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Rosso-Yamane Theorem on PBW Basis of $U_q(A_N)$ *

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ABSTRACT

Let $U_q(A_N)$ be the Drinfeld-Jimbo quantum group of type A_N. In this paper, by using Gröbner-Shirshov bases, we give a simple (but not short) proof of the Rosso-Yamane Theorem on PBW basis of $U_q(A_N)$.

RESUMEN

Sea $U_q(A_N)$ el grupo cuántico de Drinfel-Jimbo de tipo A_N. En este artículo, usando bases de Gröbner-Shirshov damos una demonstración simple (pero no corta) del Teorema de Rosso-Yamane sobre bases PBW de $U_q(A_N)$.

Key words and phrases: *Quantum group, Quantum enveloping algebra, Gröbner-Shirshov basis.*

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1 Introduction

Since any algebra (commutative, associative, Lie), as well as any module over an algebra, can be presented by generators and defining relations, it is important to have a general method to deal with these presentations. Such a method now exists and is called the Gröbner bases method (due to B. Buchberger [18], [19]), or standand bases method (due to H. Hironaka [21]), or Gröbner-Shirshov bases method (due to A. I. Shirshov [35]). The original Shirshov's paper [35] is for Lie algebra presentations, but it can be easily adopted for associative algebra presentations as well, see L. A. Bokut [3] and G. Bergman [1].

Let, for example, $L = Lie(X|[x_ix_j] - \sum \alpha_{ij}^k x_k,\ i > j,\ x_i, x_j, x_k \in X)$ be a Lie algebra over a field (or a commutative ring) k presented by a k-basis X and the multiplication table. Then $S = \{[x_ix_j] - \sum \alpha_{ij}^k x_k |\ i > j,\ x_i, x_j, x_k \in X\}$ is a Gröbner-Shirshov basis (subset) of the free Lie algebra $Lie(X)$ over k. On the other hand, the universal enveloping algebra $U(L) = k\langle X|x_ix_j - x_jx_i - \sum \alpha_{ij}^k x_k,\ i > j,\ x_i, x_j, x_k \in X\rangle$ is the associative algebra presented by the same set X and the defining relations $S^{(-)}$ (we rewrite S using $[xy] = xy - yx$). There is a general but not difficult result that for any $S \subset Lie(X)$, S is a Gröbner-Shirshov basis in the sense of Lie algebras if and only if $S^{(-)} \subset k\langle X\rangle$ is a Gröbner-Shirshov basis in the sense of associative algebras (see, for example, [9] and [7]). This means that in our case, $S^{(-)}$ is a Gröbner-Shirshov basis (subset) in $k\langle X\rangle$. By Composition-Diamond lemma (see below), the S-irreducible words on X, $Irr(S) = \{x_{i_1} \dots x_{i_k},\ i_1 \leq \dots \leq i_k,\ k \geq 0\}$ form a k-basis of $U(L)$. This is a conceptional proof of the PBW-Theorem by using Gröbner-Shirshov bases theory.

There are many results on Gröbner-Shirshov bases for associative and Lie algebras, as well as for semigroups, groups, conformal algebras, dialgebras, and so on, see, for example, surveys [14], [15], [25] and [8]. Let us mention those for simple Lie algebras and Lie superalgebras via Serre's presentations ([10], [11], [12], [13], [9]), for modules over simple Lie algebras and Iwahori-Hecke algebras ([23], [24], [25]), for Kac-Moody algebras of types $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$ ([31], [32], [33]), for Coxeter groups ([17]), for braid groups via Artin-Burau, Artin-Garside and Briman-Ko-Lee presentations ([4], [5] and [6]).

Drinfeld-Jimbo ([20], [22]) presentations for quantized enveloping algebras $U_q(g)$, where g is a semisimple Lie algebra, are a natural source of associative presentations. M. Rosso [34] and I. Yamane [36] found the PBW-basis of $U_q(A_N)$. G. Lusztig [29] and [30], and M. Kashiwara [26] and [27] found the bases of $U_q(g)$ for any semisimple algebra g, as well as for their representations. Their approach work equally well for quantized enveloping algebras associated with arbitrary symmetrizable Cartan matrix, not just those corresponding to finite dimensional Lie algebras. V. K. Kharchenko [28] found the approach to linear bases of quantized enveloping algebras via the so called character Hopf algebras.

It the paper [16], Gröbner-Shirshov bases approach was applied to study $U_q(g)$ for any symmetrizable Cartan matrix. Using this approach, they got a new proof of the triangular decomposition of $U_q(g)$ (see, for example, Jantzen [37]). For $U_q(A_N)$, it was proved by Bokut and Malcolmson [16] that the Jimbo relations (see [36]) of $U_q^+(A_N)$ constitute a Gröbner-Shirshov basis of $U_q^+(A_N)$ in Jimbo generators $x_{ij}, 1 \le i, j \le N+1$ (see below).

In this paper, we give an elementary proof that Jimbo relations S is a Gröbner-Shirshov basis of $U_q^+(A_N)$. For such a purpose, we just check all possible compositions of polynomials from S and proved that all them can be resolved. Also in §1 in this paper, we are giving necessary definitions and Composition-Diamond lemma following Shirshov [35].

2 Preliminaries

We first cite some concepts and results from the literature which are related to the Gröbner-Shirshov bases for associative algebras.

Let k be a field, $k\langle X\rangle$ the free associative algebra over k generated by X and X^* the free monoid generated by X, where the empty word is the identity which is denoted by 1. For a word $w \in X^*$, we denote the length of w by $deg(w)$. Let X^* be a well ordered set. Let $f \in k\langle X\rangle$ with the leading word $\bar{f}$. Then we call f monic if $\bar{f}$ has coefficient 1.

Definition 2.1. *([35], see also [2], [3]) Let f and g be two monic polynomials in $k\langle X\rangle$ and $<$ a well ordering on X^*. Then, there are two kinds of compositions:*

(i) If w is a word such that $w = \bar{f}b = a\bar{g}$ for some $a, b \in X^*$ with $deg(\bar{f})+deg(\bar{g}) > deg(w)$, then the polynomial $(f,g)_w = fb - ag$ is called the intersection composition of f and g with respect to w.

(ii) If $w = \bar{f} = a\bar{g}b$ for some $a, b \in X^*$, then the polynomial $(f,g)_w = f - agb$ is called the inclusion composition of f and g with respect to w.

Definition 2.2. *([2], [3], cf. [35]) Let $S \subset k\langle X\rangle$ such that every $s \in S$ is monic. Then the composition $(f,g)_w$ is called trivial modulo (S,w) if $(f,g)_w = \sum \alpha_i a_i s_i b_i$, where each $\alpha_i \in k$, $a_i, b_i \in X^*$, $s_i \in S$ and $\overline{a_i s_i b_i} < w$. If this is the case, then we write*

$$(f,g)_w \equiv 0 \quad mod(S,w).$$

In general, for $p, q \in k\langle X\rangle$, we write $p \equiv q \quad mod(S,w)$ which means that $p - q = \sum \alpha_i a_i s_i b_i$, where each $\alpha_i \in k, a_i, b_i \in X^*$, $s_i \in S$ and $\overline{a_i s_i b_i} < w$.

Definition 2.3. *([2], [3], cf. [35]) We call the set S with respect to the well ordering $<$ a Gröbner-Shirshov set (basis) in $k\langle X\rangle$ if any composition of polynomials in S is trivial modulo S.*

If a subset S of $k\langle X\rangle$ is not a Gröbner-Shirshov basis, then we can add to S all nontrivial compositions of polynomials of S, and by continuing this process (maybe infinitely) many times, we eventually obtain a Gröbner-Shirshov basis S^c. Such a process is called the Shirshov algorithm.

A well ordering $>$ on X^* is called a monomial order if it is compatible with the multiplication of words, that is, for $u, v \in X^*$, we have

$$u > v \Rightarrow w_1uw_2 > w_1vw_2, \ for\ all\ w_1,\ w_2 \in X^*.$$

A standard example of monomial order on X^* is the deg-lex order to compare two words first by degree and then lexicographically, where X is a well ordered set.

The following lemma was first proved by Shirshov [35] for free Lie algebras (with deg-lex order) in 1962 (see also Bokut [2]). In 1976, Bokut [3] specialized the approach of Shirshov to associative algebras (see also Bergman [1]). For the case of commutative polynomials, this lemma is known as the Buchberger's Theorem in [18] and [19].

Lemma 2.4. *(Composition-Diamond Lemma) Let k be a field, $k\langle X|S\rangle = k\langle X\rangle/Id(S)$ and $<$ a monomial order on X^*, where $Id(S)$ is the ideal of $k\langle X\rangle$ generated by S. Then the following statements are equivalent:*

(i) S is a Gröbner-Shirshov basis.

(ii) $f \in Id(S) \Rightarrow \bar{f} = a\bar{s}b$ for some $s \in S$ and $a, b \in X^*$.

(iii) $Irr(S) = \{u \in X^* | u \neq a\bar{s}b, s \in S, a, b \in X^*\}$ is a basis of the algebra $k\langle X|S\rangle$. □

3 Rosso-Yamane theorem on PBW basis of $U_q(A_N)$

Let k be a field, $A = (a_{ij})$ an integral symmetrizable $N \times N$ Cartan matrix so that $a_{ii} = 2$, $a_{ij} \leq 0$ $(i \neq j)$ and there exists a diagonal matrix D with diagonal entries d_i which are nonzero integers such that the product DA is symmetric. Let q be a nonzero element of k such that $q^{4d_i} \neq 1$ for each i. Then the quantum enveloping algebra is (see [20], [22])

$$U_q(A) = k\langle X \cup H \cup Y | S^+ \cup K \cup T \cup S^- \rangle,$$

where

$$\begin{array}{rcl}
X &=& \{x_i\},\\
H &=& \{h_i^{\pm 1}\},\\
Y &=& \{y_i\},\\
S^+ &=& \{\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \left(\begin{array}{c} 1-a_{ij} \\ \nu \end{array}\right)_t x_i^{1-a_{ij}-\nu} x_j x_i^\nu,\ where\ i \neq j,\ t = q^{2d_i}\},\\
S^- &=& \{\sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \left(\begin{array}{c} 1-a_{ij} \\ \nu \end{array}\right)_t y_i^{1-a_{ij}-\nu} y_j y_i^\nu,\ where\ i \neq j,\ t = q^{2d_i}\},\\
K &=& \{h_i h_j - h_j h_i, h_i h_i^{-1} - 1, h_i^{-1} h_i - 1, x_j h_i^{\pm 1} - q^{\mp d_i a_{ij}} h^{\pm 1} x_j, h_i^{\pm 1} y_j - y_j h^{\pm 1}\},\\
T &=& \{x_i y_j - y_j x_i - \delta_{ij} \frac{h_i^2 - h_i^{-2}}{q^{2d_i} - q^{-2d_i}}\} \quad \text{and}\\
\left(\begin{array}{c} m \\ n \end{array}\right)_t &=& \begin{cases} \prod_{i=1}^{n} \frac{t^{m-i+1} - t^{i-m-1}}{t^i - t^{-i}} & m > n > 0, \\ 1 & n = 0 \quad \text{or} \quad m = n. \end{cases}
\end{array}$$

Let

$$A = A_N = \left(\begin{array}{ccccc} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & \cdots & 2 \end{array}\right) \quad \text{and} \quad q^8 \neq 1.$$

It is reminded that in this case, the diagonal matrix D is identity.

We introduce some new variables defined by Jimbo (see [36]) which generate $U_q(A_N)$:

$$\widetilde{X} = \{x_{ij}, 1 \leq i < j \leq N+1\},$$

where

$$x_{ij} = \begin{cases} x_i & j = i+1, \\ q x_{i,j-1} x_{j-1,j} - q^{-1} x_{j-1,j} x_{i,j-1} & j > i+1. \end{cases}$$

We now order the set $\widetilde{X}$ in the following way.

$$x_{mn} > x_{ij} \Longleftrightarrow (m,n) >_{lex} (i,j).$$

Let us recall from Yamane [36] the following notation:

$$
\begin{aligned}
C_1 &= \{((i,j),(m,n))|i=m<j<n\},\\
C_2 &= \{((i,j),(m,n))|i<m<n<j\},\\
C_3 &= \{((i,j),(m,n))|i<m<j=n\},\\
C_4 &= \{((i,j),(m,n))|i<m<j<n\},\\
C_5 &= \{((i,j),(m,n))|i<j=m<n\},\\
C_6 &= \{((i,j),(m,n))|i<j<m<n\}.
\end{aligned}
$$

Let the set $\widetilde{S}^{+}$ consist of Jimbo relations:

$$
\begin{aligned}
x_{mn}x_{ij} &- q^{-2}x_{ij}x_{mn} && ((i,j),(m,n))\in C_1\cup C_3,\\
x_{mn}x_{ij} &- x_{ij}x_{mn} && ((i,j),(m,n))\in C_2\cup C_6,\\
x_{mn}x_{ij} &- x_{ij}x_{mn}+(q^2-q^{-2})x_{in}x_{mj} && ((i,j),(m,n))\in C_4,\\
x_{mn}x_{ij} &- q^2x_{ij}x_{mn}+qx_{in} && ((i,j),(m,n))\in C_5.
\end{aligned}
$$

It is easily seen that $U_q^{+}(A_N)=k\langle \widetilde{X}|\widetilde{S^{+}}\rangle$.

The following theorem is from [16].

Theorem 3.1. *([16] Theorem 4.1) Let the notation be as before. Then, with the deg-lex order on* $\widetilde{X}^*$, $\widetilde{S}^{+}$ *is a Gröbner-Shirshov basis for* $k\langle \widetilde{X}|\widetilde{S^{+}}\rangle = U_q^{+}(A_N)$.

Proof. We will prove that all compositions in $\widetilde{S}^{+}$ are trivial modulo $\widetilde{S}^{+}$. We consider the following cases.

Case 1. $f=x_{mn}x_{ij}-q^{-2}x_{ij}x_{mn},\ g=x_{ij}x_{kl}-q^{-2}x_{kl}x_{ij},\ w=x_{mn}x_{ij}x_{kl}$.

In the case, we have

$$(f,g)_w=-q^{-2}x_{ij}x_{mn}x_{kl}+q^{-2}x_{mn}x_{kl}x_{ij}.$$

There are four subcases to consider.

	$((i,j),(m,n))\in C_1$	$((i,j),(m,n))\in C_3$
$((k,l),(i,j))\in C_1$	1.1. $((k,l),(m,n))\in C_1$	1.3. $((k,l),(m,n))\in C_4,C_5$ *or* C_6
$((k,l),(i,j))\in C_3$	1.2. $((k,l),(m,n))\in C_4$	1.4. $((k,l),(m,n))\in C_3$

1.1. $((i,j),(m,n))\in C_1,\ ((k,l),(i,j))\in C_1$ and $((k,l),(m,n))\in C_1$.

Then, we have

$$
\begin{aligned}
(f,g)_w &\equiv -q^{-4}x_{ij}x_{kl}x_{mn}+q^{-4}x_{kl}x_{mn}x_{ij}\\
&\equiv -q^{-6}x_{kl}x_{ij}x_{mn}+q^{-6}x_{kl}x_{ij}x_{mn}\\
&\equiv 0.
\end{aligned}
$$

1.2. $((i,j),(m,n)) \in C_1,\ ((k,l),(i,j)) \in C_3$ and $((k,l),(m,n)) \in C_4$.

Then, we have $(i,j) = (m,l),\ ((k,n),(i,j)) \in C_2$ and

$$\begin{aligned}(f,g)_w &\equiv -q^{-2}x_{ij}[x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}] + q^{-2}[x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}]x_{ij} \\ &\equiv -q^{-4}x_{kl}x_{ij}x_{mn} + q^{-2}(q^2 - q^{-2})x_{kn}x_{ij}x_{ml} \\ &\quad +q^{-4}x_{kl}x_{ij}x_{mn} - q^{-2}(q^2 - q^{-2})x_{kn}x_{ij}x_{ml} \\ &\equiv 0.\end{aligned}$$

1.3. $((i,j),(m,n)) \in C_3,\ ((k,l),(i,j)) \in C_1$ and $((k,l),(m,n)) \in C_4,\ C_5$ or C_6.

1.3.1. If $((k,l),(m,n)) \in C_4$ $(m < l)$, then $(k,n) = (i,j),\ ((i,j),(m,l)) \in C_2$ and

$$\begin{aligned}(f,g)_w &\equiv -q^{-2}x_{ij}[x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}] + q^{-2}[x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}]x_{ij} \\ &\equiv -q^{-4}x_{kl}x_{ij}x_{mn} + q^{-2}(q^2 - q^{-2})x_{ij}x_{kn}x_{ml} + q^{-4}x_{kl}x_{ij}x_{mn} \\ &\quad -q^{-2}(q^2 - q^{-2})x_{kn}x_{ij}x_{ml} \\ &\equiv 0.\end{aligned}$$

1.3.2. If $((k,l),(m,n)) \in C_5$ $(m = l)$, then $(k,n) = (i,j)$ and

$$\begin{aligned}(f,g)_w &\equiv -q^{-2}x_{ij}(q^2x_{kl}x_{mn} - qx_{kn}) + q^{-2}(q^2x_{kl}x_{mn} - qx_{kn})x_{ij} \\ &\equiv -x_{ij}x_{kl}x_{mn} + q^{-1}x_{ij}x_{kn} + x_{kl}x_{mn}x_{ij} - q^{-1}x_{kn}x_{ij} \\ &\equiv -q^{-2}x_{kl}x_{ij}x_{mn} + q^{-2}x_{kl}x_{ij}x_{mn} \\ &\equiv 0.\end{aligned}$$

1.3.3. If $((k,l),(m,n)) \in C_6$ $(m > l)$, then

$$\begin{aligned}(f,g)_w &\equiv -q^{-2}x_{ij}x_{kl}x_{mn} + q^{-2}x_{kl}x_{mn}x_{ij} \\ &\equiv -q^{-4}x_{kl}x_{ij}x_{mn} + q^{-4}x_{kl}x_{ij}x_{mn} \\ &\equiv 0.\end{aligned}$$

1.4. $((i,j),(m,n)) \in C_3,\ ((k,l),(i,j)) \in C_3$ and $((k,l),(m,n)) \in C_3$.

This case is similar to 1.1.

Case 2. $f = x_{mn}x_{ij} - q^{-2}x_{ij}x_{mn},\ g = x_{ij}x_{kl} - x_{kl}x_{ij},\ w = x_{mn}x_{ij}x_{kl}$.

In the case, we have

$$(f,g)_w = -q^{-2}x_{ij}x_{mn}x_{kl} + x_{mn}x_{kl}x_{ij}.$$

There are also four subcases to consider.

	$((i,j),(m,n)) \in C_1$	$((i,j),(m,n)) \in C_3$
$((k,l),(i,j)) \in C_2$	2.1. $((k,l),(m,n)) \in C_2, C_3$ *or* C_4	2.3. $((k,l),(m,n)) \in C_2$
$((k,l),(i,j)) \in C_6$	2.2. $((k,l),(m,n)) \in C_6$	2.4. $((k,l),(m,n)) \in C_6$

2.1. $((i,j),(m,n)) \in C_1,\ ((k,l),(i,j)) \in C_2$ and $((k,l),(m,n)) \in C_2, C_3$ or C_4.

2.1.1. If $((k,l),(m,n)) \in C_2\ (n < l)$, then

$$\begin{aligned}(f,g)_w &\equiv -q^{-2}x_{ij}x_{kl}x_{mn} + x_{kl}x_{mn}x_{ij}\\ &\equiv -q^{-2}x_{kl}x_{ij}x_{mn} + q^{-2}x_{kl}x_{ij}x_{mn}\\ &\equiv 0.\end{aligned}$$

2.1.2. If $((k,l),(m,n)) \in C_3\ (n = l)$, then

$$\begin{aligned}(f,g)_w &\equiv -q^{-4}x_{ij}x_{kl}x_{mn} + q^{-2}x_{kl}x_{mn}x_{ij}\\ &\equiv -q^{-4}x_{kl}x_{ij}x_{mn} + q^{-4}x_{kl}x_{ij}x_{mn}\\ &\equiv 0.\end{aligned}$$

2.1.3. If $((k,l),(m,n)) \in C_4\ (n > l)$, then $((k,n),(i,j)) \in C_2,\ ((i,j),(m,l)) \in C_1$ and

$$\begin{aligned}(f,g)_w &\equiv -q^{-2}x_{ij}[x_{kl}x_{mn} - (q^2-q^{-2})x_{kn}x_{ml}] + [x_{kl}x_{mn} - (q^2-q^{-2})x_{kn}x_{ml}]x_{ij}\\ &\equiv -q^{-2}x_{kl}x_{ij}x_{mn} + q^{-2}(q^2-q^{-2})x_{kn}x_{ij}x_{ml} + q^{-2}x_{kl}x_{ij}x_{mn}\\ &\quad -q^{-2}(q^2-q^{-2})x_{kn}x_{ij}x_{ml}\\ &\equiv 0.\end{aligned}$$

For the cases 2.2, 2.3 and 2.4, the proofs are similar to 2.1.1.

Case 3. $f = x_{mn}x_{ij} - q^{-2}x_{ij}x_{mn},\ g = x_{ij}x_{kl} - x_{kl}x_{ij} + (q^2-q^{-2})x_{kj}x_{il},\ w = x_{mn}x_{ij}x_{kl}$. In the case, we have

$$(f,g)_w = -q^{-2}x_{ij}x_{mn}x_{kl} + x_{mn}x_{kl}x_{ij} - (q^2-q^{-2})x_{mn}x_{kj}x_{il}.$$

There are two subcases to consider.

	$((i,j),(m,n)) \in C_1$	$((i,j),(m,n)) \in C_3$
$((k,l),(i,j)) \in C_4$	3.1. $((k,l),(m,n)),((k,j),(m,n)) \in C_4$	3.2. $((k,l),(m,n)) \in C_4, C_5$ *or* C_6 $((k,j),(m,n)) \in C_3$

3.1. $((i,j),(m,n)) \in C_1,\ ((k,l),(i,j)) \in C_4$ and $(k,l),(m,n)),((k,j),(m,n)) \in C_4$. Then, we have $((k,n),(i,j)) \in C_2,\ ((i,l),(m,n)) \in C_1,\ ((i,l),(m,j)) \in C_1,\ ((m,l),(i,j))$

$\in C_1$ and

$$\begin{aligned}
(f,g)_w &\equiv -q^{-2}x_{ij}[x_{kl}x_{mn}-(q^2-q^{-2})x_{kn}x_{ml}]+[x_{kl}x_{mn}-(q^2-q^{-2})x_{kn}x_{ml}]x_{ij}\\
&\quad -(q^2-q^{-2})[x_{kj}x_{mn}-(q^2-q^{-2})x_{kn}x_{mj}]x_{il}\\
&\equiv -q^{-2}[x_{kl}x_{ij}-(q^2-q^{-2})x_{kj}x_{il}]x_{mn}+q^{-2}(q^2-q^{-2})x_{kn}x_{ij}x_{ml}+q^{-2}x_{kl}x_{ij}x_{mn}\\
&\quad -(q^2-q^{-2})x_{kn}x_{ml}x_{ij}-q^{-2}(q^2-q^{-2})x_{kj}x_{il}x_{mn}+q^{-2}(q^2-q^{-2})^2x_{kn}x_{il}x_{mj}\\
&\equiv q^{-4}(q^2-q^{-2})x_{kn}x_{ml}x_{ij}-(q^2-q^{-2})x_{kn}x_{ml}x_{ij}+q^{-2}(q^2-q^{-2})x_{kn}x_{ml}x_{ij}\\
&\equiv 0.
\end{aligned}$$

3.2. $((i,j),(m,n))\in C_3,\ ((k,l),(i,j))\in C_4,\ (k,l),(m,n))\in C_4,C_5$ *or* C_6 and $((k,j),(m,n))\in C_3$.

3.2.1. If $((k,l),(m,n))\in C_4$ $(l>m)$ and $((k,j),(m,n))\in C_3$, then $((k,n),(i,j))\in C_3,\ ((i,j),(m,l))\in C_2,\ ((i,l),(m,n))\in C_4$ and

$$\begin{aligned}
(f,g)_w &\equiv -q^{-2}x_{ij}[x_{kl}x_{mn}-(q^2-q^{-2})x_{kn}x_{ml}]+[x_{kl}x_{mn}-(q^2-q^{-2})x_{kn}x_{ml}]x_{ij}\\
&\quad -q^{-2}(q^2-q^{-2})x_{kj}x_{mn}x_{il}\\
&\equiv -q^{-2}[x_{kl}x_{ij}-(q^2-q^{-2})x_{kj}x_{il}]x_{mn}+q^{-4}(q^2-q^{-2})x_{kn}x_{ij}x_{ml}+q^{-2}x_{kl}x_{ij}x_{mn}\\
&\quad -(q^2-q^{-2})x_{kn}x_{ij}x_{ml}-q^{-2}(q^2-q^{-2})x_{kj}[x_{il}x_{mn}-(q^2-q^{-2})x_{in}x_{ml}]\\
&\equiv 0.
\end{aligned}$$

3.2.2. If $((k,l),(m,n))\in C_5$ $(l=m)$ and $((k,j),(m,n))\in C_3$, then $((k,l),(i,j))\in C_4,\ ((k,n),(i,j))\in C_3,\ ((i,l),(m,n))\in C_5$ and

$$\begin{aligned}
(f,g)_w &\equiv -q^{-2}x_{ij}(q^2x_{kl}x_{mn}-qx_{kn})+(q^2x_{kl}x_{mn}-qx_{kn})x_{ij}-q^{-2}(q^2-q^{-2})x_{kj}x_{mn}x_{il}\\
&\equiv -[x_{kl}x_{ij}-(q^2-q^{-2})x_{kj}x_{il}]x_{mn}+q^{-3}x_{kn}x_{ij}+x_{kl}x_{ij}x_{mn}-qx_{kn}x_{ij}\\
&\quad -q^{-2}(q^2-q^{-2})x_{kj}[q^2x_{il}x_{mn}-qx_{in}]\\
&\equiv q^{-3}x_{kn}x_{ij}-qx_{kn}x_{ij}+q^{-1}(q^2-q^{-2})x_{kn}x_{ij}\\
&\equiv 0.
\end{aligned}$$

3.2.3. If $((k,l),(m,n))\in C_6$ $(l<m)$ and $((k,j),(m,n))\in C_3$, then $((i,l),(m,n))\in C_6$ and

$$\begin{aligned}
(f,g)_w &\equiv -q^{-2}x_{ij}x_{kl}x_{mn}+x_{kl}x_{mn}x_{ij}-q^{-2}(q^2-q^{-2})x_{kj}x_{mn}x_{il}\\
&\equiv -q^{-2}[x_{kl}x_{ij}-(q^2-q^{-2})x_{kj}x_{il}]x_{mn}+q^{-2}x_{kl}x_{ij}x_{mn}-q^{-2}(q^2-q^{-2})x_{kj}x_{il}x_{mn}\\
&\equiv 0.
\end{aligned}$$

Case 4. $f=x_{mn}x_{ij}-q^{-2}x_{ij}x_{mn},\ g=x_{ij}x_{kl}-q^2x_{kl}x_{ij}+qx_{kj},\ w=x_{mn}x_{ij}x_{kl}$.

In the case, we have

$$(f,g)_w=-q^{-2}x_{ij}x_{mn}x_{kl}+q^2x_{mn}x_{kl}x_{ij}-qx_{mn}x_{kj}.$$

There are two subcases to consider.

	$((i,j),(m,n)) \in C_1$	$((i,j),(m,n)) \in C_3$
$((k,l),(i,j)) \in C_5$	4.1. $((k,l),(m,n)) \in C_5$ $((k,j),(m,n)) \in C_4$	4.2. $((k,l),(m,n)) \in C_6$ $((k,j),(m,n)) \in C_3$

4.1. $((i,j),(m,n)) \in C_1$, $((k,l),(i,j)) \in C_5, ((k,l),(m,n)) \in C_5$ and $((k,j),(m,n)) \in C_4$. Then, we have $((k,n),(i,j)) \in C_2$ $(m = i)$ and

$$\begin{aligned}(f,g)_w &\equiv -q^{-2}x_{ij}(q^2x_{kl}x_{mn} - qx_{kn}) + q^2(q^2x_{kl}x_{mn} - qx_{kn})x_{ij} \\ &\quad -q[x_{kj}x_{mn} - (q^2 - q^{-2})x_{kn}x_{mj}] \\ &\equiv -(q^2x_{kl}x_{ij} - qx_{kj})x_{mn} + q^{-1}x_{kn}x_{ij} + q^2x_{kl}x_{ij}x_{mn} \\ &\quad -q^3x_{kn}x_{ij} - qx_{kj}x_{mn} + q(q^2 - q^{-2})x_{kn}x_{mj} \\ &\equiv 0.\end{aligned}$$

4.2. $((i,j),(m,n)) \in C_3$, $((k,l),(i,j)) \in C_5, ((k,l),(m,n)) \in C_6$ and $((k,j),(m,n)) \in C_3$. Then, we have

$$\begin{aligned}(f,g)_w &\equiv -q^{-2}x_{ij}x_{kl}x_{mn} + q^2x_{kl}x_{mn}x_{ij} - q^{-1}x_{kj}x_{mn} \\ &\equiv -q^{-2}(q^2x_{kl}x_{ij} - qx_{kj})x_{mn} + x_{kl}x_{ij}x_{mn} - q^{-1}x_{kj}x_{mn} \\ &\equiv 0.\end{aligned}$$

Case 5. $f = x_{mn}x_{ij} - x_{ij}x_{mn}$, $g = x_{ij}x_{kl} - q^{-2}x_{kl}x_{ij}$, $w = x_{mn}x_{ij}x_{kl}$.

In the case, we have

$$(f,g)_w = -x_{ij}x_{mn}x_{kl} + q^{-2}x_{mn}x_{kl}x_{ij}.$$

There are four subcases to consider.

	$((i,j),(m,n)) \in C_2$	$((i,j),(m,n)) \in C_6$
$((k,l),(i,j)) \in C_1$	5.1. $((k,l),(m,n)) \in C_2, C_3, C_4, C_5$ *or* C_6	5.3. $((k,l),(m,n)) \in C_6$
$((k,l),(i,j)) \in C_3$	5.2. $((k,l),(m,n)) \in C_2$	5.4. $((k,l),(m,n)) \in C_6$

5.1. $((i,j),(m,n)) \in C_2$, $((k,l),(i,j)) \in C_1$, and $((k,l),(m,n)) \in C_2, C_3, C_4, C_5$ or C_6.

5.1.1. If $((k,l),(m,n)) \in C_2$ $(l > n)$, then we have $((k,l),(i,j)) \in C_1$ and

$$\begin{aligned}(f,g)_w &\equiv -x_{ij}x_{kl}x_{mn} + q^{-2}x_{kl}x_{mn}x_{ij} \\ &\equiv -q^{-2}x_{kl}x_{ij}x_{mn} + q^{-2}x_{kl}x_{ij}x_{mn} \\ &\equiv 0.\end{aligned}$$

5.1.2. If $((k,l),(m,n)) \in C_3$ $(l = n)$, then

$$\begin{aligned}(f,g)_w &\equiv -q^{-2}x_{ij}x_{kl}x_{mn} + q^{-4}x_{kl}x_{mn}x_{ij}\\ &\equiv -q^{-4}x_{kl}x_{ij}x_{mn} + q^{-4}x_{kl}x_{ij}x_{mn}\\ &\equiv 0.\end{aligned}$$

5.1.3. If $((k,l),(m,n)) \in C_4$ $(m < l < n)$, then we have $((k,l),(i,j)) \in C_1$, $((k,n),(i,j)) \in C_1$, $((i,j),(m,l)) \in C_2$ and

$$\begin{aligned}(f,g)_w &\equiv -x_{ij}[x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}] + q^{-2}[x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}]x_{ij}\\ &\equiv -x_{ij}x_{kl}x_{mn} + (q^2 - q^{-2})x_{ij}x_{kn}x_{ml} + q^{-2}x_{kl}x_{mn}x_{ij} - q^{-2}(q^2 - q^{-2})x_{kn}x_{ml}x_{ij}\\ &\equiv -q^{-2}x_{kl}x_{ij}x_{mn} + q^{-2}(q^2 - q^{-2})x_{kn}x_{ij}x_{ml} + q^{-2}x_{kl}x_{ij}x_{mn}\\ &\quad -q^{-2}(q^2 - q^{-2})x_{kn}x_{ij}x_{ml}\\ &\equiv 0.\end{aligned}$$

5.1.4. If $((k,l),(m,n)) \in C_5$ $(m = l)$, then we have $((k,n),(i,j)) \in C_1$ and

$$\begin{aligned}(f,g)_w &\equiv -x_{ij}(q^2x_{kl}x_{mn} - qx_{kn}) + q^{-2}(q^2x_{kl}x_{mn} - qx_{kn})x_{ij}\\ &\equiv -q^2x_{ij}x_{kl}x_{mn} + qx_{ij}x_{kn} + x_{kl}x_{mn}x_{ij} - q^{-1}x_{kn}x_{ij}\\ &\equiv -x_{kl}x_{ij}x_{mn} + q^{-1}x_{kn}x_{ij} + x_{kl}x_{ij}x_{mn} - q^{-1}x_{kn}x_{ij}\\ &\equiv 0.\end{aligned}$$

5.1.5. If $((k,l),(m,n)) \in C_6$ $(l < m)$, the proof is similar to 5.1.1.

For the cases of 5.2, 5.3 and 5.4, the proofs are also similar to 5.1.1.

Case 6. $f = x_{mn}x_{ij} - x_{ij}x_{mn}$, $g = x_{ij}x_{kl} - x_{kl}x_{ij}$, $w = x_{mn}x_{ij}x_{kl}$.

In the case, we have

$$(f,g)_w = -x_{ij}x_{mn}x_{kl} + x_{mn}x_{kl}x_{ij}.$$

There are four subcases to consider.

	$((i,j),(m,n)) \in C_2$	$((i,j),(m,n)) \in C_6$
$((k,l),(i,j)) \in C_2$	6.1. $((k,l),(m,n)) \in C_2$	6.3. $((k,l),(m,n)) \in C_2, C_3, C_4, C_5$ *or* C_6
$((k,l),(i,j)) \in C_6$	6.2. $((k,l),(m,n)) \in C_6$	6.4. $((k,l),(m,n)) \in C_6$

6.1. $((i,j),(m,n)) \in C_2$, $((k,l),(i,j)) \in C_2$ and $((k,l),(m,n)) \in C_2$.

Then, we have

$$\begin{aligned}(f,g)_w &\equiv -x_{ij}x_{kl}x_{mn} + x_{kl}x_{mn}x_{ij}\\ &\equiv -x_{kl}x_{ij}x_{mn} + x_{kl}x_{ij}x_{mn}\\ &\equiv 0.\end{aligned}$$

6.2. $((i,j),(m,n)) \in C_2,\ ((k,l),(i,j)) \in C_6$ and $((k,l),(m,n)) \in C_6$.

This case is similar to 6.1.

6.3. $((i,j),(m,n)) \in C_6,\ ((k,l),(i,j)) \in C_2$ and $((k,l),(m,n)) \in C_2,, C_3, C_4, C_5$ or C_6.

6.3.1. If $((k,l),(m,n)) \in C_2\ (l > n)$, the proof is similar to 6.1.

6.3.2. If $((k,l),(m,n)) \in C_3\ (l = n)$, then

$$\begin{aligned}(f,g)_w &\equiv -q^{-2}x_{ij}x_{kl}x_{mn} + q^{-2}x_{kl}x_{mn}x_{ij}\\ &\equiv -q^{-2}x_{kl}x_{ij}x_{mn} + q^{-2}x_{kl}x_{ij}x_{mn}\\ &\equiv 0.\end{aligned}$$

6.3.3. If $((k,l),(m,n)) \in C_4\ (m < l < n)$, then we have $((k,n),(i,j)) \in C_2,\ ((i,j),(m,n)) \in C_6$ and

$$\begin{aligned}(f,g)_w &\equiv -x_{ij}[x_{kl}x_{mn} - (q^2-q^{-2})x_{kn}x_{ml}] + [x_{kl}x_{mn} - (q^2-q^{-2})x_{kn}x_{ml}]x_{ij}\\ &\equiv -x_{ij}x_{kl}x_{mn} + (q^2-q^{-2})x_{ij}x_{kn}x_{ml} + x_{kl}x_{mn}x_{ij} - (q^2-q^{-2})x_{kn}x_{ml}x_{ij}\\ &\equiv -x_{kl}x_{ij}x_{mn} + (q^2-q^{-2})x_{kn}x_{ij}x_{ml} + x_{kl}x_{ij}x_{mn} - (q^2-q^{-2})x_{kn}x_{ij}x_{ml}\\ &\equiv 0.\end{aligned}$$

6.3.4. If $((k,l),(m,n)) \in C_5\ (m = l)$, then we have $((k,n),(i,j)) \in C_2$ and

$$\begin{aligned}(f,g)_w &\equiv -x_{ij}(q^2x_{kl}x_{mn} - qx_{kn}) + (q^2x_{kl}x_{mn} - qx_{kn})x_{ij}\\ &\equiv -q^2x_{ij}x_{kl}x_{mn} + qx_{ij}x_{kn} + q^2x_{kl}x_{mn}x_{ij} - qx_{kn}x_{ij}\\ &\equiv -q^2x_{kl}x_{ij}x_{mn} + qx_{kn}x_{ij} + q^2x_{kl}x_{ij}x_{mn} - qx_{kn}x_{ij}\\ &\equiv 0.\end{aligned}$$

6.3.5. If $((k,l),(m,n)) \in C_6\ (l < m)$, the proof is similar to 6.1.

6.4. $((i,j),(m,n)) \in C_6,\ ((k,l),(i,j)) \in C_6$ and $((k,l),(m,n)) \in C_6$.

This case is also similar to 6.1.

Case 7. $f = x_{mn}x_{ij} - x_{ij}x_{mn},\ g = x_{ij}x_{kl} - x_{kl}x_{ij} + (q^2-q^{-2})x_{kj}x_{il},\ w = x_{mn}x_{ij}x_{kl}$.

In the case, we have

$$(f,g)_w = -x_{ij}x_{mn}x_{kl} + x_{mn}x_{kl}x_{ij} - (q^2-q^{-2})x_{mn}x_{kj}x_{il}.$$

There are two subcases to consider.

	$((i,j),(m,n)) \in C_2$	$((i,j),(m,n)) \in C_6$
$((k,l),(i,j)) \in C_4$	7.1. $((k,l),(m,n)) \in C_2, C_3, C_4, C_5$ *or* C_6 $((k,j),(m,n)) \in C_2$	7.2. $((k,l),(m,n))$, $((k,j),(m,n)) \in C_6$

7.1. $((i,j),(m,n)) \in C_2$, $((k,l),(i,j)) \in C_4$, $((k,l),(m,n)) \in C_2, C_3, C_4, C_5$ or C_6 and $((k,j),(m,n)) \in C_2$.

7.1.1. If $((k,l),(m,n)) \in C_2$ $(n < l)$ and $((k,j),(m,n)) \in C_2$, then we have $((i,l),(m,n)) \in C_2$ and

$$\begin{aligned}(f,g)_w &\equiv -x_{ij}x_{kl}x_{mn} + x_{kl}x_{mn}x_{ij} - (q^2-q^{-2})x_{kj}x_{mn}x_{il}\\ &\equiv -[x_{kl}x_{ij} - (q^2-q^{-2})x_{kj}x_{il}]x_{mn} + x_{kl}x_{ij}x_{mn} - (q^2-q^{-2})x_{kj}x_{il}x_{mn}\\ &\equiv 0.\end{aligned}$$

7.1.2. If $((k,l),(m,n)) \in C_3$ $(n = l)$ and $((k,j),(m,n)) \in C_2$, then $((i,l),(m,n)) \in C_3$ and

$$\begin{aligned}(f,g)_w &\equiv -q^{-2}x_{ij}x_{kl}x_{mn} + q^{-2}x_{kl}x_{mn}x_{ij} - (q^2-q^{-2})x_{kj}x_{mn}x_{il}\\ &\equiv -q^{-2}[x_{kl}x_{ij} - (q^2-q^{-2})x_{kj}x_{il}]x_{mn} + q^{-2}x_{kl}x_{ij}x_{mn} - q^{-2}(q^2-q^{-2})x_{kj}x_{il}x_{mn}\\ &\equiv 0.\end{aligned}$$

7.1.3. If $((k,l),(m,n)) \in C_4$ $(m < l < n)$ and $((k,j),(m,n)) \in C_2$, then we obtain $((k,n),(i,j)) \in C_4$, $((i,j),(m,l)) \in C_2$, $((i,l),(m,n)) \in C_4$ and

$$\begin{aligned}(f,g)_w &\equiv -x_{ij}[x_{kl}x_{mn} - (q^2-q^{-2})x_{kn}x_{ml}] + [x_{kl}x_{mn} - (q^2-q^{-2})x_{kn}x_{ml}]x_{ij}\\ &\quad -(q^2-q^{-2})x_{kj}x_{mn}x_{il}\\ &\equiv -x_{ij}x_{kl}x_{mn} + (q^2-q^{-2})x_{ij}x_{kn}x_{ml} + x_{kl}x_{mn}x_{ij} - (q^2-q^{-2})x_{kn}x_{ml}x_{ij}\\ &\quad -(q^2-q^{-2})x_{kj}[x_{il}x_{mn} - (q^2-q^{-2})x_{in}x_{ml}]\\ &\equiv -[x_{kl}x_{ij} - (q^2-q^{-2})x_{kj}x_{il}]x_{mn} + (q^2-q^{-2})[x_{kn}x_{ij} - (q^2-q^{-2})x_{kj}x_{in}]x_{ml}\\ &\quad +x_{kl}x_{ij}x_{mn} - (q^2-q^{-2})x_{kn}x_{ij}x_{ml}\\ &\equiv 0.\end{aligned}$$

7.1.4. If $((k,l),(m,n)) \in C_5$ $(m = l)$ and $((k,j),(m,n)) \in C_2$, then $((k,n),(i,j)) \in C_4$, $((i,l),(m,n)) \in C_5$ and

$$\begin{aligned}(f,g)_w &\equiv -x_{ij}(q^2x_{kl}x_{mn} - qx_{kn}) + (q^2x_{kl}x_{mn} - qx_{kn})x_{ij} - (q^2-q^{-2})x_{kj}x_{mn}x_{il}\\ &\equiv -q^2x_{ij}x_{kl}x_{mn} + qx_{ij}x_{kn} + q^2x_{kl}x_{mn}x_{ij} - qx_{kn}x_{ij}\\ &\quad -(q^2-q^{-2})x_{kj}(q^2x_{il}x_{mn} - qx_{in})\\ &\equiv -q^2[x_{kl}x_{ij} - (q^2-q^{-2})x_{kj}x_{il}]x_{mn} + q[x_{kn}x_{ij} - (q^2-q^{-2})x_{kj}x_{in}]\\ &\quad +q^2x_{kl}x_{ij}x_{mn} - qx_{kn}x_{ij} - q^2(q^2-q^{-2})x_{kj}x_{il}x_{mn} + q(q^2-q^{-2})x_{kj}x_{in}\\ &\equiv 0.\end{aligned}$$

7.1.5. If $((k,l),(m,n)) \in C_6$ $(l < m)$ and $((k,j),(m,n)) \in C_2$, then $((i,l),(m,n)) \in C_6$. This case is similar to 7.1.1.

7.2. $((i,j),(m,n)) \in C_6,\ ((k,l),(i,j)) \in C_4,\ ((k,l),(m,n)),((k,j),(m,n)) \in C_6.$

This case is also similar to 7.1.1.

Case 8. $f = x_{mn}x_{ij} - x_{ij}x_{mn},\ g = x_{ij}x_{kl} - q^2 x_{kl}x_{ij} + q x_{kj},\ w = x_{mn}x_{ij}x_{kl}.$

In the case, we have

$$(f,g)_w = -x_{ij}x_{mn}x_{kl} + q^2 x_{mn}x_{kl}x_{ij} + q x_{mn}x_{kj}.$$

There are two subcases to consider.

	$((i,j),(m,n)) \in C_2$	$((i,j),(m,n)) \in C_6$
$((k,l),(i,j)) \in C_5$	8.1. $((k,l),(m,n)) \in C_6$ $((k,j),(m,n)) \in C_2$	8.2. $((k,l),(m,n)),((k,j),(m,n)) \in C_6$

8.1. $((i,j),(m,n)) \in C_2,\ ((k,l),(i,j)) \in C_5,\ ((k,l),(m,n)) \in C_6$ and $((k,j),(m,n)) \in C_2$. Then, we have

$$\begin{aligned}(f,g)_w &\equiv -x_{ij}x_{kl}x_{mn} + q^2 x_{kl}x_{mn}x_{ij} + q x_{kj}x_{mn}\\ &\equiv -(q^2 x_{kl}x_{ij} - q x_{kj})x_{mn} + q^2 x_{kl}x_{ij}x_{mn} + q x_{kj}x_{mn}\\ &\equiv 0.\end{aligned}$$

8.2. $((i,j),(m,n)) \in C_6,\ ((k,l),(i,j)) \in C_5,\ ((k,l),(m,n)),((k,j),(m,n)) \in C_6.$

This case is similar to 8.1.

Case 9. $f = x_{mn}x_{ij} - x_{ij}x_{mn} + (q^2 - q^{-2})x_{in}x_{mj},\ g = x_{ij}x_{kl} - q^{-2}x_{kl}x_{ij},\ w = x_{mn}x_{ij}x_{kl}.$

In the case, we have

$$(f,g)_w = -x_{ij}x_{mn}x_{kl} + (q^2 - q^{-2})x_{in}x_{mj}x_{kl} + q^{-2}x_{mn}x_{kl}x_{ij}.$$

There are two subcases to consider.

	$((i,j),(m,n)) \in C_4$
$((k,l),(i,j)) \in C_1$	9.1. $((k,l),(m,n)),((k,l),(m,j)) \in C_4,\ C_5$ *or* C_6
$((k,l),(i,j)) \in C_3$	9.2. $((k,l),(m,n)) \in C_4\ \ ((k,l),(m,j)) \in C_3$

9.1. $((i,j),(m,n)) \in C_4,\ ((k,l),(i,j)) \in C_1$ and $((k,l),(m,n)),((k,l),(m,j)) \in C_4,\ C_5$ or C_6.

9.1.1. If $((k,l),(m,n)),((k,l),(m,j)) \in C_4$ $(l > m)$, then we have $((i,j),(k,n)) \in C_1,$ $((k,n),(m,l)) \in C_2,$ $((k,j),(i,n)) \in C_1,$ $((k,l),(i,n)) \in C_1,$ $((i,j),(m,l)) \in C_2$ and

$$\begin{aligned}
(f,g)_w &\equiv -x_{ij}[x_{kl}x_{mn} - (q^2-q^{-2})x_{kn}x_{ml}] + (q^2-q^{-2})x_{in}[x_{kl}x_{mj} - (q^2-q^{-2})x_{kj}x_{ml}] \\
&\quad + q^{-2}[x_{kl}x_{mn} - (q^2-q^{-2})x_{kn}x_{ml}]x_{ij} \\
&\equiv -x_{ij}x_{kl}x_{mn} + (q^2-q^{-2})x_{ij}x_{kn}x_{ml} + (q^2-q^{-2})x_{in}x_{kl}x_{mj} \\
&\quad -(q^2-q^{-2})^2x_{in}x_{kj}x_{ml} + q^{-2}x_{kl}x_{mn}x_{ij} - q^{-2}(q^2-q^{-2})x_{kn}x_{ml}x_{ij} \\
&\equiv -q^{-2}x_{kl}x_{ij}x_{mn} + (q^2-q^{-2})x_{ij}x_{kn}x_{ml} + q^{-2}(q^2-q^{-2})x_{kl}x_{in}x_{mj} \\
&\quad -q^{-2}(q^2-q^{-2})^2x_{kj}x_{in}x_{ml} + q^{-2}x_{kl}[x_{ij}x_{mn} - (q^2-q^{-2})x_{in}x_{mj}] \\
&\quad -q^{-2}(q^2-q^{-2})x_{kn}x_{ij}x_{ml} \\
&\equiv (q^2-q^{-2})x_{ij}x_{kn}x_{ml} - q^{-2}(q^2-q^{-2})^2x_{kj}x_{in}x_{ml} - q^{-4}(q^2-q^{-2})x_{ij}x_{kn}x_{ml} \\
&\equiv 0.
\end{aligned}$$

9.1.2. If $((k,l),(m,n)),((k,l),(m,j)) \in C_5$ $(l = m)$, then we have $((i,j),(k,n)) \in C_1,$ $((k,l),(i,n),((k,j),(i,n)) \in C_1$ and

$$\begin{aligned}
(f,g)_w &\equiv -x_{ij}(q^2x_{kl}x_{mn} - qx_{kn}) + (q^2-q^{-2})x_{in}(q^2x_{kl}x_{mj} - qx_{kj}) \\
&\quad + q^{-2}(q^2x_{kl}x_{mn} - qx_{kn})x_{ij} \\
&\equiv -q^2x_{ij}x_{kl}x_{mn} + qx_{ij}x_{kn} + q^2(q^2-q^{-2})x_{in}x_{kl}x_{mj} - q(q^2-q^{-2})x_{in}x_{kj} \\
&\quad + x_{kl}x_{mn}x_{ij} - q^{-1}x_{kn}x_{ij} \\
&\equiv -x_{kl}x_{ij}x_{mn} + qx_{ij}x_{kn} + (q^2-q^{-2})x_{kl}x_{in}x_{mj} - q^{-1}(q^2-q^{-2})x_{kj}x_{in} \\
&\quad + x_{kl}[x_{ij}x_{mn} - (q^2-q^{-2})x_{in}x_{mj}] - q^{-3}x_{ij}x_{kn} \\
&\equiv qx_{ij}x_{kn} - qx_{kj}x_{in} + q^{-3}x_{kj}x_{in} - q^{-3}x_{ij}x_{kn} \\
&\equiv 0.
\end{aligned}$$

9.1.3. If $((k,l),(m,n)),((k,l),(m,j)) \in C_6$ $(l < m)$, then we have $((k,l),(i,n)) \in C_1$ and

$$\begin{aligned}
(f,g)_w &\equiv -x_{ij}x_{kl}x_{mn} - (q^2-q^{-2})x_{in}x_{kl}x_{mj} + q^{-2}x_{kl}x_{mn}x_{ij} \\
&\equiv -q^{-2}x_{kl}x_{ij}x_{mn} - q^{-2}(q^2-q^{-2})x_{kl}x_{in}x_{mj} + q^{-2}x_{kl}[x_{ij}x_{mn} - (q^2-q^{-2})x_{in}x_{mj}] \\
&\equiv 0.
\end{aligned}$$

9.2. $((i,j),(m,n)) \in C_4,$ $((k,l),(i,j)) \in C_3,$ $((k,l),(m,n)) \in C_4$ and $((k,l),(m,j)) \in C_3.$

Then, we have $((k,n),(i,j)) \in C_2,\ ((k,l),(i,n)) \in C_4,\ ((i,j),(m,l)) \in C_3$ and

$$\begin{aligned}
(f,g)_w &\equiv -x_{ij}[x_{kl}x_{mn} - (q^2-q^{-2})x_{kn}x_{ml}] + q^{-2}(q^2-q^{-2})x_{in}x_{kl}x_{mj} \\
&\quad + q^{-2}[x_{kl}x_{mn} - (q^2-q^{-2})x_{kn}x_{ml}]x_{ij} \\
&\equiv -x_{ij}x_{kl}x_{mn} + (q^2-q^{-2})x_{ij}x_{kn}x_{ml} + q^{-2}(q^2-q^{-2})x_{in}x_{kl}x_{mj} + q^{-2}x_{kl}x_{mn}x_{ij} \\
&\quad - q^{-2}(q^2-q^{-2})x_{kn}x_{ml}x_{ij} \\
&\equiv -q^{-2}x_{kl}x_{ij}x_{mn} + (q^2-q^{-2})x_{kn}x_{ij}x_{ml} + q^{-2}(q^2-q^{-2})[x_{kl}x_{in} \\
&\quad - (q^2-q^{-2})x_{kn}x_{il}]x_{mj} + q^{-2}x_{kl}[x_{ij}x_{mn} - (q^2-q^{-2})x_{in}x_{mj}] \\
&\quad - q^{-4}(q^2-q^{-2})x_{kn}x_{ij}x_{ml} \\
&\equiv (q^2-q^{-2})x_{kn}x_{ij}x_{ml} - q^{-2}(q^2-q^{-2})x_{kn}x_{il}x_{mj} - q^{-4}(q^2-q^{-2})x_{kn}x_{ij}x_{ml} \\
&\equiv 0.
\end{aligned}$$

Case 10. $f = x_{mn}x_{ij} - x_{ij}x_{mn} + (q^2-q^{-2})x_{in}x_{mj},\ g = x_{ij}x_{kl} - x_{kl}x_{ij},\ w = x_{mn}x_{ij}x_{kl}$.

In the case, we have

$$(f,g)_w = -x_{ij}x_{mn}x_{kl} + (q^2-q^{-2})x_{in}x_{mj}x_{kl} + x_{mn}x_{kl}x_{ij}.$$

There are two subcases to consider.

	$((i,j),(m,n)) \in C_4$
$((k,l),(i,j)) \in C_2$	10.1. $((k,l),(m,n)) \in C_2, C_3$ *or* C_4 $((k,l),(m,j)) \in C_2$
$((k,l),(i,j)) \in C_6$	10.2. $((k,l),(m,n)), ((k,l),(m,j)) \in C_6$

10.1. $((i,j),(m,n)) \in C_4,\ ((k,l),(i,j)) \in C_2,\ ((k,l),(m,n)) \in C_2, C_3$ or C_4 and $((k,l),(m,j)) \in C_2$.

10.1.1. If $((k,l),(m,n)) \in C_2$ $(l > n)$, then we have $((k,l),(i,n)) \in C_2$ and

$$\begin{aligned}
(f,g)_w &\equiv -x_{ij}x_{kl}x_{mn} + (q^2-q^{-2})x_{in}x_{kl}x_{mj} + x_{kl}x_{mn}x_{ij} \\
&\equiv -x_{kl}x_{ij}x_{mn} + (q^2-q^{-2})x_{kl}x_{in}x_{mj} + x_{kl}[x_{ij}x_{mn} - (q^2-q^{-2})x_{in}x_{mj}] \\
&\equiv 0.
\end{aligned}$$

10.1.2. If $((k,l),(m,n)) \in C_3$ $(l = n)$, then we have $((k,l),(i,n)) \in C_3$ and

$$\begin{aligned}
(f,g)_w &\equiv -q^{-2}x_{ij}x_{kl}x_{mn} + (q^2-q^{-2})x_{in}x_{kl}x_{mj} + q^{-2}x_{kl}x_{mn}x_{ij} \\
&\equiv -q^{-2}x_{kl}x_{ij}x_{mn} + q^{-2}(q^2-q^{-2})x_{kl}x_{in}x_{mj} + q^{-2}x_{kl}[x_{ij}x_{mn} - (q^2-q^{-2})x_{in}x_{mj}] \\
&\equiv 0.
\end{aligned}$$

10.1.3. If $((k,l),(m,n)) \in C_4$ $(l < n)$, then we have $((k,n),(i,j)) \in C_2$, $((k,l),(i,n)) \in C_4$, $((i,j),(m,l)) \in C_4$ and

$$\begin{aligned}
(f,g)_w &\equiv -x_{ij}[x_{kl}x_{mn} - (q^2-q^{-2})x_{kn}x_{ml}] + (q^2-q^{-2})x_{in}x_{kl}x_{mj} \\
&\quad +[x_{kl}x_{mn} - (q^2-q^{-2})x_{kn}x_{ml}]x_{ij} \\
&\equiv -x_{ij}x_{kl}x_{mn} + (q^2-q^{-2})x_{ij}x_{kn}x_{ml} + (q^2-q^{-2})x_{in}x_{kl}x_{mj} \\
&\quad +x_{kl}x_{mn}x_{ij} - (q^2-q^{-2})x_{kn}x_{ml}x_{ij} \\
&\equiv -x_{kl}x_{ij}x_{mn} + (q^2-q^{-2})x_{kn}x_{ij}x_{ml} + (q^2-q^{-2})[x_{kl}x_{in} - (q^2-q^{-2})x_{kn}x_{il}]x_{mj} \\
&\quad +x_{kl}[x_{ij}x_{mn} - (q^2-q^{-2})x_{in}x_{mj}] - (q^2-q^{-2})x_{kn}[x_{ij}x_{ml} - (q^2-q^{-2})x_{il}x_{mj}] \\
&\equiv 0.
\end{aligned}$$

10.2. $((i,j),(m,n)) \in C_4$, $((k,l),(i,j)) \in C_6$, $((k,l),(m,n)),(k,l),(m,j)) \in C_6$.

This case is similar to 10.1.

Case 11. $f = x_{mn}x_{ij} - x_{ij}x_{mn} + (q^2-q^{-2})x_{in}x_{mj}$, $g = x_{ij}x_{kl} - x_{kl}x_{ij} + (q^2-q^{-2})x_{kj}x_{il}$, $w = x_{mn}x_{ij}x_{kl}$.

In the case, we have

$$(f,g)_w = -x_{ij}x_{mn}x_{kl} + (q^2-q^{-2})x_{in}x_{mj}x_{kl} + x_{mn}x_{kl}x_{ij} - (q^2-q^{-2})x_{mn}x_{kj}x_{il},$$

with

	$((i,j),(m,n)) \in C_4$
$((k,l),(i,j)) \in C_4$	$((k,l),(m,n)),((k,l),(m,j)) \in C_4, C_5$ *or* C_6

11.1. If $((k,l),(m,n))$, $((k,l),(m,j)) \in C_4$ $(l > m)$, then we have $((k,n),(i,j)) \in C_2$, $((k,l),(i,n)) \in C_4$, $((k,j),(i,n)) \in C_4$, $((i,j),(m,l)) \in C_2$, $((i,l),(m,n)) \in C_4$, $((i,l),(m,j)) \in C_4$ and

$$\begin{aligned}
(f,g)_w &\equiv -x_{ij}[x_{kl}x_{mn} - (q^2-q^{-2})x_{kn}x_{ml}] + (q^2-q^{-2})x_{in}[x_{kl}x_{mj} - (q^2-q^{-2})x_{kj}x_{ml}] \\
&\quad +[x_{kl}x_{mn} - (q^2-q^{-2})x_{kn}x_{ml}]x_{ij} - (q^2-q^{-2})[x_{kj}x_{mn} - (q^2-q^{-2})x_{kn}x_{mj}]x_{il} \\
&\equiv -x_{ij}x_{kl}x_{mn} + (q^2-q^{-2})x_{ij}x_{kn}x_{ml} + (q^2-q^{-2})x_{in}x_{kl}x_{mj} \\
&\quad -(q^2-q^{-2})x_{in}x_{kj}x_{ml} + x_{kl}x_{mn}x_{ij} - (q^2-q^{-2})x_{kn}x_{ml}x_{ij} \\
&\quad -(q^2-q^{-2})x_{kj}x_{mn}x_{il} + (q^2-q^{-2})^2x_{kn}x_{mj}x_{il} \\
&\equiv -[x_{kl}x_{ij} - (q^2-q^{-2})x_{kj}x_{il}]x_{mn} + (q^2-q^{-2})x_{kn}x_{ij}x_{ml} + (q^2-q^{-2})[x_{kj}x_{in} \\
&\quad -(q^2-q^{-2})x_{kn}x_{il}]x_{mj} - (q^2-q^{-2})[x_{kj}x_{in} - (q^2-q^{-2})x_{kn}x_{ij}]x_{ml} \\
&\quad +x_{kl}[x_{ij}x_{mn} - (q^2-q^{-2})x_{in}x_{mj}] - (q^2-q^{-2})x_{kn}x_{ij}x_{ml} \\
&\quad -(q^2-q^{-2})x_{kj}[x_{il}x_{mn} - (q^2-q^{-2})x_{in}x_{ml}] \\
&\quad +(q^2-q^{-2})^2x_{kn}[x_{il}x_{mj} - (q^2-q^{-2})x_{ij}x_{ml}] \\
&\equiv 0.
\end{aligned}$$

11.2. If $((k,l),(m,n))$, $((k,l),(m,j)) \in C_5$ $(l = m)$, then we have $((k,n),(i,j)) \in C_2$, $((k,l),(i,n)) \in C_4$, $((k,j),(i,n)) \in C_4$, $((i,l),(m,n)) \in C_5$, $((i,l),(m,j)) \in C_5$ and

$$\begin{aligned}
(f,g)_w &\equiv -x_{ij}(q^2x_{kl}x_{mn} - qx_{kn}) + (q^2 - q^{-2})x_{in}(q^2x_{kl}x_{mj} - qx_{kj}) + (q^2x_{kl}x_{mn} - qx_{kn})x_{ij} \\
&\quad -(q^2 - q^{-2})[x_{kj}x_{mn} - (q^2 - q^{-2})x_{kn}x_{mj}]x_{il} \\
&\equiv -q^2x_{ij}x_{kl}x_{mn} + qx_{ij}x_{kn} + q^2(q^2 - q^{-2})x_{in}x_{kl}x_{mj} - q(q^2 - q^{-2})x_{in}x_{kj} \\
&\quad +q^2x_{kl}x_{mn}x_{ij} - qx_{kn}x_{ij} - (q^2 - q^{-2})x_{kj}x_{mn}x_{il} + (q^2 - q^{-2})^2x_{kn}x_{mj}x_{il} \\
&\equiv -q^2[x_{kl}x_{ij} - (q^2 - q^{-2})x_{kj}x_{il}]x_{mn} + qx_{kn}x_{ij} + q^2(q^2 - q^{-2})[x_{kl}x_{in} \\
&\quad -(q^2 - q^{-2})x_{kn}x_{il}]x_{mj} - q(q^2 - q^{-2})[x_{kj}x_{in} - (q^2 - q^{-2})x_{kn}x_{ij}] \\
&\quad +q^2x_{kl}[x_{ij}x_{mn} - (q^2 - q^{-2})x_{in}x_{mj}] - qx_{kn}x_{ij} \\
&\quad -(q^2 - q^{-2})x_{kj}[q^2x_{il}x_{mn} - qx_{in}] + (q^2 - q^{-2})^2x_{kn}[q^2x_{il}x_{mj} - qx_{ij}] \\
&\equiv 0.
\end{aligned}$$

11.3. If $((k,l),(m,n))$, $((k,l),(m,j)) \in C_6$ $(l < m)$, then $((k,j),(m,n)) \in C_4$, $((k,l),(i,n)) \in C_4$, $((i,l),(m,n)),((i,l),(m,j)) \in C_6$ and

$$\begin{aligned}
(f,g)_w &\equiv -x_{ij}x_{kl}x_{mn} + (q^2 - q^{-2})x_{in}x_{kl}x_{mj} + x_{kl}x_{mn}x_{ij} \\
&\quad -(q^2 - q^{-2})[x_{kj}x_{mn} - (q^2 - q^{-2})x_{kn}x_{mj}]x_{il} \\
&\equiv -[x_{kl}x_{ij} - (q^2 - q^{-2})x_{kj}x_{il}]x_{mn} + (q^2 - q^{-2})[x_{kl}x_{in} - (q^2 - q^{-2})x_{kn}x_{il}]x_{mj} \\
&\quad +x_{kl}[x_{ij}x_{mn} - (q^2 - q^{-2})x_{in}x_{mj}] - (q^2 - q^{-2})x_{kj}x_{il}x_{mn} + (q^2 - q^{-2})^2x_{kn}x_{il}x_{mj} \\
&\equiv 0.
\end{aligned}$$

Case 12. $f = x_{mn}x_{ij} - x_{ij}x_{mn} + (q^2 - q^{-2})x_{in}x_{mj}$, $g = x_{ij}x_{kl} - q^2x_{kl}x_{ij} + qx_{kj}$, $w = x_{mn}x_{ij}x_{kl}$, with

	$((i,j),(m,n)) \in C_4$
$((k,l),(i,j)) \in C_5$	$((k,l),(m,n)),((k,l),(m,j)) \in C_6$ $((k,j),(m,n)) \in C_4$ $((k,l),(i,n)) \in C_5$

In the case, we can deduce that

$$\begin{aligned}
(f,g)_w &= -x_{ij}x_{mn}x_{kl} + (q^2 - q^{-2})x_{in}x_{mj}x_{kl} + q^2x_{mn}x_{kl}x_{ij} - qx_{mn}x_{kj} \\
&\equiv -x_{ij}x_{kl}x_{mn} + (q^2 - q^{-2})x_{in}x_{kl}x_{mj} + q^2x_{kj}x_{mn}x_{ij} \\
&\quad -q[x_{kj}x_{mn} - (q^2 - q^{-2})x_{kn}x_{mj}] \\
&\equiv -(q^2x_{kl}x_{ij} - qx_{kj})x_{mn} + (q^2 - q^{-2})(q^2x_{kl}x_{in} - qx_{kn})x_{mj} \\
&\quad +q^2x_{kl}[x_{ij}x_{mn} - (q^2 - q^{-2})x_{in}x_{mj}] - qx_{kj}x_{mn} + q(q^2 - q^{-2})x_{kn}x_{mj} \\
&\equiv -q^2x_{kl}x_{ij}x_{mn} + qx_{kj}x_{mn} + q^2(q^2 - q^{-2})x_{kl}x_{in}x_{mj} - q(q^2 - q^{-2})x_{kn}x_{mj} \\
&\quad +q^2x_{kl}x_{ij}x_{mn} - q^2(q^2 - q^{-2})x_{kl}x_{in}x_{mj} - qx_{kj}x_{mn} + q(q^2 - q^{-2})x_{kn}x_{mj} \\
&\equiv 0.
\end{aligned}$$

Case 13. $f = x_{mn}x_{ij} - q^2x_{ij}x_{mn} + qx_{in},\ g = x_{ij}x_{kl} - q^{-2}x_{kl}x_{ij},\ w = x_{mn}x_{ij}x_{kl}.$

In the case, we have

$$(f,g)_w = -q^2x_{ij}x_{mn}x_{kl} + qx_{in}x_{kl} + q^{-2}x_{mn}x_{kl}x_{ij}.$$

There are two subcases to consider.

	$((i,j),(m,n)) \in C_5$
$((k,l),(i,j)) \in C_1$	13.1. $((k,l),(m,n)) \in C_6 \quad ((k,l),(i,n)) \in C_1$
$((k,l),(i,j)) \in C_3$	13.2. $((k,l),(m,n)) \in C_5 \quad ((k,l),(i,n)) \in C_4$

13.1. $((i,j),(m,n)) \in C_5,\ ((k,l),(i,j)) \in C_1,\ ((k,l),(m,n)) \in C_6$ and $((k,l),(i,n)) \in C_1$. Then, we have

$$\begin{aligned} (f,g)_w &= -q^2x_{ij}x_{kl}x_{mn} + q^{-1}x_{kl}x_{in} + q^{-2}x_{kl}x_{mn}x_{ij} \\ &\equiv -x_{kl}x_{ij}x_{mn} + q^{-1}x_{kl}x_{in} + q^{-2}x_{kl}(q^2x_{ij}x_{mn} - qx_{in}) \\ &\equiv -x_{kl}x_{ij}x_{mn} + q^{-1}x_{kl}x_{in} + x_{kl}x_{ij}x_{mn} - q^{-1}x_{kl}x_{in} \\ &\equiv 0. \end{aligned}$$

13.2. $((i,j),(m,n)) \in C_5,\ ((k,l),(i,j)) \in C_3,\ ((k,l),(m,n)) \in C_5$ and $((k,l),(i,n)) \in C_4$. Then, we have $((k,n),(i,j)) \in C_2$ and

$$\begin{aligned} (f,g)_w &\equiv -q^2x_{ij}(q^2x_{kl}x_{mn} - qx_{kn}) + q[x_{kl}x_{in} - (q^2 - q^{-2})x_{kn}x_{il}] \\ &\quad -q^{-2}(q^2x_{kl}x_{mn} - qx_{kn})x_{ij} \\ &\equiv -q^4x_{ij}x_{kl}x_{mn} + q^3x_{ij}x_{kn} + qx_{kl}x_{in} - q(q^2 - q^{-2})x_{kn}x_{il} + x_{kl}x_{mn}x_{ij} \\ &\quad -q^{-1}x_{kn}x_{ij} \\ &\equiv -q^2x_{kl}x_{ij}x_{mn} + q^3x_{kn}x_{ij} + qx_{kl}x_{in} - q^3x_{kn}x_{il} \\ &\quad +q^{-1}x_{kn}x_{il} + q^2x_{kl}x_{ij}x_{mn} - qx_{kl}x_{in} - q^{-1}x_{kn}x_{ij} \\ &\equiv 0. \end{aligned}$$

Case 14. $f = x_{mn}x_{ij} - q^2x_{ij}x_{mn} + qx_{in},\ g = x_{ij}x_{kl} - x_{kl}x_{ij},\ w = x_{mn}x_{ij}x_{kl}.$

In the case, we have

$$(f,g)_w = -q^2x_{ij}x_{mn}x_{kl} + qx_{in}x_{kl} + x_{mn}x_{kl}x_{ij}.$$

There are two subcases to consider.

	$((i,j),(m,n)) \in C_5$
$((k,l),(i,j)) \in C_2$	14.1. $((k,l),(m,n)),((k,l),(i,n)) \in C_2, C_3$ *or* C_4
$((k,l),(i,j)) \in C_6$	14.2. $((k,l),(m,n)),((k,l),(i,n)) \in C_6$

14.1. $((i,j),(m,n)) \in C_5,\ ((k,l),(i,j)) \in C_2$ and $((k,l),(m,n)),((k,l),(i,n)) \in C_2, C_3$ or C_4.

14.1.1. If $((k,l),(m,n))$ and $((k,l),(i,n)) \in C_2\ (l > n)$, then

$$\begin{aligned}(f,g)_w &= -q^2 x_{ij}x_{kl}x_{mn} + qx_{kl}x_{in} + x_{kl}x_{mn}x_{ij}\\ &\equiv -q^2 x_{kl}x_{ij}x_{mn} + qx_{kl}x_{in} + x_{kl}(q^2 x_{ij}x_{mn} - qx_{in})\\ &\equiv 0.\end{aligned}$$

14.1.2. If $((k,l),(m,n))$ and $((k,l),(i,n)) \in C_3\ (l = n)$, then

$$\begin{aligned}(f,g)_w &= -x_{ij}x_{kl}x_{mn} + q^{-1}x_{kl}x_{in} + q^{-2}x_{kl}x_{mn}x_{ij}\\ &\equiv -x_{kl}x_{ij}x_{mn} + q^{-1}x_{kl}x_{in} + x_{kl}x_{ij}x_{mn} - q^{-1}x_{kl}x_{in}\\ &\equiv 0.\end{aligned}$$

14.1.3. If $((k,l),(m,n)),\ ((k,l),(i,n)) \in C_4\ (l < n)$, then we have $((k,n),(i,j)) \in C_2,\ ((i,j),(m,l)) \in C_5$ and

$$\begin{aligned}(f,g)_w &\equiv -q^2 x_{ij}[x_{kl}x_{mn} - (q^2-q^{-2})x_{kn}x_{ml}] + q[x_{kl}x_{in} - (q^2-q^{-2})x_{kn}x_{il}]\\ &\quad +[x_{kl}x_{mn} - (q^2-q^{-2})x_{kn}x_{ml}]x_{ij}\\ &\equiv -q^2 x_{ij}x_{kl}x_{mn} + q^2(q^2-q^{-2})x_{ij}x_{kn}x_{ml} + qx_{kl}x_{in} - q(q^2-q^{-2})x_{kn}x_{il}\\ &\quad +x_{kl}x_{mn}x_{ij} - (q^2-q^{-2})x_{kn}x_{ml}x_{ij}\\ &\equiv -q^2 x_{kl}x_{ij}x_{mn} + q^2(q^2-q^{-2})x_{kn}x_{ij}x_{ml} + qx_{kl}x_{in} - q(q^2-q^{-2})x_{kn}x_{il}\\ &\quad +x_{kn}(q^2x_{ij}x_{mn} - qx_{in}) - (q^2-q^{-2})x_{kn}(q^2x_{ij}x_{mn} - qx_{il})\\ &\equiv 0.\end{aligned}$$

14.2. $((i,j),(m,n)) \in C_5,\ ((k,l),(i,j)) \in C_6$ and $((k,l),(m,n)),((k,l),(i,n)) \in C_6$.

This case is similar to 14.1.1.

Case 15. $f = x_{mn}x_{ij} - q^2x_{ij}x_{mn} + qx_{in},\ g = x_{ij}x_{kl} - x_{kl}x_{ij} + (q^2-q^{-2})x_{kj}x_{il},\ w = x_{mn}x_{ij}x_{kl}$, with

	$((i,j),(m,n)) \in C_5$
$((k,l),(i,j)) \in C_4$	$((k,l),(m,n)) \in C_6$ $((k,l),(i,n)) \in C_4$ $((k,j),(m,n)) \in C_5$ $((i,l),(m,n)) \in C_6$

Then, we have

$$\begin{aligned}
(f,g)_w &= -q^2 x_{ij}x_{mn}x_{kl} + qx_{in}x_{kl} + x_{mn}x_{kl}x_{ij} - (q^2-q^{-2})x_{mn}x_{kj}x_{il}\\
&\equiv -q^2 x_{ij}x_{kl}x_{mn} + q[x_{kl}x_{in} - (q^2-q^{-2})x_{kn}x_{il}] + x_{kl}x_{mn}x_{ij}\\
&\quad -(q^2-q^{-2})(q^2x_{kj}x_{mn} - qx_{kn})x_{il}\\
&\equiv -q^2[x_{kl}x_{ij} - (q^2-q^{-2})x_{kj}x_{il}]x_{mn} + qx_{kl}x_{in} - q(q^2-q^{-2})x_{kn}x_{il}\\
&\quad +x_{kl}(q^2x_{ij}x_{mn} - qx_{in}) - q^2(q^2-q^{-2})x_{kj}x_{mn}x_{il} + q(q^2-q^{-2})x_{kn}x_{il}\\
&\equiv -q^2x_{kl}x_{ij}x_{mn} + q^2(q^2-q^{-2})x_{kj}x_{il}x_{mn} + qx_{kl}x_{in} - q(q^2-q^{-2})x_{kn}x_{il}\\
&\quad +q^2x_{kl}x_{ij}x_{mn} - qx_{kl}x_{in} - q^2(q^2-q^{-2})x_{kj}x_{il}x_{mn} + q(q^2-q^{-2})x_{kn}x_{il}\\
&\equiv 0.
\end{aligned}$$

Case 16. $f = x_{mn}x_{ij} - q^2x_{ij}x_{mn} + qx_{in},\ g = x_{ij}x_{kl} - q^2x_{kl}x_{ij} + qx_{kj},\ w = x_{mn}x_{ij}x_{kl}$, with

	$((i,j),(m,n)) \in C_5$
$((k,l),(i,j)) \in C_5$	$((k,l),(m,n)) \in C_6$ $((k,l),(i,n)),((k,j),(m,n)) \in C_5$

In the case, we have

$$\begin{aligned}
(f,g)_w &= -q^2x_{ij}x_{mn}x_{kl} + qx_{in}x_{kl} + q^2x_{mn}x_{kl}x_{ij} - qx_{mn}x_{kj}\\
&\equiv -q^2x_{ij}x_{kl}x_{mn} + q(q^2x_{kl}x_{in} - qx_{kn}) + q^2x_{kl}x_{mn}x_{ij} - q(q^2x_{kj}x_{mn} - qx_{kn})\\
&\equiv -q^2(q^2x_{kl}x_{ij} - qx_{kj})x_{mn} + q^3x_{kl}x_{in} - q^2x_{kn} + q^2x_{kl}(q^2x_{ij}x_{mn} - qx_{in})\\
&\quad -q^3x_{kj}x_{mn} + q^2x_{kn}\\
&\equiv 0.
\end{aligned}$$

Thus, $\widetilde{S}^+$ is a Gröbner-Shirshov basis. This completes the proof of Theorem 3.1. □

Similarly, with the deg-lex order on $\widetilde{Y}^*$, $\widetilde{S}^-$ is a Gröbner-Shirshov basis for $U_q^-(A_N) = k\langle\widetilde{Y}|\widetilde{S}^-\rangle$.

We now use the same notation as before. Order the generators by: $x_i > x_j,\ h_i > h_i^{-1} > h_j > h_j^{-1},\ y_i > y_j$ if $i > j$, and $x_i > h_j^{\pm1} > y_m$ for all $i,\ j,\ m$. Then we obtain a well ordering (deg-lex) on $\widetilde{X} \cup H \cup \widetilde{Y}$. Thus, by Theorem 3.1, we re-obtain the following theorem in [16].

Theorem 3.2. *([16] Theorem 2.7) Let the notation be as before. Then with the deg-lex order on* $\{\widetilde{X}\cup H\cup\widetilde{Y}\}^*$, $\widetilde{S}^+\cup T\cup K\cup\widetilde{S}^-$ *is a Gröbner-Shirshov basis for* $U_q(A_N) = k\langle\widetilde{X}\cup H\cup\widetilde{Y}|\widetilde{S}^+\cup T\cup K\cup\widetilde{S}^-\rangle$.

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