeta}}{|\underline{\xi} - \underline{\zeta}|^m} d\sigma_{\underline{\zeta}} = 1, \quad \underline{\xi} \in \partial\Omega.$$

Similarly we find for the exterior NTBV of the Cauchy integral:

$$\lim_{\Omega^- \ni \underline{x} \to \underline{\xi}} \mathcal{C}[u](\underline{x}) = -\frac{1}{2} u(\underline{\xi}) + \frac{1}{2} H[u](\underline{\xi}), \quad \underline{\xi} \in \partial\Omega. \tag{3.2}$$

The obtained results (3.1)–(3.2) are the so–called Plemelj–Sokhotzki formulae, leading to the Cauchy transforms defined on $\mathcal{C}_\infty(\partial\Omega)$ by

$$\mathcal{C}^+[u] = \frac{1}{2} u + \frac{1}{2} H[u], \qquad \mathcal{C}^-[u] = -\frac{1}{2} u + \frac{1}{2} H[u].$$

It follows that

$$u = \mathcal{C}^+[u] - \mathcal{C}^-[u], \qquad H[u] = \mathcal{C}^+[u] + \mathcal{C}^-[u],$$

expressing the function $u \in C_\infty(\partial\Omega)$ as the *jump* of its Cauchy integral over the boundary $\partial\Omega$. In section 5 the operators H and $\mathcal{C}^\pm$ will be extended to operators on $L_2(\partial\Omega)$.

4 The double–layer potential

There is a nice connection between the Cauchy integral and the related operators H and $\mathcal{C}^\pm$ on the one side and the double–layer potential on $\partial\Omega$ on the other. Indeed, the splitting of the product of two Clifford vectors into the scalar valued dot product and the bivector valued wedge product allows for rewriting the Cauchy integral of the function $u \in \mathcal{C}_\infty(\partial\Omega)$ as

$$\mathcal{C}[u](\underline{x}) = \frac{1}{a_m} \int_{\partial\Omega} \frac{(\overline{\underline{\zeta}} - \overline{\underline{x}}) \circ \nu(\underline{\zeta})}{|\underline{\zeta} - \underline{x}|^m} u(\underline{\zeta}) dS(\underline{\zeta}) + \frac{1}{a_m} \int_{\partial\Omega} \frac{(\overline{\underline{\zeta}} - \overline{\underline{x}}) \wedge \nu(\underline{\zeta})}{|\underline{\zeta} - \underline{x}|^m} u(\underline{\zeta}) dS(\underline{\zeta}) \tag{4.1}$$

Denoting by $\overrightarrow{x}$ the geometric vector associated with the Clifford vector $\underline{x}$, we have that $(\overline{\underline{\zeta}} - \overline{\underline{x}}) \circ \nu(\underline{\zeta}) = \langle \overrightarrow{\zeta} - \overrightarrow{x}, \overrightarrow{\nu}(\underline{\zeta}) \rangle$, where $\langle ., . \rangle$ here denotes the standard Euclidean scalar product. As

$$\overrightarrow{\nabla}_{\overrightarrow{\zeta}} \frac{1}{|\overrightarrow{\zeta} - \overrightarrow{x}|^{m-2}} = -\,(m-2)\, \frac{\overrightarrow{\zeta} - \overrightarrow{x}}{|\overrightarrow{\zeta} - \overrightarrow{x}|^m}$$

we have

$$\frac{(\overline{\underline{\zeta}} - \overline{\underline{x}}) \ \circ \ \nu(\underline{\zeta})}{|\underline{\zeta} - \underline{x}|^m} = -\frac{1}{m-2}\langle \overrightarrow{\nabla}_{\underline{\zeta}} \frac{1}{|\overrightarrow{\zeta} - \overrightarrow{x}|^{m-2}}, \overrightarrow{\nu}(\zeta)\rangle = -\frac{1}{m-2}\frac{\partial}{\partial \overrightarrow{\nu}}\left(\frac{1}{|\overrightarrow{\zeta} - \overrightarrow{x}|^{m-2}}\right)$$

This first term in the Cauchy integral (4.1) then takes the form

$$-\frac{1}{m-2}\frac{1}{a_m}\int_{\partial\Omega}\frac{\partial}{\partial \overrightarrow{\nu}}\left(\frac{1}{|\overrightarrow{\zeta} - \overrightarrow{x}|^{m-2}}\right) \ u(\underline{\zeta}) \ dS(\underline{\zeta})$$

in which one recognizes, up to constants, the double–layer potential with density $u(\underline{\zeta})$ on $\partial\Omega$. Note that for the constant density $u = 1$ on $\partial\Omega$, one has

$$\int_{\partial\Omega}\frac{(\underline{\zeta} - \underline{x}) \wedge \nu(\underline{\zeta})}{|\underline{\zeta} - \underline{x}|^m} dS(\underline{\zeta}) = 0, \quad \underline{x} \in \Omega^+ \cup \Omega^- \tag{4.2}$$

whence

$$\int_{\partial\Omega}\frac{\partial}{\partial \overrightarrow{\nu}}\left(\frac{1}{|\underline{\zeta} - \underline{x}|^{m-2}}\right) dS(\underline{\zeta}) \quad = \quad -(m-2)a_m, \quad \underline{x} \in \Omega^+ \tag{4.3}$$

$$\int_{\partial\Omega}\frac{\partial}{\partial \overrightarrow{\nu}}\left(\frac{1}{|\underline{\zeta} - \underline{x}|^{m-2}}\right) dS(\underline{\zeta}) \quad = \quad 0, \qquad \underline{x} \in \Omega^- \tag{4.4}$$

confirming known results about Gauß's Integral (see e.g. [23, p.360]).

It is well–known from classical potential theory that the double–layer potential is harmonic in $\Omega^+ \cup \Omega^-$. Under the assumptions made on the region Ω and the function u, the double–layer potential is even defined for $\underline{\xi} \in \partial\Omega$; the value at $\underline{\xi} \in \partial\Omega$ is called its *direct value* and denoted by $\widetilde{W}(\underline{\xi})$. This function $\widetilde{W}(\underline{\xi})$ is a continuous function on $\partial\Omega$ and moreover one has (see e.g. [23, p.360]):

$$\begin{aligned}
\lim_{\Omega^+ \ni \ \underline{x} \to \underline{\xi}} \int_{\partial\Omega}\frac{\partial}{\partial \overrightarrow{\nu}}\left(\frac{1}{|\underline{\zeta} - \underline{x}|^{m-2}}\right) u(\underline{\zeta})dS(\underline{\zeta}) \quad &= \quad -\frac{1}{2}(m-2)a_m u(\underline{\xi}) + \widetilde{W}(\underline{\xi}) \\
\lim_{\Omega^- \ni \ \underline{x} \to \underline{\xi}} \int_{\partial\Omega}\frac{\partial}{\partial \overrightarrow{\nu}}\left(\frac{1}{|\underline{\zeta} - \underline{x}|^{m-2}}\right) u(\underline{\zeta})dS(\underline{\zeta}) \quad &= \quad \frac{1}{2}(m-2)a_m u(\underline{\xi}) + \widetilde{W}(\underline{\xi})
\end{aligned}$$

with

$$\widetilde{W}(\underline{\xi}) = -(m-2)\int_{\partial\Omega}\frac{(\underline{\xi} - \underline{\zeta}) \circ \nu(\underline{\zeta})}{|\underline{\xi} - \underline{\zeta}|^m} \ u(\underline{\zeta}) \ dS(\underline{\zeta}), \quad \underline{\xi} \in \partial\Omega.$$

It follows that the Hilbert transform of a scalar valued function contains a scalar and a bivector part:

$$H[u](\xi) = -\frac{1}{m-2}\frac{2}{a_m}\widetilde{W}(\xi) + \frac{2}{a_m}\mathrm{Pv}\int_{\partial\Omega}\frac{(\underline{\xi} - \underline{\zeta}) \wedge \nu(\underline{\zeta})}{|\underline{\xi} - \underline{\zeta}|^m} u(\underline{\zeta})dS(\underline{\zeta}), \quad \underline{\xi} \in \partial\Omega,$$

and that the Principal Value has to be taken only of the bivector part. As we know from the above example that $H[1] = 1$, we obtain the following formulae completing (4.2)–(4.4):

$$\begin{aligned} \mathrm{Pv} \int_{\partial\Omega} \frac{(\underline{\zeta} - \underline{\xi}) \wedge \nu(\underline{\zeta})}{|\underline{\zeta} - \underline{\xi}|^m} \, dS(\underline{\zeta}) &= 0, & \underline{\xi} \in \partial\Omega, \quad (4.5) \\ \int_{\partial\Omega} \frac{\partial}{\partial \overrightarrow{\nu}} \left(\frac{1}{|\underline{\zeta} - \underline{\xi}|^{m-2}} \right) dS(\underline{\zeta}) &= -\frac{1}{2}(m-2) a_m, & \underline{\xi} \in \partial\Omega, \quad (4.6) \end{aligned}$$

again confirming well–known properties of Gauß's Integral.

5 The Hardy spaces $H_2^{\pm}(\partial\Omega)$

By $M_\infty(\Omega^+)$ we denote the space of left–monogenic functions in Ω^+ which are moreover $C_\infty(\overline{\Omega^+})$. Similarly $M_\infty(\Omega^-)$ denotes the space of left–monogenic functions in Ω^-, moreover being $\mathcal{C}_\infty(\overline{\Omega^-})$ ánd vanishing at infinity. The Cauchy integral operator $\mathcal{C}$ maps $C_\infty(\partial\Omega)$ into $M_\infty(\Omega^+)$ as well as into $M_\infty(\Omega^-)$, while the operators H and $\mathcal{C}^\pm$ map $\mathcal{C}_\infty(\partial\Omega)$ into itself. We call $M_\infty^\pm(\partial\Omega)$ the spaces of functions on $\partial\Omega$ which are the NTBVs of the functions in $M_\infty(\Omega^\pm)$ respectively, and we define the Hardy spaces $H_2^\pm(\partial\Omega)$ as the closure in $L_2(\partial\Omega)$ of $M_\infty^\pm(\partial\Omega)$. It should be emphasized that the usual notation for $H_2^+(\partial\Omega)$ is $H^2(\partial\Omega)$, and that $H_2^-(\partial\Omega)$ is mostly not considered. Our notation however reflects the symmetry in the properties of both Hardy spaces. The operators $\mathcal{C}, H$ and $\mathcal{C}^\pm$ may be extended, through a density argument, to operators defined on $L_2(\partial\Omega)$. Introducing the Hardy spaces $H_2(\Omega^\pm)$ of left–monogenic functions in $\Omega^\pm$, which do have NTBVs in $L_2(\partial\Omega)$, and, in the case of Ω^-, also vanish at infinity, we have the following properties of those operators.

Theorem 5.1.

(i) The Cauchy integral operator $\mathcal{C}$ maps $L_2(\partial\Omega)$ into $H_2(\Omega^\pm)$ and the NTBVs of $\mathcal{C}[f], f \in L_2(\partial\Omega)$, are given by $\mathcal{C}^\pm[f] = \pm\frac{1}{2}f + \frac{1}{2}H[f]$;
(ii) The Cauchy transforms $\mathcal{C}^\pm$ are bounded linear operators from $L_2(\partial\Omega)$ into $H_2^\pm(\partial\Omega)$;
(iii) The Hilbert transform H is a bounded linear operator from $L_2(\partial\Omega)$ onto $L_2(\partial\Omega)$;
(iv) $H^2 = \mathbf{1}$ or $H^{-1} = H$ on $L_2(\partial\Omega)$;
(v) The adjoint operator H^* of H is given by $H^* = \nu H \nu$ and $H^{*2} = \mathbf{1}$ on $L_2(\partial\Omega)$;
(vi) $H_2^\pm(\partial\Omega)$ are eigenspaces of H with respective eigenvalues ± 1.

It is important to note that a function $g \in L_2(\partial\Omega)$ belongs to the Hardy space $H_2^+(\partial\Omega)$ if and only if $\mathcal{C}^+[g] = g$, which is equivalent with $\mathcal{C}^-[g] = 0$ and still with $H[g] = g$. Thus a function $g \in H_2^+(\partial\Omega)$ may be identified with its left–monogenic extension $\mathcal{C}[g] \in H_2(\Omega^+)$, which is tacitly done most of the time. On the other hand, due to Cauchy's Theorem, $\mathcal{C}[g] = 0$ in Ω^- for each $g \in H_2^+(\partial\Omega)$. Similarly a function $\widetilde{g} \in L_2(\partial\Omega)$ belongs to $H_2^-(\partial\Omega)$ if and only if $\mathcal{C}^-[\widetilde{g}] = -\widetilde{g}$ or $\mathcal{C}^+[\widetilde{g}] = 0$ or still $H[\widetilde{g}] = -\widetilde{g}$. A function $\widetilde{g} \in H_2^-(\partial\Omega)$ may thus be identified with its left–monogenic extension $\mathcal{C}[-\widetilde{g}] \in H_2(\Omega^-)$, while here $\mathcal{C}[\widetilde{g}] = 0$ in Ω^+ for all $\widetilde{g} \in H_2^-(\partial\Omega)$.

Clearly the Cauchy transforms $\pm\mathcal{C}^{\pm}$ are (skew) projection operators on $L_2(\partial\Omega)$, sometimes called the Hardy projections, since

$$(\pm\mathcal{C}^{\pm})^2[f] = \frac{1}{4}(\mathbf{1} \pm H)^2[f] = \frac{1}{2}(\mathbf{1} \pm H)[f] = (\pm\mathcal{C}^{\pm})[f]$$

and

$$\mathcal{C}^{+}(-\mathcal{C}^{-})[f] = \frac{1}{4}(\mathbf{1} + H)(\mathbf{1} - H)[f] = 0 = (-\mathcal{C}^{-})\mathcal{C}^{+}[f]$$

By means of the Hardy projections a skew direct sum decomposition of $L_2(\partial\Omega)$ is obtained at once:

$$L_2(\partial\Omega) = H_2^{+}(\partial\Omega) \oplus H_2^{-}(\partial\Omega)$$

with

$$f = \mathcal{C}^{+}[f] + (-\mathcal{C}^{-})[f] = \frac{1}{2}(\mathbf{1} + H)[f] + \frac{1}{2}(\mathbf{1} - H)[f]$$

and

$$H[f] = \mathcal{C}^{+}[f] + \mathcal{C}^{-}[f] = \frac{1}{2}(\mathbf{1} + H)[f] - \frac{1}{2}(\mathbf{1} - H)[f]$$

Naturally we have

$$H_2^{+}(\partial\Omega) = \text{im } \mathcal{C}^{+} = \ker \mathcal{C}^{-}$$

and

$$H_2^{-}(\partial\Omega) = \text{im } \mathcal{C}^{-} = \ker \mathcal{C}^{+}$$

expressing the fact that $\mathcal{C}^{\pm}$ are projections *parallel* to $H_2^{\mp}(\partial\Omega)$ (see also Figure 1).

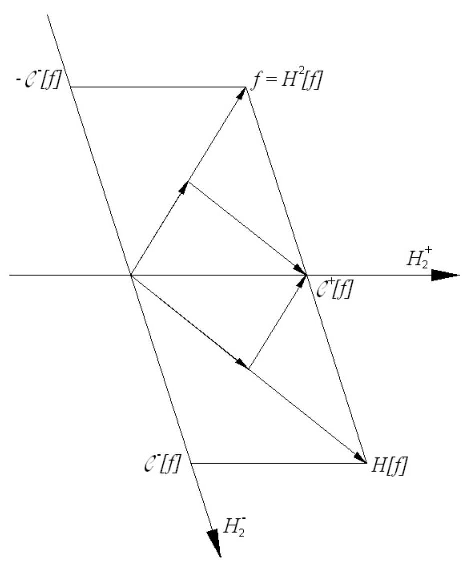

Figure 1

Although this decomposition of $L_2(\partial\Omega)$ is rather immediate, it is an important result. In fact it states that an $L_2(\partial\Omega)$–function f may be split into a sum of a function $\mathcal{C}^+[f] \in L_2(\partial\Omega)$ with left–monogenic extension to Ω^+ and a function $(-\mathcal{C}^-)[f] \in L_2(\partial\Omega)$ with left–monogenic extension to Ω^- vanishing at infinity. This result is the general counterpart of the classical result in Clifford analysis stating that each spherical harmonic may be split into the sum of an inner and an outer spherical monogenic (see e.g. [13]).

We conclude this section by giving two examples.

As already mentioned before $H[1] = 1$, which means that the constant function 1 belongs to $H_2^+(\partial\Omega)$ with $\mathcal{C}[1] = 1 \in H_2(\Omega^+)$, $\mathcal{C}[1] = 0$ in Ω^- and $\langle 1, 1\rangle = \text{area}(\partial\Omega)$.

The function $\frac{\underline{x}}{|\underline{x}|^m}$ is left–monogenic in $\mathbb{R}^m \setminus \{0\}$ and vanishes at infinity. Its restriction to $\partial\Omega$, given by $\frac{\underline{x}}{|\underline{x}|^m}|_{\partial\Omega} = \frac{\underline{\zeta}}{|\underline{\zeta}|^m}$, belongs to $H_2^-(\partial\Omega)$ with

$$\mathcal{C}[\frac{\underline{\zeta}}{|\underline{\zeta}|^m}] \;=\; \frac{1}{a_m}\int_{\partial\Omega} \frac{\underline{x}-\underline{\zeta}}{|\underline{x}-\underline{\zeta}|^m} d\sigma_{\underline{\zeta}} \frac{\underline{\zeta}}{|\underline{\zeta}|^m} \;=\; \begin{cases} \dfrac{\underline{x}}{|\underline{x}|^m} & , \quad \underline{x} \in \Omega^-, \\ 0 & , \quad \underline{x} \in \Omega^+. \end{cases}$$

and also

$$H[\frac{\underline{\zeta}}{|\underline{\zeta}|^m}] = -\frac{\underline{\zeta}}{|\underline{\zeta}|^m}, \; \underline{\zeta} \in \partial\Omega.$$

6 The orthogonal decomposition of $L_2(\partial\Omega)$

As the Hardy space $H_2^+(\partial\Omega)$ is a closed subspace of $L_2(\partial\Omega)$, it is itself a Hilbert space and it induces the following orthogonal direct sum decomposition of $L_2(\partial\Omega)$:

$$L_2(\partial\Omega) = H_2^+(\partial\Omega) \oplus H_2^+(\partial\Omega)^\perp.$$

The orthogonal projections $\mathbb{P}$ and $\mathbb{P}^\perp$ on $H_2^+(\partial\Omega)$ and $H_2^+(\partial\Omega)^\perp$ respectively are called the Szegö projections.

The Hilbert space $H_2^+(\partial\Omega)$ possesses a reproducing kernel $S(\underline{\zeta}, \underline{x})$, $\underline{\zeta} \in \partial\Omega$, $\underline{x} \in \Omega^+$, the so–called Szegö kernel, for which

$$\langle S(\underline{\zeta}, \underline{x}), g(\underline{\zeta})\rangle = \mathcal{C}[g](\underline{x}), \quad \underline{x} \in \Omega^+$$

for all $g \in H_2^+(\partial\Omega)$. Stricly speaking the reproducing character is only obtained by identifying the function $g \in H_2^+(\partial\Omega)$ with its left–monogenic extension $\mathcal{C}[g]$ to Ω^+. Note that the Szegö kernel

$S(\underline{\zeta}, \underline{x})$ is only defined for $\underline{x} \in \Omega^+$. It is the kernel function of the integral transform expressing the projection $\mathbb{P}$ of $L_2(\partial\Omega)$ on $H_2^+(\partial\Omega)$:

$$\langle S(\underline{\zeta}, \underline{x}), f(\underline{\zeta})\rangle = \mathbb{P}[f](\underline{x}), \quad f \in L_2(\partial\Omega), \quad \underline{x} \in \Omega^+.$$

There is an intimate relationship between the Cauchy and Szegö kernels as established in the following proposition.

Proposition 6.1. *The Szegö kernel is the Szegö projection of the Cauchy kernel on the Hardy space $H_2^+(\partial\Omega)$, i.e. for all $\underline{x} \in \Omega^+$ holds*

$$S(\underline{\zeta}, \underline{x}) = \mathbb{P}[C(\underline{\zeta}, \underline{x})] = \mathbb{P}[\frac{1}{a_m}\nu(\underline{\zeta})\frac{\underline{\zeta} - \underline{x}}{|\underline{\zeta} - \underline{x}|^m}], \quad \underline{\zeta} \in \partial\Omega.$$

Proof. Take $g \in H_2^+(\partial\Omega)$ and $\underline{x} \in \Omega^+$. Then

$$\langle S(\underline{\zeta}, \underline{x}), g(\underline{\zeta})\rangle = \mathcal{C}[g](\underline{x}) = \langle C(\underline{\zeta}, \underline{x}), g(\underline{\zeta})\rangle = \langle \mathbb{P}[C(\underline{\zeta}, \underline{x})], g(\underline{\zeta})\rangle.$$

■

Proposition 6.2. *The Szegö kernel is Hermitean symmetric, i.e. for all $\underline{x}, \underline{y} \in \Omega^+$ it holds that $S(\underline{x}, \underline{y}) = \overline{S(\underline{y}, \underline{x})}$.*

Proof. Take $\underline{x}, \underline{y} \in \Omega^+$. Then, with $\underline{\zeta} \in \partial\Omega$,

$$\begin{aligned}\langle S(\underline{\zeta}, \underline{x}), S(\underline{\zeta}, \underline{y})\rangle &= \langle S(\underline{\zeta}, \underline{x}), \mathbb{P}[C(\underline{\zeta}, \underline{y})]\rangle = \langle S(\underline{\zeta}, \underline{x}), C(\underline{\zeta}, \underline{x})\rangle \\ &= \overline{\langle C(\underline{\zeta}, \underline{x}), S(\underline{\zeta}, \underline{x})\rangle} = \overline{\mathcal{C}[S(\underline{\zeta}, \underline{x})](\underline{y})}\end{aligned}$$

The result follows in view of the identifications

$$\mathcal{C}[S(\underline{\zeta}, \underline{y})](\underline{x}) \approx S(\underline{x}, \underline{y}), \qquad \mathcal{C}[S(\underline{\zeta}, \underline{x})](\underline{y}) \approx S(\underline{y}, \underline{x}).$$

■

Proposition 6.3. *One has $S(\underline{x}, \underline{x}) > 0$ for all $\underline{x} \in \Omega^+$.*

Proof. First observe that it is impossible that $S(\underline{\zeta}, \underline{x}) = 0$ for a.e. $\underline{\zeta} \in \partial\Omega$, since for all $\underline{x} \in \Omega^+$:

$$\int_{\partial\Omega} \overline{S(\underline{\zeta}, \underline{x})}\, dS(\underline{\zeta}) = \langle S(\underline{\zeta}, \underline{x}), 1_{\underline{\zeta}}\rangle = \mathcal{C}[1](\underline{x}) = 1.$$

As a consequence of Proposition 6.2 one has for all $\underline{x} \in \Omega^+$:

$$S(\underline{x}, \underline{x}) = \overline{S(\underline{x}, \underline{x})} = \langle S(\underline{\zeta}, \underline{x}), S(\underline{\zeta}, \underline{x})\rangle = \int_{\partial\Omega} \overline{S(\underline{\zeta}, \underline{x})} S(\underline{\zeta}, \underline{x}) dS(\underline{\zeta})$$

or, as the Szegö kernel is parabivector valued (i.e. the sum of a scalar and a bivector):

$$S(\underline{x},\underline{x}) = \overline{S(\underline{x},\underline{x})} = \int_{\partial\Omega} |S(\underline{\zeta},\underline{x})|^2 dS(\underline{\zeta})$$

from which the result follows. ■

Using the Szegö kernel the Cauchy integral of a function $f \in L_2(\partial\Omega)$ may now be expressed as follows:

$$\mathcal{C}[f](\underline{x}) = \langle C(\underline{\zeta},\underline{x}), f(\underline{\zeta})\rangle = \langle \mathbb{P}[C(\underline{\zeta},\underline{x})], \mathbb{P}[f]\rangle + \langle \mathbb{P}^{\perp}[C(\underline{\zeta},\underline{x})], \mathbb{P}^{\perp}[f]\rangle. \tag{6.1}$$

For $\underline{x} \in \Omega^-$ this reduces to

$$\mathcal{C}[f](\underline{x}) = \langle \mathbb{P}^{\perp}[C(\underline{\zeta},\underline{x})], \mathbb{P}^{\perp}[f]\rangle$$

since $\mathcal{C}[\mathbb{P}[f]] = 0$ in Ω^-. In particular for a function $g \in H_2^+(\partial\Omega)$, and still with $\underline{x} \in \Omega^-$, we obtain

$$\mathcal{C}[g] = \langle C(\underline{\zeta},\underline{x}), g(\underline{\zeta})\rangle = 0$$

showing that for $\underline{x} \in \Omega^-$ the Cauchy kernel $\mathcal{C}(\underline{\zeta},\underline{x}) = \nu(\underline{\zeta})E(\underline{\zeta}-\underline{x})$ belongs to $H_2^+(\partial\Omega)^{\perp}$ and hence coincides with $\mathbb{P}^{\perp}[C(\underline{\zeta},\underline{x})]$, while $\mathbb{P}[C(\underline{\zeta},\underline{x})] = 0$. This is confirmed by the fact that for $\underline{x} \in \Omega^-$ the fundamental solution $E(\underline{\zeta}-\underline{x}) \in H_2^+(\partial\Omega)$, since it may be extended left–monogenically to Ω^+ by the function $E(\underline{y}-\underline{x})$. For $\underline{x} \in \Omega^+$ the expression (6.1) for the Cauchy integral reduces to

$$\mathcal{C}[f](\underline{x}) = \langle S(\underline{\zeta},\underline{x}), \mathbb{P}[f]\rangle + \langle \mathbb{P}^{\perp}[C(\underline{\zeta},\underline{x})], \mathbb{P}^{\perp}[f]\rangle$$

which in general cannot be simplified further.

From the previous section we know that for $f \in L_2(\partial\Omega)$ the Hardy projection $\mathbb{P}[f] \in H_2^+(\partial\Omega)$ possesses a left–monogenic extension $\mathcal{C}[\mathbb{P}[f]] \in H_2(\Omega^+)$ with $\mathcal{C}[\mathbb{P}[f]] = 0$ in Ω^-, and also that $\mathbb{P}[f] = H[\mathbb{P}[f]] = \mathcal{C}^+[\mathbb{P}[f]]$, while $\mathcal{C}^-[\mathbb{P}[f]] = 0$.

Now we search for similar properties of the other Hardy projection $\mathbb{P}^{\perp}[f] \in H_2^+(\partial\Omega)^{\perp}$. In any case its Cauchy integral $\mathcal{C}[\mathbb{P}^{\perp}[f]]$, though left–monogenic in Ω^+ and in Ω^-, is not an extension to Ω^- of $\mathbb{P}^{\perp}[f]$.

Proposition 6.4. *For a function $h \in L_2(\partial\Omega)$ to belong to $H_2^+(\partial\Omega)^{\perp}$ it is necessary and sufficient that $H^*[h] = -h$.*

Proof. If $h \in H_2^+(\partial\Omega)^{\perp}$ then $\langle g, h\rangle = \langle H[g], h\rangle = 0$ for all $g \in H_2^+(\partial\Omega)$ and conversely. This is equivalent with $\langle g, H^*[h]\rangle = 0$ for all $g \in H_2^+(\partial\Omega)$ and so $H^*[h] \in H_2^+(\partial\Omega)^{\perp}$. We also have that for all $f \in L_2(\partial\Omega)$:

$$\langle \mathcal{C}^+[f], h\rangle = \frac{1}{2}\ \langle f + H[f], h\rangle = 0$$

or

$$\langle f, h\rangle + \langle f, H^*[h]\rangle = 0 = \langle f, h + H^*[h]\rangle$$

which means that $h + H^*[h] = 0$. Conversely, if $H^*[h] = -h$, then for all $g \in H_2^+(\partial\Omega)$:

$$\langle g, h\rangle = \langle H[g], h\rangle = \langle g, H^*[h]\rangle = \langle g, -h\rangle$$

and hence $\langle g, 2h\rangle = 0$, which means that $h \in H_2^+(\partial\Omega)^\perp$. ■

Note that for a non–zero function $h \in H_2^+(\partial\Omega)^\perp$ there cannot exist a left–monogenic function in Ω^+ such that its NTBV is h. However there is a one–to–one correspondence between $H_2^+(\partial\Omega)$ and $H_2^+(\partial\Omega)^\perp$, which is easily expressed by means of the unit normal vector $\nu(\underline{\xi})$, $\underline{\xi} \in \partial\Omega$.

Proposition 6.5. *A function $g \in L_2(\partial\Omega)$ belongs to $H_2^+(\partial\Omega)$ if and only if $\nu g \in H_2^+(\partial\Omega)^\perp$, and vice–versa.*

Proof. If $g \in H_2^+(\partial\Omega)$ then $H[g] = g$ and so $H^*[\nu g] = \nu H \nu[\nu g] = -\nu H[g] = -\nu g$, from which it follows that $\nu g \in H_2^+(\partial\Omega)^\perp$, and conversely. If $h \in H_2^+(\partial\Omega)$ then $-\nu\nu h \in H_2^+(\partial\Omega)^\perp$ and so $\nu h \in H_2^+(\partial\Omega)$, and conversely. ■

Corollary 6.6. *The orthogonal direct sum decomposition of $L_2(\partial\Omega)$ takes the form*

$$L_2(\partial\Omega) = H_2^+(\partial\Omega) \oplus \nu\ H_2^+(\partial\Omega) = \nu\ H_2^+(\partial\Omega)^\perp \oplus H_2^+(\partial\Omega)^\perp.$$

7 The Kerzman–Stein operator

The Hilbert operator H on $L_2(\partial\Omega)$ is not self–adjoint. The so–called Kerzman–Stein operator, defined by

$$\mathcal{A} = \frac{1}{2}\,(H - H^*)$$

measures the "non–selfadjointness" of the Hilbert transform. We will find alternative expressions for this operator at the end of this section. To this end, we first introduce four self–adjoint bounded operators on $L_2(\partial\Omega)$, by means of the unit normal function ν on $\partial\Omega$.

Proposition 7.1. *The operators $H\nu, \nu H, \nu H^*$ and $H^*\nu$ are self–adjoint bounded operators on $L_2(\partial\Omega)$ moreover satisfying*

(i) $(\nu H)^2 = (H^*\nu)^2 = H^*H$;
(ii) $(\nu H^*)^2 = (H\nu)^2 = HH^*$;
(iii) $(\nu H)(H\nu) = -\mathbf{1} = (H\nu)(\nu H)$;
(iv) $(\nu H^*)(H^*\nu) = -\mathbf{1} = (H^*\nu)(\nu H^*)$;
(v) $\langle H\nu f, H\nu g\rangle = \langle H^* f, H^* g\rangle = \langle \nu H^* f, \nu H^* g\rangle,\ f, g \in L_2(\partial\Omega)$;
(vi) $\langle H^*\nu f, H^*\nu g\rangle = \langle Hf, Hg\rangle = \langle \nu Hf, \nu Hg\rangle,\ f, g \in L_2(\partial\Omega)$.

Note that the function ν belongs to $H_2^+(\partial\Omega)^\perp$ since the constant function $1 \in H_2^+(\partial\Omega)$. It thus follows that $H^*[\nu] = -\nu$. Moreover one has

(i) $||\nu||^2 = \langle \nu, \nu \rangle = \langle 1, 1 \rangle = \text{area}(\partial\Omega)$;

(ii) $\langle \nu, H\nu \rangle = \langle H^*\nu, \nu \rangle = -\langle \nu, \nu \rangle = -\text{area}(\partial\Omega)$;

(iii) $||H\nu||^2 = \langle H\nu, H\nu \rangle = \langle H^*1, H^*1 \rangle = ||H^*1||^2$.

As we have seen in the previous section the unit normal vector function ν allows for an alternative form of the orthogonal decomposition of $L_2(\partial\Omega)$. Take $f \in L_2(\partial\Omega)$ then also $\nu f \in L_2(\partial\Omega)$ and we have on the one side $f = \mathbb{P}[f] + \mathbb{P}^\perp[f]$ so that

$$\nu f = \nu \mathbb{P}^\perp[f] + \nu \mathbb{P}[f]$$

and on the other

$$\nu f = \mathbb{P}[\nu f] + \mathbb{P}^\perp[\nu f].$$

Hence $\mathbb{P}[\nu f] = \nu \mathbb{P}^\perp[f]$ and $\mathbb{P}^\perp[\nu f] = \nu \mathbb{P}[f]$ while also $\mathbb{P}[f] = -\nu \mathbb{P}^\perp[\nu f]$ and $\mathbb{P}^\perp[f] = -\nu \mathbb{P}[\nu f]$. This leads to

$$f = \mathbb{P}[f] - \nu \mathbb{P}[\nu f] = -\nu \mathbb{P}^\perp[\nu f] + \mathbb{P}^\perp[f]$$

$$\nu f = \mathbb{P}[\nu f] + \nu \mathbb{P}[f] = \nu \mathbb{P}^\perp[f] + \mathbb{P}^\perp[\nu f].$$

Taking the Hilbert transform into account we obtain

$$H[f] = H[\mathbb{P}[f]] + H[\mathbb{P}^\perp[f]] = \mathbb{P}[f] + H[\mathbb{P}^\perp[f]]$$

from which it follows that

$$\frac{1}{2}(\mathbf{1} - H)[f] = \frac{1}{2}(\mathbf{1} - H)[\mathbb{P}^\perp[f]].$$

By a similar argument we find

$$\frac{1}{2}(\mathbf{1} + H^*)[f] = \frac{1}{2}(\mathbf{1} + H^*)[\mathbb{P}[f]].$$

In the operator $\frac{1}{2}(\mathbf{1} - H)$ we clearly recognize the Cauchy transform $(-\mathcal{C}^-)$ for which indeed $\left(\frac{1}{2}(\mathbf{1} - H)\right)^2 = \frac{1}{2}(\mathbf{1} - H)$ and $\frac{1}{2}(\mathbf{1} - H)\frac{1}{2}(\mathbf{1} + H) = 0$, with $\frac{1}{2}(\mathbf{1} + H) = \mathcal{C}^+$. On grounds of analogy we put

$$\frac{1}{2}(\mathbf{1} + H^*) = \mathcal{D}^- \text{ and } \frac{1}{2}(-\mathbf{1} + H^*) = \mathcal{D}^+$$

defining in this way two bounded linear operators on $L_2(\partial\Omega)$ which moreover satisfy

(i) $(\mathcal{D}^-)^2 = \mathcal{D}^-$;

(ii) $(-\mathcal{D}^+)^2 = (-\mathcal{D}^+)$;

(iii) $\mathcal{D}^+\mathcal{D}^- = \mathcal{D}^-\mathcal{D}^+ = 0$.

In a similar way as the Hardy space $H_2^+(\partial\Omega)$ is characterized by $\mathcal{C}^\pm$, we may now characterize its orthogonal complement $H_2^+(\partial\Omega)^\perp$ by means of the operators $\mathcal{D}^\pm$:

$$h \in H_2^+(\partial\Omega)^\perp \Longleftrightarrow \mathcal{D}^-[h] = 0 \Longleftrightarrow \mathcal{D}^+[h] = -h \Longleftrightarrow H^*[h] = -h.$$

Notice that these newly introduced operators are the adjoints of the Cauchy transforms, i.e. $(\mathcal{C}^+)^* = \mathcal{D}^-$ and $(\mathcal{C}^-)^* = \mathcal{D}^+$, and for each function $f \in L_2(\partial\Omega)$

$$\langle \mathcal{C}^+[f], \mathcal{D}^+[f] \rangle = 0 \text{ and } \langle \mathcal{C}^-[f], \mathcal{D}^-[f] \rangle = 0$$

meaning that $\mathcal{D}^+[f]$ belongs to $H_2^+(\partial\Omega)^\perp$, while $\mathcal{D}^-[f]$ belongs to $H_2^-(\partial\Omega)^\perp$. To the authors' knowledge no integral transform, similar to the Cauchy integral, has $\mathcal{D}^\pm$ as its NTBV.

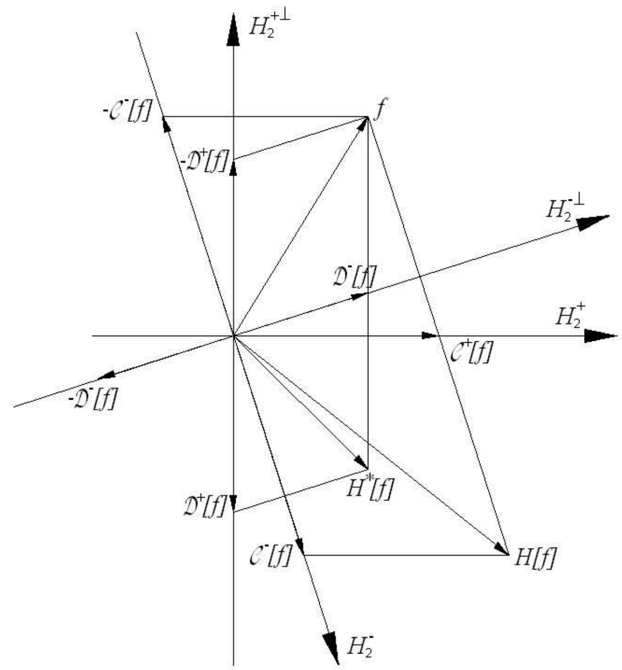

Figure 2

The four operators $\mathcal{C}^\pm$ and $\mathcal{D}^\pm$ are really fundamental; they are the building blocks of the operators $\mathbf{1}$, H and H^* (see Figure 2): $\mathbf{1} = \mathcal{C}^+ - \mathcal{C}^- = \mathcal{D}^- - \mathcal{D}^+$, while $H = \mathcal{C}^+ + \mathcal{C}^-$ and $H^* = \mathcal{D}^+ + \mathcal{D}^-$.

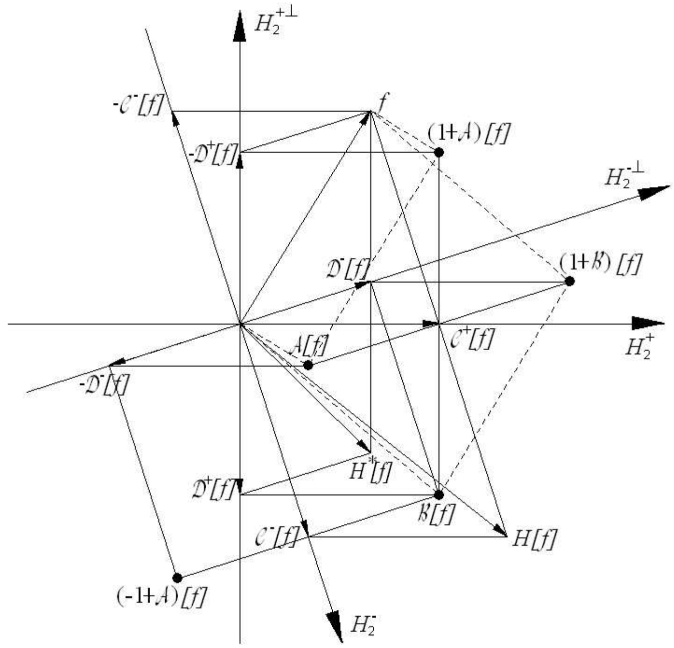

Figure 3

Moreover coming back now to the Kerzman–Stein operator, we observe that

$$\mathcal{A} = \frac{1}{2}(H - H^*) = \frac{1}{2}(\mathbf{1} + H) - \frac{1}{2}(\mathbf{1} + H^*) = \mathcal{C}^+ - (\mathcal{C}^+)^* = \mathcal{C}^+ - \mathcal{D}^-$$

as well as

$$\mathcal{A} = \frac{1}{2}(H - H^*) = \frac{1}{2}(-\mathbf{1} + H) - \frac{1}{2}(-\mathbf{1} + H^*) = \mathcal{C}^- - (\mathcal{C}^-)^* = \mathcal{C}^- - \mathcal{D}^+.$$

In a similar way we define the operator $\mathcal{B}$ by

$$\mathcal{C}^+ + \mathcal{D}^+ = \mathcal{C}^- + \mathcal{D}^- = \frac{1}{2}(H + H^*) = \mathcal{B}$$

which clearly is a self–adjoint bounded operator on $L_2(\partial\Omega)$ as well. Next

$$\begin{aligned} \mathbf{1} + \mathcal{A} &= \frac{1}{2}(\mathbf{1} + H) + \frac{1}{2}(\mathbf{1} - H^*) \;=\; \mathcal{C}^+ - \mathcal{D}^+, \\ -\mathbf{1} + \mathcal{A} &= -\frac{1}{2}(\mathbf{1} - H) - \frac{1}{2}(\mathbf{1} + H^*) \;=\; \mathcal{C}^- - \mathcal{D}^-, \end{aligned}$$

and

$$\begin{aligned} \mathbf{1} + \mathcal{B} &= \frac{1}{2}(\mathbf{1} + H) + \frac{1}{2}(\mathbf{1} + H^*) \;=\; \mathcal{C}^+ + \mathcal{D}^-, \\ -\mathbf{1} + \mathcal{B} &= -\frac{1}{2}(\mathbf{1} - H) - \frac{1}{2}(\mathbf{1} - H^*) \;=\; \mathcal{C}^- + \mathcal{D}^+. \end{aligned}$$

It follows that

$$\mathbb{P}(\mathbf{1}+\mathcal{A}) = \mathbb{P}\mathcal{C}^+ = \mathcal{C}^+ = \mathbb{P}\mathcal{B}, \quad \mathbb{P}^\perp(\mathbf{1}+\mathcal{A}) = -\mathbb{P}^\perp\mathcal{D}^+ = -\mathcal{D}^+ = -\mathbb{P}^\perp\mathcal{B}$$

since $\mathcal{C}^+[f] \in H_2^+(\partial\Omega)$ and $\mathcal{D}^+[f] \in H_2^+(\partial\Omega)^\perp$ for all $f \in L_2(\partial\Omega)$, which means that $(\mathbf{1}+\mathcal{A})[f]$ and $\mathcal{B}[f]$ lie "symmetric" w.r.t $H_2^+(\partial\Omega)$ (see Figure 3).

It should be noted that one of the above formulae relating the Hardy and Szegö projections to each other is the famous Kerzman–Stein Formula

$$\mathbb{P}(\mathbf{1}+\mathcal{A}) = \mathcal{C}^+$$

which in fact allows for proving the boundedness of the Cauchy operator $\mathcal{C}^+$ on $L_2(\Omega)$, since the Kerzman–Stein operator $\mathcal{A}$ may be expressed as an integral operator which is no longer singular:

$$\begin{aligned}
\mathcal{A}[f](\underline{\xi}) &= \frac{1}{2}(H - H^*)[f](\underline{\xi}) \\
&= \frac{1}{a_m}\ \mathrm{Pv} \int_{\partial\Omega} \frac{\underline{\xi}-\underline{\zeta}}{|\underline{\xi}-\underline{\zeta}|^m}\nu(\underline{\zeta})f(\underline{\zeta})dS(\underline{\zeta}) \\
&\qquad -\frac{1}{a_m}\ \mathrm{Pv} \int_{\partial\Omega} \nu(\underline{\xi})\frac{\underline{\xi}-\underline{\zeta}}{|\underline{\xi}-\underline{\zeta}|^m}\nu(\underline{\zeta})\nu(\underline{\zeta})f(\underline{\zeta})dS(\underline{\zeta}) \\
&= \frac{1}{a_m} \int_{\partial\Omega} \frac{(\underline{\xi}-\underline{\zeta})\nu(\underline{\zeta}) + \nu(\underline{\xi})(\underline{\xi}-\underline{\zeta})}{|\underline{\xi}-\underline{\zeta}|^m} f(\underline{\zeta})dS(\underline{\zeta}) \\
&= \frac{1}{a_m} \int_{\partial\Omega} \frac{(\underline{\xi}-\underline{\zeta}) \circ (\nu(\underline{\zeta}) + \nu(\underline{\xi}))}{|\underline{\xi}-\underline{\zeta}|^m} f(\underline{\zeta})dS(\underline{\zeta}) \\
&\qquad +\frac{1}{a_m} \int_{\partial\Omega} \frac{(\underline{\xi}-\underline{\zeta}) \wedge (\nu(\underline{\zeta}) - \nu(\underline{\xi}))}{|\underline{\xi}-\underline{\zeta}|^m} f(\underline{\zeta})dS(\underline{\zeta}).
\end{aligned}$$

Moreover we have

$$\begin{aligned}
\mathcal{A}\mathbb{P} &= \mathcal{C}^-\mathbb{P} - \mathcal{D}^+\mathbb{P} = -\mathcal{D}^+\mathbb{P} \\
\mathcal{B}\mathbb{P} &= \mathcal{C}^-\mathbb{P} + \mathcal{D}^-\mathbb{P} = \mathcal{D}^-\mathbb{P}
\end{aligned}$$

implying that for each function $g \in H_2^+(\partial\Omega)$, $\mathcal{A}[g]$ belongs to $H_2^+(\partial\Omega)^\perp$ and $\mathcal{B}[g]$ belongs to $H_2^-(\partial\Omega)^\perp$, and similarly

$$\begin{aligned}
\mathcal{A}\mathbb{P}^\perp &= \mathcal{C}^+\mathbb{P}^\perp - \mathcal{D}^-\mathbb{P}^\perp = \mathcal{C}^+\mathbb{P}^\perp = (\mathbf{1}+\mathcal{B})\mathbb{P}^\perp \\
\mathcal{B}\mathbb{P}^\perp &= \mathcal{C}^-\mathbb{P}^\perp + \mathcal{D}^-\mathbb{P}^\perp = \mathcal{C}^-\mathbb{P}^\perp = (-\mathbf{1}+\mathcal{A})\mathbb{P}^\perp
\end{aligned}$$

implying that for each function $h \in H_2^+(\partial\Omega)^\perp$, $\mathcal{A}[h] = (\mathbf{1}+\mathcal{B})[h]$ belongs to $H_2^+(\partial\Omega)$ and $\mathcal{B}[h] = (-\mathbf{1}+\mathcal{A})[h]$ belongs to $H_2^-(\partial\Omega)$.

8 A second orthogonal decomposition of $L_2(\partial\Omega)$

Making use of the Hardy space $H_2^-(\partial\Omega)$, introduced in section 5, a second orthogonal direct sum decomposition of $L_2(\partial\Omega)$ is at hand:

$$L_2(\partial\Omega) = H_2^-(\partial\Omega) \oplus H_2^-(\partial\Omega)^\perp.$$

Both components may be characterized in a similar way as was done for $H_2^+(\partial\Omega)$ and $H_2^+(\partial\Omega)^\perp$.

Proposition 8.1.

(i) A function $\widetilde{g}$ belongs to $H_2^-(\partial\Omega)$ if and only if $H[\widetilde{g}] = -\widetilde{g}$ or $\mathcal{C}^+[\widetilde{g}] = 0$, or still $\mathcal{C}^-[\widetilde{g}] = -\widetilde{g}$.

(ii) A function $\widetilde{h}$ belongs to $H_2^-(\partial\Omega)^\perp$ if and only if $H^*[\widetilde{h}] = \widetilde{h}$ or $\mathcal{D}^+[\widetilde{h}] = 0$, or still $\mathcal{D}^-[\widetilde{h}] = \widetilde{h}$.

Note that for a non–zero function $\widetilde{h} \in H_2^-(\partial\Omega)^\perp$ there cannot exist a left–monogenic function in Ω^-, vanishing at infinity, such that its NTBV is $\widetilde{h}$. However there again is a one–to–one correspondence, now between $H_2^-(\partial\Omega)$ and $H_2^-(\partial\Omega)^\perp$, established by means of the unit normal vector $\nu(\underline{\xi})$, $\underline{\xi} \in \partial\Omega$.

Proposition 8.2. *A function $\widetilde{g} \in L_2(\partial\Omega)$ belongs to $H_2^-(\partial\Omega)$ if and only if $\nu\widetilde{g} \in H_2^-(\partial\Omega)^\perp$, and vice–versa.*

Proof. Similar to the proof of Proposition 6.5. ■

Corollary 8.3. *The second orthogonal direct sum decomposition of $L_2(\partial\Omega)$ takes the form*

$$L_2(\partial\Omega) = H_2^-(\partial\Omega) \oplus \nu\ H_2^-(\partial\Omega) = \nu\ H_2^-(\partial\Omega)^\perp \oplus H_2^-(\partial\Omega)^\perp.$$

We denote the orthogonal projections on $H_2^-(\partial\Omega)$ and $H_2^-(\partial\Omega)^\perp$ by $\mathbb{Q}$ and $\mathbb{Q}^\perp$ respectively, and we put for $\underline{x} \in \Omega^-$

$$T(\underline{\zeta}, \underline{x}) = -\mathbb{Q}[C(\underline{\zeta}, \underline{x})], \quad \underline{\zeta} \in \partial\Omega$$

clearly the counterpart of the Szegö kernel for the Hilbert space $H_2^-(\partial\Omega)$. The function $T(\underline{\zeta}, \underline{x})$, $\underline{x} \in \Omega^-$, possesses the reproducing property since for each $\widetilde{g} \in H_2^-(\partial\Omega)$

$$\langle T(\underline{\zeta}, \underline{x}), \widetilde{g}(\underline{\zeta})\rangle = \langle -\mathbb{Q}[C(\underline{\zeta}, \underline{x})], \widetilde{g}(\underline{\zeta})\rangle = \langle C(\underline{\zeta}, \underline{x}), -\widetilde{g}(\underline{\zeta})\rangle = \mathcal{C}[-\widetilde{g}](\underline{x})$$

where at the utmost right hand side the functions $\widetilde{g} \in H_2^-(\partial\Omega)$ and $\mathcal{C}[-\widetilde{g}]$ are identified. In the same order of ideas it is also the kernel function of the integral transform expressing the projection $\mathbb{Q}$ of $L_2(\partial\Omega)$ on $H_2^-(\partial\Omega)$:

$$\langle T(\underline{\zeta}, \underline{x}), f(\underline{\zeta})\rangle = \mathbb{Q}[f](\underline{x}), \quad f \in L_2(\partial\Omega), \quad \underline{x} \in \Omega^-.$$

On the other hand for $\underline{x} \in \Omega^+$ we obtain for the Cauchy integral of an arbitrary function $f \in L_2(\partial\Omega)$,

$$\begin{aligned} \mathcal{C}[f] &= \langle C(\underline{\zeta}, \underline{x}), f(\underline{\zeta})\rangle = \langle \mathbb{Q}[C(\underline{\zeta}, \underline{x})], \mathbb{Q}[f(\underline{\zeta})]\rangle + \langle \mathbb{Q}^\perp[C(\underline{\zeta}, \underline{x})], \mathbb{Q}^\perp[f(\underline{\zeta})]\rangle \\ &= \mathcal{C}[\mathbb{Q}[f]] + \langle \mathbb{Q}^\perp[C(\underline{\zeta}, \underline{x})], \mathbb{Q}^\perp[f(\underline{\zeta})]\rangle = \langle \mathbb{Q}^\perp[C(\underline{\zeta}, \underline{x})], \mathbb{Q}^\perp[f(\underline{\zeta})]\rangle \end{aligned}$$

since the Cauchy integral of $\mathbb{Q}[f] \in H_2^-(\partial\Omega)$ vanishes in Ω^+. For a function $\widetilde{g} \in H_2^-(\partial\Omega)$ this leads in particular to

$$0 = \mathcal{C}[\widetilde{g}] = \langle C(\underline{\zeta}, \underline{x}), \widetilde{g}(\underline{\zeta}) \rangle, \quad \underline{x} \in \Omega^+$$

which means that for $\underline{x} \in \Omega^+$ the Cauchy kernel $C(\underline{\zeta}, \underline{x}) = \nu(\underline{\zeta}) E(\underline{\zeta} - \underline{x})$ belongs to $H_2^-(\partial\Omega)^\perp$, which is confirmed by the fact that for $\underline{x} \in \Omega^+$ the fundamental solution $E(\underline{\zeta} - \underline{x}) \in H_2^-(\partial\Omega)$, since it may be extended left–monogenically to Ω^- by the function $E(\underline{y} - \underline{x})$.

Returning to the Kerzman–Stein operator $\mathcal{A}$ and the related operator $\mathcal{B}$, it also follows that

$$\mathbb{Q}(-\mathbf{1} + \mathcal{A}) = \mathbb{Q}\mathcal{C}^- = \mathcal{C}^- = \mathbb{Q}\mathcal{B}, \quad \mathbb{Q}^\perp(-\mathbf{1} + \mathcal{A}) = -\mathbb{Q}^\perp \mathcal{D}^- = -\mathcal{D}^- = -\mathbb{Q}^\perp \mathcal{B}$$

since $\mathcal{C}^-[f] \in H_2^-(\partial\Omega)$ and $\mathcal{D}^-[f] \in H_2^-(\partial\Omega)^\perp$ for all $f \in L_2(\partial\Omega)$, which means that $(-\mathbf{1} + \mathcal{A})[f]$ and $\mathcal{B}[f]$ lie "symmetric" w.r.t $H_2^-(\partial\Omega)$ (see again Figure 3). Moreover

$$\begin{aligned} \mathcal{A}\mathbb{Q} &= \mathcal{C}^+\mathbb{Q} - \mathcal{D}^-\mathbb{Q} = -\mathcal{D}^-\mathbb{Q} \\ \mathcal{B}\mathbb{Q} &= \mathcal{C}^+\mathbb{Q} + \mathcal{D}^+\mathbb{Q} = \mathcal{D}^+\mathbb{Q} \end{aligned}$$

implying that for each function $\widetilde{g} \in H_2^-(\partial\Omega)$, $\mathcal{A}[\widetilde{g}]$ belongs to $H_2^-(\partial\Omega)^\perp$ and $\mathcal{B}[\widetilde{g}]$ belongs to $H_2^+(\partial\Omega)^\perp$, and similarly

$$\begin{aligned} \mathcal{A}\mathbb{Q}^\perp &= \mathcal{C}^-\mathbb{Q}^\perp - \mathcal{D}^+\mathbb{Q}^\perp = \mathcal{C}^-\mathbb{Q}^\perp = -(\mathbf{1} - \mathcal{B})\mathbb{Q}^\perp \\ \mathcal{B}\mathbb{Q}^\perp &= \mathcal{C}^+\mathbb{Q}^\perp + \mathcal{D}^+\mathbb{Q}^\perp = \mathcal{C}^+\mathbb{Q}^\perp = (\mathbf{1} + \mathcal{A})\mathbb{Q}^\perp \end{aligned}$$

implying that for each function $\widetilde{h} \in H_2^-(\partial\Omega)^\perp$, $\mathcal{A}[\widetilde{h}] = (-\mathbf{1} + \mathcal{B})[\widetilde{h}]$ belongs to $H_2^-(\partial\Omega)$ and $\mathcal{B}[\widetilde{h}] = (\mathbf{1} + \mathcal{A})[\widetilde{h}]$ belongs to $H_2^+(\partial\Omega)$.

9 Extension of the unit normal function ν

As the boundary $\partial\Omega$ is assumed to be C_∞–smooth, it is always possible to introduce the vector function $N(\underline{x})$ in an open neighbourhood $\widetilde{\partial\Omega}$ of $\partial\Omega$ such that

(i) $N(\underline{x})$ is a smooth function

(ii) $|N(\underline{x})| = 1$ for all $\underline{x} \in \widetilde{\partial\Omega}$

(iii) the restriction of $N(\underline{x})$ to $\partial\Omega$ is precisely $\nu(\underline{\xi})$, $\underline{\xi} \in \partial\Omega$.

If the closed surface $\partial\Omega$ has a defining C_∞–function $\rho(\underline{x})$, i.e. $\partial\Omega = \{\underline{x} : \rho(\underline{x}) = 0\}$, while $\Omega^+ = \{\underline{x} : \rho(\underline{x}) < 0\}$ and $\Omega^- = \{\underline{x} : \rho(\underline{x}) > 0\}$, then

$$\nu(\underline{\xi}) = \frac{\underline{\partial}\rho(\underline{\xi})}{|\underline{\partial}\rho(\underline{\xi})|}$$

for all $\underline{\xi} \in \partial\Omega$ and the function

$$N(\underline{x}) = \frac{\underline{\partial}\rho(\underline{x})}{|\underline{\partial}\rho(\underline{x})|}, \quad \underline{x} \in \widetilde{\partial\Omega}$$

satisfies all above requirements. Note that certainly $|\underline{\partial}\rho(\underline{\xi})| \neq 0$ for all $\underline{\xi} \in \partial\Omega$ due to the supposed smoothness of $\partial\Omega$, so that $|\underline{\partial}\rho(\underline{x})| \neq 0$ in an appropriate open neighbourhood $\widetilde{\partial\Omega}$ of $\partial\Omega$.

For a given function $f \in L_2(\partial\Omega)$ we consider in $\widetilde{\partial\Omega}$ the function

$$F(\underline{x}) = \mathcal{C}[\mathbb{P}[f]](\underline{x}) - N(\underline{x})\mathcal{C}[\mathbb{P}[\nu f]](\underline{x}).$$

The first term $F_1(\underline{x}) = \mathcal{C}[\mathbb{P}[f]](\underline{x})$ is left–monogenic in Ω^+ while vanishing in Ω^- and moreover for $\underline{\xi} \in \partial\Omega$ it holds that

$$\begin{aligned} \lim_{\Omega^+ \ni \underline{x} \to \underline{\xi}} F_1(\underline{x}) &= \mathbb{P}[f](\underline{\xi}) \\ \lim_{\Omega^- \ni \underline{x} \to \underline{\xi}} F_1(\underline{x}) &= 0. \end{aligned}$$

The function $F_2(\underline{x}) = N(\underline{x})\ \mathcal{C}[\mathbb{P}[\nu f]](\underline{x})$ apparently is not left–monogenic in $\widetilde{\partial\Omega}^+ = \widetilde{\partial\Omega} \cap \Omega^+$, but still vanishes in $\widetilde{\partial\Omega}^- = \widetilde{\partial\Omega} \cap \Omega^-$, and for $\underline{\xi} \in \partial\Omega$ it holds that

$$\begin{aligned} \lim_{\widetilde{\partial\Omega}^+ \ni \underline{x} \to \underline{\xi}} F_2(\underline{x}) &= \nu(\underline{\xi})\mathbb{P}[\nu f](\underline{\xi}) = -\mathbb{P}^{\perp}[f](\underline{\xi}) \\ \lim_{\widetilde{\partial\Omega}^- \ni \underline{x} \to \underline{\xi}} F_2(\underline{x}) &= 0. \end{aligned}$$

It follows that $F(\underline{x})$ is not left–monogenic in $\widetilde{\partial\Omega}$, but for $\underline{\xi} \in \partial\Omega$ it holds that

$$\begin{aligned} \lim_{\widetilde{\partial\Omega}^+ \ni \underline{x} \to \underline{\xi}} F(\underline{x}) &= \mathbb{P}[f](\underline{\xi}) - \nu(\underline{\xi})\mathbb{P}[\nu f](\underline{\xi}) = f(\underline{\xi}) \\ \lim_{\widetilde{\partial\Omega}^- \ni \underline{x} \to \underline{\xi}} F(\underline{x}) &= 0. \end{aligned}$$

We will now show that $F(\underline{x})$ is harmonic in $\widetilde{\partial\Omega} \backslash \partial\Omega$. To that end consider the operator $\underline{\partial}^* = N\ \underline{\partial}\ N$ for which also holds $\underline{\partial}^* N = -N\underline{\partial}$ and $N\underline{\partial}^* = -\underline{\partial}N$. For this operator, the following lemmata are easily proved.

Lemma 9.1. *If F is sufficiently smooth then $\underline{\partial}F = 0$ if and only if $\underline{\partial}^*(Nf) = 0$.*

Lemma 9.2. *The operator $\underline{\partial}^*$ factorizes the Laplace operator:*

$$(\underline{\partial}^*)^2 = -\Delta.$$

We then arrive at the desired result.

Proposition 9.3. *In $\widetilde{\partial\Omega} \setminus \partial\Omega$ one has:*

(i) $F_1(\underline{x}) = \mathcal{C}[\mathbb{P}[f]](\underline{x})$ *is left–monogenic with* $F_1 = 0$ *in* $\widetilde{\partial\Omega}^-$*;*

(ii) $F_2(\underline{x}) = N(\underline{x})\ \mathcal{C}[\mathbb{P}[\nu f]](\underline{x})$ *is a null solution of* $\underline{\partial}^*$ *with* $F_2 = 0$ *in* $\widetilde{\partial\Omega}^-$*;*

(iii) $F(\underline{x}) = F_1(\underline{x}) - F_2(\underline{x})$ *is harmonic with* $F = 0$ *in* $\widetilde{\partial\Omega}^-$*.*

Proof.

(i) This is a property of the Cauchy integral in $\mathbb{R}^m \setminus \partial\Omega$.

(ii) Follows from Lemma 9.1 since a Cauchy integral is left–monogenic in $\mathbb{R}^m \setminus \partial\Omega$.

(iii) $\Delta F = \Delta F_1 - \Delta F_2 = \underline{\partial}^2 F_1 - \underline{\partial}^{*2} F_2 = 0$. ■

We thus have proved that, given a function $f \in L_2(\partial\Omega)$, there exists a function F in $\widetilde{\partial\Omega} \setminus \partial\Omega$, namely $F(\underline{x}) = \mathcal{C}[\mathbb{P}[f]](\underline{x}) - N(\underline{x})\mathcal{C}[\mathbb{P}[\nu f]](\underline{x})$, which is harmonic in $\widetilde{\partial\Omega} \setminus \partial\Omega$, vanishes in $\widetilde{\partial\Omega}^-$ and for which one has $\lim_{\widetilde{\partial\Omega}^+ \ni \underline{x} \to \underline{\xi}} F(\underline{x}) = f(\underline{\xi})$.

Remark 9.4. *Unfortunately the function $N(\underline{x})$ is only defined in an open neighbourhood of $\partial\Omega$. Solving the Dirichlet problem and constructing the associated Poisson kernel by means of the Szegö projections and the Cauchy integral, remains an open problem. This problem can be reformulated as follows. Let $\mathcal{H}_2(\Omega^+)$ be the Hardy space of harmonic functions in Ω^+ with a NTBV in $L_2(\partial\Omega)$. Clearly $H_2(\Omega^+)$ is a closed subspace of $\mathcal{H}_2(\Omega^+)$ leading to the direct sum decomposition*

$$\mathcal{H}_2(\Omega^+) = H_2(\Omega^+) \oplus H_2(\Omega^+)^\perp$$

the orthogonal complement being taken in $\mathcal{H}_2(\Omega^+)$. The question now is: what is $\mathcal{H}_2(\Omega^+)^\perp$? In the specific case where Ω is the unit ball the answer is known (see e.g. [11]), in general it is not.

Finally the unit vector function N may also be used in the construction of a reproducing kernel for the Hilbert space $H_2^+(\partial\Omega)^\perp$. Indeed, take $h \in H_2^+(\partial\Omega)^\perp$, then $\nu h \in H_2^+(\partial\Omega)$ and by means of the Szegö kernel we obtain for $\underline{x} \in \Omega^+$:

$$\mathcal{C}[\nu h](\underline{x}) = \langle S(\underline{\zeta}, \underline{x}), \nu(\underline{\zeta})h(\underline{\zeta})\rangle$$

and hence for $\underline{x} \in \widetilde{\partial\Omega}^+$:

$$N(\underline{x})\ \mathcal{C}[\nu h](\underline{x}) = -N(\underline{x})\ \langle \nu(\underline{\zeta})S(\underline{\zeta}, \underline{x}), h(\underline{\zeta})\rangle$$

or

$$-N(\underline{x})\ \mathcal{C}[\nu h](\underline{x}) = N(\underline{x})\ \langle L(\underline{\zeta}, \underline{x}), h(\underline{\zeta})\rangle$$

where we have introduced to so–called Garabedian kernel for $H_2^+(\partial\Omega)^\perp$:

$$L(\underline{\zeta}, \underline{x}) = \nu(\underline{\zeta})S(\underline{\zeta}, \underline{x}), \quad \underline{\zeta} \in \partial\Omega, \quad \underline{x} \in \Omega^+.$$

This Garabedian kernel is reproducing for $H_2^+(\partial\Omega)^\perp$ in the sense that for $h \in H_2^+(\partial\Omega)^\perp$ and for $\underline{x} \in \widetilde{\partial\Omega}^+$, the function $N(\underline{x})\ \langle L(\underline{\zeta}, \underline{x}), h(\underline{\zeta})\rangle$ equals $-N(\underline{x})\ \mathcal{C}[\nu h](\underline{x})$ which in $\widetilde{\partial\Omega}^+$ is a null solution of the operator $\underline{\partial}^*$ and has NTBV $h(\underline{\xi})$ for $\underline{x} \in \widetilde{\partial\Omega}^+$ tending to $\underline{\xi} \in \partial\Omega$.

As $S(\underline{\zeta}, \underline{x}) = \mathbb{P}[C(\underline{\zeta}, \underline{x})]$ we also have for $\underline{x} \in \Omega^+$ and $\underline{\zeta} \in \partial\Omega$ that

$$L(\underline{\zeta}, \underline{x}) = \nu(\underline{\zeta})\mathbb{P}[C(\underline{\zeta}, \underline{x})] = \mathbb{P}^{\perp}[\nu(\underline{\zeta})C(\underline{\zeta}, \underline{x})] = \mathbb{P}^{\perp}[E(\underline{x} - \underline{\zeta})].$$

Note that the translated fundamental solution $E(\underline{x}-\underline{\zeta}) = \nu(\underline{\zeta})C(\underline{\zeta}, \underline{x})$ of the Dirac operator belongs to $H_2^-(\partial\Omega)$.

10 Conclusions

The central notion in this paper is the Hilbert transform on the smooth boundary $\partial\Omega$ of a bounded domain Ω in Euclidean space, which has been defined quite naturally as a part of the inner and outer NTBVs of the Cauchy integral of an L_2–function on $\partial\Omega$, the success of this approach being entirely due to the powerful concept of monogenic function in Clifford analysis.

At the same time we have devoted some attention to the concept of Hardy space, to which the Hilbert transform is closely related. In this we have treated the inner and the outer region determined by the considered closed hypersurface $\partial\Omega$ on equal footing, enabling us to obtain new bounded linear operators on $\partial\Omega$, similar to the Cauchy transforms, as well as to derive new relations in between those operators, similar to the traditional Kerzman–Stein formula.

Finally, we have also paid attention to the Dirichlet problem which, in its turn, is intimately related to the concepts of Hilbert transform and Hardy space. We have succeeded in constructing a harmonic function in a neighborhood of the boundary $\partial\Omega$, tending to the given function on $\partial\Omega$ itself, but we have not obtained an expression for the Poisson kernel in this general setting.

It goes without saying that the study of the triptych Hilbert transform – Hardy space – Dirichlet problem in the particular case of the unit sphere (see [5]) has much more concrete results to offer, in particular w.r.t. this last issue. However, on the unit sphere, some interesting concepts, features and insights are inevitably lost, since the Hilbert transform becomes a self–adjoint operator.

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