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Spectral Rank for C^*-Algebras

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ABSTRACT

We introduce a notion of dimension for C^*-algebras that we call spectral rank, based on spectrums of generators of C^*-algebras. We study some basic properties for this new rank and establish its fundamental theory.

RESUMEN

Introducimos la noción de dimensión para C^*-algebras que llamamos rango espectral; esta noción es basada en el espectro de los generadores de C^*-algebras. Estudiamos algunas propriedades básicas para este nuevo rango y establecemos su teoria fundamental.

Key words and phrases: *C^*-algebra, Rank, Spectrum.*

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Introduction

There have been several attempts to introduce suitable ranks for C^*-algebras; the stable rank (and connected stable rank) of Rieffel [4], the real rank of Brown and Pedersen [1], the completely positive (or decomposition) rank (or covering dimension) of Winter [7], and the topological rank and (another) covering dimension of the author [5], [6], etc. For reference, see [2] or [3].

In this paper we introduce a yet another notion of dimension for C^*-algebras that we call spectral rank. This rank is based on spectrums of generators of C^*-algebras. We study some basic properties for this new rank that might become an interesting new invariant for C^*-algebras. In Section 1, some basic properties of the spectral rank for C^*-algebras concerning their fundamental algebraic structures are discussed. In Section 2, introduced is an approximate version of the spectral rank that we call approximate spectral rank.

1 Spectral rank

Let $\mathfrak{A}$ be a C^*-algebra. The spectrum of an element a of $\mathfrak{A}$ is defined by

$$\mathrm{sp}(a) = \{\lambda \in \mathbb{C} \mid \ a - \lambda 1 \text{ is not invertible in } \mathfrak{A}^+\},$$

where $\mathfrak{A}^+ = \mathfrak{A}$ when $\mathfrak{A}$ is unital, and $\mathfrak{A}^+$ is the unitization of $\mathfrak{A}$ by $\mathbb{C}$ of complex numbers when $\mathfrak{A}$ is non-unital. Note that the spectrum $\mathrm{sp}(a)$ is a non-empty closed subset of $\mathbb{C}$ and bounded by the norm $\|a\|$.

Definition 1.1. Let $\mathfrak{A}$ be a C^*-algebra with (specific and initial) generators a_j. Define the spectral rank of $\mathfrak{A}$ to be

$$\mathrm{spr}(\mathfrak{A}) = \inf \left\{ \sum_j \dim \mathrm{sp}(a_j) \mid a_j \text{ are generators of } \mathfrak{A} \right\}$$

where $\dim(\cdot)$ is the (covering) dimension for spaces.

Remark. This notion should not depend on generators but do depend in a sense, and certainly does not depend on their certain equivalences like adjoint unitary operations since $\mathrm{sp}(a_j) = \mathrm{sp}(\mathrm{Ad}(u)a_j)$ where $\mathrm{Ad}(u)a_j = ua_ju^*$ for some unitary u, and some cancellative terms. In another view, we just look at generators of the algebraic part of $\mathfrak{A}$ and ignore some unknown ones in the C^*-closure of the part in $\mathfrak{A}$, and always consider such a situation in what follows.

Proposition 1.2. *Let $\mathfrak{A}$ be a C^*-algebra with unitary generators u_j for $j \in J$ a set. Then*

$$\mathrm{spr}(\mathfrak{A}) \le \sum_j \dim \mathrm{sp}(u_j) \le \sum_j 1 = |J|,$$

where the second inequality is equality if $\mathfrak{A}$ is universal.

Proof. Note that $\mathrm{sp}(u_j)$ is a closed subset of the torus $\mathbb{T}$ with $\dim \mathbb{T} = 1$. Thus, $\dim \mathrm{sp}(u_j) \leq 1$. If $\mathfrak{A}$ is universal, we have $\mathrm{sp}(u_j) = \mathbb{T}$. □

Proposition 1.3. *Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{B}$ its quotient C^*-algebra, where generators of $\mathfrak{B}$ are mapped from those of $\mathfrak{A}$ by the quotient map. Then*

$$\mathrm{spr}(\mathfrak{A}) \geq \mathrm{spr}(\mathfrak{B}).$$

Proof. Let π be the quotient map from $\mathfrak{A}$ to $\mathfrak{B}$. Let a_j be generators of $\mathfrak{A}$. Then $\pi(a_j)$ are generators of $\mathfrak{B}$ and note that $\mathrm{sp}(\pi(a_j)) \subset \mathrm{sp}(a_j)$. □

Example 1.4. Let $C(\mathbb{T}^n)$ be the C^*-algebra of all continuous functions on the n-torus $\mathbb{T}^n$. This is the universal C^*-algebra generated by mutually commuting n unitaries. Hence, $\mathrm{spr}(C(\mathbb{T}^n)) = n$.

Let $\mathbb{T}^n_\theta$ be the noncommutative n-torus, which is defined to be the universal C^*-algebra generated by n unitaries u_j such that $u_j u_i = e^{2\pi i \theta_{ij}} u_i u_j$ for $1 \leq i, j \leq n$, where $\theta = (\theta_{ij})$ is a skew-adjoint $n \times n$ matrix over $\mathbb{R}$. Thus, $\mathrm{spr}(\mathbb{T}^n_\theta) = n$.

Let $C^*(F_n)$ be the full group C^*-algebra of the free group F_n with n generators. This is the universal C^*-algebra generated by n unitaries with no relations. Hence, $\mathrm{spr}(C^*(F_n)) = n$.

Proposition 1.5. *Let $\mathfrak{A}$ be a C^*-algebra with isometry generators s_j for $j \in J$ a set. Then*

$$\mathrm{spr}(\mathfrak{A}) \leq \sum_j \dim \mathrm{sp}(s_j) = \sum_j 2 = 2|J|,$$

and the inequality $\mathrm{spr}(\mathfrak{A}) \leq 2|J|$ *holds in general, with* $\{a_j\}_{j \in J}$ *generators of* $\mathfrak{A}$.

Proof. Note that $\mathrm{sp}(s_j)$ is the unit disk $\mathbb{D}$ with $\dim \mathbb{D} = 2$. Thus, $\dim \mathrm{sp}(s_j) = 2$.

In general, $\mathrm{sp}(a_j)$ is a closed subset of $\mathbb{C}$, so that $\dim \mathrm{sp}(a_j) \leq 2$. □

Example 1.6. Let $\mathfrak{F}$ be the Toeplitz algebra, which is the universal C^*-algebra generated by an isometry. Thus, $\mathrm{spr}(\mathfrak{F}) = 2$.

Let $C^*(N_n)$ be the full semigroup C^*-algebra of the free semigroup N_n with n generators, i.e., $N_n \cong *^n\mathbb{N}$ the n-fold free product of the semigroup $\mathbb{N}$ of natural numbers. This is the universal C^*-algebra generated by n isometries with no relations. Hence, $\mathrm{spr}(C^*(N_n)) = 2n$.

Let O_n be the Cuntz algebra $(2 \leq n < \infty)$, which is the universal C^*-algebra generated by n isometries s_j such that $\sum_{j=1}^n s_j s_j^* = 1$. Then $\mathrm{spr}(O_n) = 2n$.

Let O_∞ be the Cuntz algebra generated by isometries s_j $(j \in \mathbb{N})$ such that $\sum_{j=1}^n s_j s_j^* < 1$. Then $\mathrm{spr}(O_\infty) = \infty$.

Proposition 1.7. *Let $\mathfrak{A}$ be a C^*-algebra generated by compact operators. Then*

$$\mathrm{spr}(\mathfrak{A}) = 0.$$

Proof. Note that for a compact opearator T, we have $\dim \mathrm{sp}(T) = 0$. □

Proposition 1.8. *Let $\mathfrak{A}, \mathfrak{B}$ be C^*-algebras and $\mathfrak{A} \oplus \mathfrak{B}$ their direct sum. Then*

$$\mathrm{spr}(\mathfrak{A} \oplus \mathfrak{B}) = \mathrm{spr}(\mathfrak{A}) + \mathrm{spr}(\mathfrak{B}).$$

Proof. Suppose that a_j are generators of $\mathfrak{A}$ and b_j are those of $\mathfrak{B}$. Then $a_j \oplus 0$ and $0 \oplus b_j$ are generators of $\mathfrak{A} \oplus \mathfrak{B}$. □

Example 1.9. For $M_n(\mathbb{C})$ the $n \times n$ matrix algebra over $\mathbb{C}$, $\mathrm{spr}(M_n(\mathbb{C})) = 0$.

Let $\mathbb{K}$ be the C^*-algebra of all compact operators on a separable infinite-dimensional Hilbert space. Then $\mathrm{spr}(\mathbb{K}) = 0$.

An AF algebra that is an inductive limit of finite dimensional C^*-algebras (i.e., finite direct sums of some $M_n(\mathbb{C})$) has spectral rank zero.

Proposition 1.10. *Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{I}$ its C^*-subalgebra, where generators of $\mathfrak{I}$ can be always taken from those of $\mathfrak{A}$. Then*

$$\mathrm{spr}(\mathfrak{I}) \le \mathrm{spr}(\mathfrak{A}).$$

In particular, we may take $\mathfrak{I}$ as a closed two-sided ideal or hereditary C^*-subalgebra in this sense.

Proof. Generators of $\mathfrak{I}$ can be viewed as a part of those of $\mathfrak{A}$. □

Remark. The assumption for generators is necessary. Indeed, there exist some C^*-algebras that are embeddable into AF algebras, but they should have spectral rank non-zero, such as rotation C^*-algebras. This is an obstruction to our theory, but it seems that in this case the generators of those C^*-algebras are not to be visible in AF. □

Proposition 1.11. *Let $0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{I} \to 0$ be a short exact sequence of C^*-algebras, where generators of $\mathfrak{I}$ are always taken from those of $\mathfrak{A}$. Then*

$$\mathrm{spr}(\mathfrak{A}) \ge \max\{\mathrm{spr}(\mathfrak{I}), \mathrm{spr}(\mathfrak{A}/\mathfrak{I})\}.$$

Remark. It is likely but not true in general that

$$\mathrm{spr}(\mathfrak{A}) \le \mathrm{spr}(\mathfrak{I}) + \mathrm{spr}(\mathfrak{A}/\mathfrak{I}).$$

For instance, the Toeplitz algebra $\mathfrak{F} = C^*(S)$ is decomposed into

$$0 \to \mathbb{K} \to \mathfrak{F} \to C(\mathbb{T}) \to 0,$$

where $\mathbb{K}$ is generated by some elements like the finite rank projections $1 - S^n(S^n)^*$ for $n \in \mathbb{N}$ and S is mapped to the generator of $C(\mathbb{T})$. But $\mathrm{spr}(\mathfrak{F}) = 2$, $\mathrm{spr}(\mathbb{K}) = 0$, and $\mathrm{spr}(C(\mathbb{T})) = 1$. However, the generators of $\mathbb{K}$ are not a part of those of $\mathfrak{F}$.

Proposition 1.12. *Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{A}^+$ its unitization by $\mathbb{C}$. Then*

$$\mathrm{spr}(\mathfrak{A}) = \mathrm{spr}(\mathfrak{A}^+).$$

Proof. Note that for the unit 1, $\mathrm{sp}(1) = \{1\} \subset \mathbb{C}$. Thus, $\dim \mathrm{sp}(1) = 0$. □

Proposition 1.13. *Let $0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{B} \to 0$ be an extension of C^*-algebras. Then*

$$\mathrm{spr}(\mathfrak{A}) \leq \max\{\mathrm{spr}(M(\mathfrak{I})), \mathrm{spr}(\mathfrak{B})\}.$$

where $M(\mathfrak{I})$ is the multiplier algebra of $\mathfrak{I}$, and generators of $\mathfrak{A}$ are viewed as part of those of the direct sum $M(\mathfrak{I}) \oplus \mathfrak{B}$ containing the pullback C^*-algebra associated with the extension.

Proof. It is well known that $\mathfrak{A}$ is isomorphic to the pullback C^*-algebra in $M(\mathfrak{I}) \oplus \mathfrak{B}$ with the associated Busby map from $\mathfrak{B}$ to $M(\mathfrak{I})/\mathfrak{I}$ and the quotient map from $M(\mathfrak{I})$ to $M(\mathfrak{I})/\mathfrak{I}$. □

Proposition 1.14. *Let $\mathfrak{A}, \mathfrak{B}$ be C^*-algebras and $\mathfrak{A} \otimes \mathfrak{B}$ their C^*-tensor product with a C^*-norm. Then*

$$\max\{\mathrm{spr}(\mathfrak{A}), \mathrm{spr}(\mathfrak{B})\} \leq \mathrm{spr}(\mathfrak{A} \otimes \mathfrak{B}) \leq \mathrm{spr}(\mathfrak{A}) + \mathrm{spr}(\mathfrak{B}).$$

Proof. The left inequality is clear since $\mathfrak{A}$, $\mathfrak{B}$ are C^*-subalgebras of $\mathfrak{A} \otimes \mathfrak{B}$ preserving generators.

Assume first that $\mathfrak{A}$, $\mathfrak{B}$ are unital. Suppose that a_j are generators of $\mathfrak{A}$ and b_j are those of $\mathfrak{B}$. Then $a_j \otimes 1$ and $1 \otimes b_j$ are generators of $\mathfrak{A} \otimes \mathfrak{B}$.

If $\mathfrak{A}$ or $\mathfrak{B}$ are non-unital, then

$$\begin{aligned} \mathrm{spr}(\mathfrak{A} \otimes \mathfrak{B}) &\leq \mathrm{spr}(\mathfrak{A}^+ \otimes \mathfrak{B}^+) \\ &\leq \mathrm{spr}(\mathfrak{A}^+) + \mathrm{spr}(\mathfrak{B}^+) = \mathrm{spr}(\mathfrak{A}) + \mathrm{spr}(\mathfrak{B}) \end{aligned}$$

since $\mathfrak{A} \otimes \mathfrak{B}$ is a closed ideal of $\mathfrak{A}^+ \otimes \mathfrak{B}^+$. □

Corollary 1.15. *For $M_n(\mathfrak{A})$ the $n \times n$ matrix algebra over a C^*-algebra $\mathfrak{A}$,*

$$\mathrm{spr}(M_n(\mathfrak{A})) = \mathrm{spr}(\mathfrak{A}).$$

Furthermore, if $\mathfrak{B}$ is a C^*-algebra with $\mathrm{spr}(\mathfrak{B}) = 0$, then

$$\mathrm{spr}(\mathfrak{A} \otimes \mathfrak{B}) = \mathrm{spr}(\mathfrak{A}).$$

Proof. Note that $M_n(\mathfrak{A}) \cong \mathfrak{A} \otimes M_n(\mathbb{C})$. □

Proposition 1.16. *Let $\mathfrak{A}$ be a C^*-algebra, G a finitely generated discrete group with n generators, and $\mathfrak{A} \rtimes_\alpha G$ a (full or reduced) C^*-crossed product of $\mathfrak{A}$ by an action α of G by automorphisms. Then*

$$\mathrm{spr}(\mathfrak{A}) \leq \mathrm{spr}(\mathfrak{A} \rtimes_\alpha G) \leq \mathrm{spr}(\mathfrak{A}) + n.$$

In particular, if $G = \mathbb{Z}$, then

$$\mathrm{spr}(\mathfrak{A}) \leq \mathrm{spr}(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \leq \mathrm{spr}(\mathfrak{A}) + 1.$$

Proof. The crossed product $\mathfrak{A} \rtimes_\alpha G$ is generated by $\mathfrak{A}$ and the unitaries corresponding to the generators of G (on the universal, or a certain Hilbert space). □

Corollary 1.17. *Let G be a finitely generated discrete group with n generators. Let $C^*(G)$ and $C^*_r(G)$ be its full and reduced group C^*-algebras. Then*

$$\mathrm{spr}(C^*(G)) \leq n, \quad \textit{and} \quad \mathrm{spr}(C^*_r(G)) \leq n.$$

Proposition 1.18. *Let $\mathfrak{A}$ be a C^*-algebra, N a finitely generated discrete semi-group with n generators, and $\mathfrak{A} \rtimes_\beta N$ a (full or reduced) semi-group crossed product of $\mathfrak{A}$ by an action β of N by endomorphisms. Then*

$$\mathrm{spr}(\mathfrak{A}) \leq \mathrm{spr}(\mathfrak{A} \rtimes_\beta N) \leq \mathrm{spr}(\mathfrak{A}) + 2n.$$

In particular, if $N = \mathbb{N}$, then

$$\mathrm{spr}(\mathfrak{A}) \leq \mathrm{spr}(\mathfrak{A} \rtimes_\beta \mathbb{N}) \leq \mathrm{spr}(\mathfrak{A}) + 2.$$

Proof. The crossed product $\mathfrak{A} \rtimes_\beta N$ is generated by $\mathfrak{A}$ and the isometries corresponding to the generators of N (on the universal, or a certain Hilbert space). □

Corollary 1.19. *Let N be a finitely generated discrete semi-group with n generators. Let $C^*(N)$ and $C^*_r(N)$ be its full and reduced semi-group C^*-algebras. Then*

$$\mathrm{spr}(C^*(N)) \leq 2n, \quad \textit{and} \quad \mathrm{spr}(C^*_r(N)) \leq 2n.$$

Proposition 1.20. *Let $\mathfrak{A}$ be a continuous field C^*-algebra on a locally compact Hausdorff space X with fibers the same C^*-algebra $\mathfrak{B}$. Then*

$$\mathrm{spr}(\mathfrak{A}) \leq \mathrm{spr}(C_0(X)) + \mathrm{spr}(\mathfrak{B}),$$

where $C_0(X)$ is the C^*-algebra of all continuous functions on X vanishing at infinity. If $\mathfrak{B}$ is unital with $\mathrm{spr}(\mathfrak{B}) = 0$*, then*

$$\mathrm{spr}(\mathfrak{A}) = \mathrm{spr}(C_0(X)).$$

Proof. Assume first that $\mathfrak{B}$ is unital. Then $\mathfrak{A}$ is assumed to be generated by generators of $C_0(X)$ and those of $\mathfrak{B}$. Also, we obtain $\mathrm{spr}(C_0(X)) \leq \mathrm{spr}(\mathfrak{A})$.

If $\mathfrak{B}$ is non-unital, we can consider the unitization $\mathfrak{A}^+$ of $\mathfrak{A}$ by adding the unit field on X taking the unit of the unitization $\mathfrak{B}^+$ of $\mathfrak{B}$. Therefore, we obtain

$$\mathrm{spr}(\mathfrak{A}) = \mathrm{spr}(\mathfrak{A}^+) \leq \mathrm{spr}(C_0(X)) + \mathrm{spr}(\mathfrak{B}^+) = \mathrm{spr}(C_0(X)) + \mathrm{spr}(\mathfrak{B}).$$

□

Example 1.21. Let $\mathbb{T}^2_\theta$ be the rational rotation C^*-algebra corresponding to a rational θ. This can be viewed as a continuous filed C^*-algebra on the 2-torus $\mathbb{T}^2$ with fibers the same $M_n(\mathbb{C})$ with n the period of θ. Thus,

$$\mathrm{spr}(\mathbb{T}^2_\theta) = \mathrm{spr}(C(\mathbb{T}^2)) = 2.$$

Proposition 1.22. *Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{B}$ a C^*-algebra deformed from $\mathfrak{A}$ with generators and relations deformed from those of $\mathfrak{A}$. Then*

$$\mathrm{spr}(\mathfrak{A}) = \mathrm{spr}(\mathfrak{B}).$$

Example 1.23. Let $\mathbb{T}^n_\theta$ be a noncommutative n-torus. This is deformed from $C(\mathbb{T}^n)$, and $\mathrm{spr}(\mathbb{T}^n_\theta) = \mathrm{spr}(C(\mathbb{T}^n)) = n$.

Proposition 1.24. *Let $\mathfrak{A}, \mathfrak{B}$ be C^*-algebras and $\mathfrak{A} * \mathfrak{B}$ their C^*-free product with a (full or reduced) C^*-norm. Then*

$$\max\{\mathrm{spr}(\mathfrak{A}), \mathrm{spr}(\mathfrak{B})\} \le \mathrm{spr}(\mathfrak{A} * \mathfrak{B}) \le \mathrm{spr}(\mathfrak{A}) + \mathrm{spr}(\mathfrak{B}).$$

Also, $\mathfrak{A} * \mathfrak{B}$ can be replaced with the unital C^*-free product $\mathfrak{A} *_{\mathbb{C}} \mathfrak{B}$.

Proof. The left inequality is clear since $\mathfrak{A}$, $\mathfrak{B}$ are C^*-subalgebras of $\mathfrak{A} * \mathfrak{B}$ preserving generators.

Suppose that a_j are generators of $\mathfrak{A}$ and b_j are those of $\mathfrak{B}$. Then a_j and b_j are generators of $\mathfrak{A} * \mathfrak{B}$. □

2 Approximate spectral rank

Definition 2.1. Let $\mathfrak{A}$ be a C^*-algebra. Define the approximate spectral rank of $\mathfrak{A}$ to be the minimum non-negative integer $n = \mathrm{aspr}(\mathfrak{A})$ such that for any $a \in \mathfrak{A}$ and $\varepsilon > 0$, there exists a C^*-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ with $\mathrm{spr}(\mathfrak{B}) \le n$ such that $\|a - b\| \le \varepsilon$ for some $b \in \mathfrak{B}$.

Proposition 2.2. *Let $\mathfrak{A}$ be an inductive limit of C^*-algebras $\mathfrak{A}_n$ with $\mathrm{spr}(\mathfrak{A}_n) \le s_n$ for some s_n. Then*

$$\mathrm{aspr}(\mathfrak{A}) \le \underline{\lim}\, s_n,$$

where $\underline{\lim}$ is the limit infimum.

Example 2.3. If $\mathfrak{A}$ is an AF-algebra, then $\mathrm{aspr}(\mathfrak{A}) = 0 = \mathrm{spr}(\mathfrak{A})$.

Let $\mathfrak{A}$ be an AT-algebra, i.e., an inductive limit of finite direct sums of matrix algebras over $C(\mathbb{T})$. If $\mathfrak{A}$ is an inductive limit of k direct sums of matrix algebras over $C(\mathbb{T})$, then $\mathrm{aspr}(\mathfrak{A}) \le k$. Indeed, for such a k direct sum,

$$\begin{aligned}\mathrm{spr}(\oplus_{j=1}^k M_{n_j}(C(\mathbb{T}))) = \mathrm{spr}(\oplus_{j=1}^k (M_{n_j}(\mathbb{C}) \otimes C(\mathbb{T}))) &= \sum_{j=1}^k \mathrm{spr}(M_{n_j}(\mathbb{C}) \otimes C(\mathbb{T})) \\ &\le \sum_{j=1}^k (\mathrm{spr}(M_{n_j}(\mathbb{C})) + \mathrm{spr}(C(\mathbb{T}))) = k.\end{aligned}$$

In particular, if $\mathbb{T}_\theta^2$ is a simple noncommutative 2-torus, then it is an inductvie limit of 2 direct sums of matrix algebras over $C(\mathbb{T})$. Hence, $\mathrm{aspr}(\mathbb{T}_\theta^2) \leq 2$.

Proposition 2.4. *Let X be a locally compact Hausdorff space with $\dim X$ finite. Then*

$$\mathrm{aspr}(C_0(X)) \leq \dim X.$$

Proof. Note that X can be viewed as a projective limit of the product spaces $[0,1]^n$, where $\dim X = n$. Thus, $C_0(X)$ is an inductive limit of $C([0,1]^n)$. Since $C([0,1]^n) \cong \otimes^n C([0,1])$, we obtain the conclusion. □

Remark. There exists a locally compact Hausdorff space X with $\dim X = 1$ but $\dim X^+ = 0$, where X^+ is the one-point compactification of X. Thus,

$$\mathrm{aspr}(C_0(X)) = 1, \quad \text{but} \quad \mathrm{spr}(C_0(X)) = \mathrm{spr}(C(X^+)) = 0,$$

where $C_0(X)^+ \cong C(X^+)$. Also, $\mathrm{aspr}(C(X^+)) = 0$.

Remark. More fundamental properties for the approximate spectral rank could be obtained as the spectral rank in Section 1, but their details would be considered somewhere else.

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