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# The Modulo Two Homotopy Groups of the $L_2$-Localization of the Ravenel Spectrum

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## ABSTRACT

The Ravenel spectra $T(m)$ for non-negative integers $m$ interpolate between the sphere spectrum and the Brown-Peterson spectrum. Let $L_2$ denote the Bousfield-Ravenel localization functor with respect to $v_2^{-1}BP$. In this paper, we determine the homotopy groups $\pi_*(L_2T(m):\mathbb{Z}/2)=[M_2,L_2T(m)]_*$ for $m>1$, where $M_2$ denotes the modulo two Moore spectrum.

## RESUMEN

El espectro de Ravenel $T(m)$ para enteros no negativos $m$ interpola entre el espectro esferico y el espectro de Brown-Peterson. Denotemos por $L_2$ el funtor de localización de Bousfield-Ravenel con respecto a $v_2^{-1}BP$. En este artículo, determinamos el grupo de homotopia $\pi_*(L_2T(m):\mathbb{Z}/2)=[M_2,L_2T(m)]_*$ para $m>1$, donde $M_2$ denota el espectro de Moor modulo dos.

**Key words and phrases:** *homotopy groups, Bousfield-Ravenel localization, Ravenel spectrum.*

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# 1 Introduction

Let $\mathcal{S}_{(2)}$ denote the stable homotopy category of 2-local spectra, and $BP \in \mathcal{S}_{(2)}$ denote the Brown-Peterson ring spectrum. Then, $BP_* = \pi_*(BP) = \mathbb{Z}_{(2)}[v_1, v_2, \dots]$ and $BP_*(BP) = \pi_*(BP \wedge BP) = BP_*[t_1, t_2, \dots]$, which form a Hopf algebroid. The Adams-Novikov spectral sequence for computing the homotopy groups $\pi_*(X)$ of a spectrum $X$ has the $E_2$-term $E_2^*(X) = \operatorname{Ext}^*_{BP_*(BP)}(BP_*, BP_*(X))$. Let $L_2\colon \mathcal{S}_{(2)} \to \mathcal{S}_{(2)}$ be the Bousfield-Ravenel localization functor with respect to $v_2^{-1}BP$. Then, the $E_2$-term $E_2^*(L_2S^0)$ for the sphere spectrum $S^0$ is determined in [12], but the homotopy groups $\pi_*(L_2S^0)$ stay undetermined. The Ravenel spectrum $T(m)$ for $m > 0$ is a ring spectrum characterized by $BP_*(T(m)) = BP_*[t_1, t_2, \dots t_m] \subset BP_*(BP)$ as a $BP_*(BP)$-comodule. The spectrum $T(m)$ interpolates between the sphere spectrum and the Brown-Peterson spectrum, and so the homotopy groups $\pi_*(L_2T(m))$ seem accessible if $m$ is sufficiently large. Indeed, $\pi_*(L_2T(\infty)) = \pi_*(L_2BP)$ is determined by Ravenel [8]. Let $M_k$ denote the mod $k$ Moore spectrum defined by the cofiber sequence

$$S^0 \xrightarrow{2} S^0 \xrightarrow{i} M_k \xrightarrow{j} S^1. \tag{1.1}$$

For $m = 1$, $T(1) \wedge M_2$ is the Mahowald spectrum $X\langle 1\rangle$ and the homotopy groups of $L_2X\langle 1\rangle$ are determined in [11]. But even the homotopy groups of $L_2T(1) \wedge M_4$ are too complicated to be determined completely (*cf.* [2], [3]). Consider a spectrum $T(m)/(v_1^a)$ defined as a cofiber of the self-map $v_1^a\colon \Sigma^{2a}T(m) \to T(m)$ defined by the generator $v_1 \in \pi_2(T(m))$. We use the notation:

$$V_m(0) = T(m) \wedge M_2 \quad \text{and} \quad V_m(1)_a = T(m)/(v_1^a) \wedge M_2, \tag{1.2}$$

and abbreviate $V_m(1)_1$ to $V_m(1)$. In this paper, we consider the case where $m > 1$, and determine $\pi_*(L_2V_m(1))$ and $\pi_*(L_2V_m(0))$. The Adams-Novikov $E_2$-term $E_2^*(L_2V_m(1))$ for $m > 1$ is determined by Ravenel [10] as follows:

$$E_2^*(L_2V_m(1)) = K_m(2)_* \otimes \wedge(h_{1,0}, h_{1,1}, h_{2,0}, h_{2,1}) \tag{1.3}$$

for generators $h_{i,j} \in E_2^{1,2^{m+i+j+1}-2^{j+1}}(L_2V_m(1))$ and $K_m(2)_* = v_2^{-1}\mathbb{Z}/2[v_2, v_3, \dots, v_{m+2}]$. We show that $V_m(1)$ is a $T(m)$-module spectrum with $M_2$-action, and then that all additive generators of the $E_2$-term are permanent cycles and the extension problem of the spectral sequence is trivial.

**Theorem 1.4.** *$\pi_*(L_2V_m(1)) = K_m(2)_* \otimes \wedge(h_{1,0}, h_{1,1}, h_{2,0}, h_{2,1})$ as a $\mathbb{Z}/2$-module.*

Let $\alpha\colon \Sigma^8M_2 \to M_2$ denote the Adams map such that $BP_*(\alpha) = v_1^4$, and $K_2^a$ denote a cofiber of $\alpha^a$. Then, we show that $V_m(1)_{4a} = T(m) \wedge K_2^a$ in Lemma 2.4 and denote the telescope of $V_m(1)_4 \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} V_m(1)_{4a} \xrightarrow{\alpha} V_m(1)_{4a+4} \xrightarrow{\alpha} \cdots$ by $V_m(1)_\infty$. By the $v_1$-Bockstein spectral sequence, we determine the Adams-Novikov $E_2$-term $E_2^*(L_2V_m(1)_\infty)$, whose structure is given in [4] without proof. Here we give a proof of it. Consider the integers $e_n$ and $a_n$ defined by

$$e_n = \frac{8^n - 1}{7} \quad \text{and} \quad a_n = \begin{cases} 1 & n = 0 \\ 3e_{k+1} - 1 & n = 3k+1 \\ 6e_{k+1} & n = 3k+2 \\ 12e_{k+1} & n = 3k+3. \end{cases} \tag{1.5}$$

We introduce modules

$$\begin{aligned} E_m(2)_* &= v_2^{-1}\mathbb{Z}_{(2)}[v_1, v_2, \ldots, v_{m+2}], \\ Q(k) &= E_{m-1}(2)_*/(2, v_1^{a_k})[x_{k+1}]\langle x_k/v_1^{a_k}\rangle, \end{aligned}$$

where $x_n \in E_m(2)_*$ is an element defined in (4.1) such that $x_n \equiv v_{m+2}^{2^n}$ modulo $(2, v_1)$, and $x_n/v_1^{a_n} \in E_2^0(L_2V_m(1)_\infty)$ by Proposition 4.3. We also introduce homology classes $\zeta$ and $\zeta_n$ of $E_2^1(V_m(0))$, which correspond to elements $v_{m+2}h_{1,1}$ and $v_{m+2}^{2^l e_k}\zeta_l \in E_2^1(L_2V_m(1))$ for $n = 3k+l$ with $l \in \{1, 2, 3\}$, respectively, where $\zeta_l$ corresponds to $h_{1,0}$ if $l = 1$, and $h_{2,l-2}$ if $l = 2, 3$.

**Proposition 1.6.** (*cf.* [4]) *The $E_2$-term of Adams-Novikov spectral sequence for computing $\pi_*(L_2V_m(1)_\infty)$ is isomorphic to the direct sum of $Q(0) \otimes \wedge(h_{1,0}, h_{2,0}, h_{2,1})$ and the tensor product of $\wedge(\zeta)$ and*

$$E_{m-1}(2)_*/(2, v_1^\infty) \oplus \bigoplus_{k>0} Q(k) \otimes \wedge(\zeta_{k+1}, \zeta_{k+2})$$

*as a $\mathbb{Z}/2[v_1]$-module.*

By noticing that $x_n \in E_2^0(L_2V_m(1)_{a_n})$ survives to $\pi_*(L_2V_m(1)_{a_n})$ in Lemma 5.1, we see that all additive generators of Proposition 1.6 are permanent cycles.

**Theorem 1.7.** *The homotopy groups $\pi_*(L_2V_m(1)_\infty)$ are isomorphic to the Adams-Novikov $E_2$-term given in Proposition 1.6.*

Consider the cofiber sequence

$$V_m(0) \xrightarrow{\eta} v_1^{-1}V_m(0) \xrightarrow{p} V_m(1)_\infty \longrightarrow \Sigma V_m(0) \tag{1.8}$$

for the localization map $\eta$. Here, we introduce algebras

$$k_m(1)_* = \mathbb{Z}/2[v_1, v_2, \ldots, v_{m+1}] \quad \text{and} \quad K_m(1)_* = v_1^{-1}k_m(1)_*.$$

Ravenel showed the following

**Proposition 1.9.** (*cf.* [10]) *The homotopy groups $\pi_*(v_1^{-1}V_m(0))$ are isomorphic to $K_m(1)_* \otimes \wedge(h_{1,0})$.*

There is a relation between $h_{1,0}$ and $\zeta$, which is shown in section four:

**Lemma 1.10.** *The induced homomorphism $p_*$ from $p$ in (1.8) assigns $h_{1,0}/v_1^j \in E_2^1(v_1^{-1}V_m(0))$ to $\zeta/v_1^{j-2} \in E_2^1(L_2V_m(1)_\infty)$.*

Observing the correspondence in the Adams-Novikov $E_2$-terms, we obtain

**Corollary 1.11.** *The homotopy groups $\pi_*(L_2V_m(0))$ are isomorphic to the direct sum of $\Sigma^{-1}Q(0)\otimes \wedge(h_{1,0}, h_{2,0}, h_{2,1})$ and the tensor product of $\wedge(\zeta)$ and*

$$k_m(1)_* \oplus \Sigma^{-1}k_m(1)_*/(2, v_1^\infty, v_2^\infty) \oplus \bigoplus_{k>0} \Sigma^{-1}Q(k) \otimes \wedge(\zeta_{k+1}, \zeta_{k+2})$$

*as a $\mathbb{Z}/2[v_1]$-module.*

In the next section, we observe about an action of the Moore spectrum $M_2$ on $V_m(1)_t$ and a ring structure of $V_m(1)_{4t}$, in order to study the Adams-Novikov differential and the extension problem of the spectral sequence in the following sections. We prove Theorem 1.4 in section three. Section four is devoted to show Proposition 1.6. We end by proving Theorem 1.7 in the last section.

## 2 The spectrum $T(m) \wedge K_k^t$

We work in the stable homotopy category of spectra localized at the prime two. Let $BP$ denote the Brown-Peterson spectrum. Then, we have the Adams-Novikov spectral sequence

$$E_2^{s,t}(X) = \mathrm{Ext}_\Gamma^{s,t}(A, BP_*(X)) \Longrightarrow \pi_*(X).$$

Here $(A, \Gamma)$ is the associated Hopf algebroid such that

$$(A, \Gamma) \ = \ (BP_*,\ BP_*(BP)) \ = \ (\mathbb{Z}_{(2)}[v_1, v_2, \dots], BP_*[t_1, t_2, \dots])$$

for the Hazewinkel generators $v_k \in BP_{2^{k+1}-2}$ and the generators $t_k \in BP_{2^{k+1}-2}(BP)$.

Let $M_k$ and $K_k^t$ for $k = 2, 4$ and $t > 0$ denote spectra defined by the cofiber sequences

$$S^0 \xrightarrow{2} S^0 \xrightarrow{i} M_k \xrightarrow{j} S^1 \quad \text{and} \quad \Sigma^{8t}M_k \xrightarrow{\alpha^t} M_k \xrightarrow{i_k^t} K_k^t \xrightarrow{j_k^t} \Sigma^{8t+1}M_k.$$

Here $\alpha$ denotes the Adams map such that $BP_*(\alpha) = v_1^4$. Note that $M_4$ and $K_4^t$ are ring spectra (*cf.* [5]). The Ravenel spectrum $T(m)$ is characterized by $BP_*(T(m)) = A[t_1, \dots, t_m] \subset \Gamma$ as $\Gamma$-comodules, and is a ring spectrum, whose multiplication and unit map we denote by $\mu$ and $\iota$, respectively. Throughout the paper, we fix a positive integer $m$. Let $(A, \Gamma_m) = (A, \Gamma/(t_1, t_2, \dots, t_m))$ be the Hopf algebroid associated with $(A, \Gamma)$, and consider a spectrum $X$ such that $BP_*(X) = M \otimes_A A[t_1, \dots, t_m]$ for a $\Gamma$-comodule $M$. Then, we have an isomorphism

$$E_2^*(X) = \mathrm{Ext}_{\Gamma_m}^*(A, M) \tag{2.1}$$

by the change of rings theorem (*cf.* [10]). By observing the reduced cobar complex for the Ext group, we have

**Lemma 2.2.** *The $E_2$-term has the vanishing line of the slope $1/(q_m - 1)$ if $M$ is $(-1)$-connected.*

Hereafter, we put

$$q_m = 2^{m+2} - 2 \tag{2.3}$$

which is the degree of $u_1 = v_{m+1}$ and $s_1 = t_{m+1}$. This shows $\pi_2(T(m)) = BP_2 = \mathbb{Z}_{(2)}\{v_1\}$ if $m > 0$. Let $T(m)/(v_1^a)$ for an integer $a > 0$ denote the cofiber of $\widetilde{v}_1^a\colon \Sigma^{8a}T(m) \to T(m)$, where $\widetilde{v}_1 : \Sigma^8 T(m) \to T(m)$ is the composite

$$\widetilde{v}_1 :\ \ \Sigma^8 T(m) = S^8 \wedge T(m) \xrightarrow{v_1 \wedge T(m)} T(m) \wedge T(m) \ \xrightarrow{\mu}\ T(m).$$

**Lemma 2.4.** *For $k = 2, 4$ and $a > 0$, $T(m)/(v_1^{4a}) \wedge M_k = T(m) \wedge K_k^a$. In particular, $T(m) \wedge K_2^a \wedge M_4 = T(m)/(v_1^{4a}) \wedge M_2 \wedge M_4 = T(m) \wedge M_2 \wedge K_4^a$.*

**Proof.** Since $\pi_8(T(m)\wedge M_k) = BP_8/(k) = \mathbb{Z}/k\{v_1^4, v_1v_2\}$ by Lemma 2.2, we see that $v_1^4 \wedge M_k = \iota \wedge \alpha i \in \pi_8(T(m)\wedge M_k)$. Indeed, both of these elements are assigned to $v_1^4 \in BP_8(T(m)\wedge M_i)$ under the homomorphism induced from the unit map of $BP$. It extends to $v_1^4 \wedge M_k = \iota\wedge\alpha : M_k \to T(m)\wedge M_k$, since $[M_k, T(m) \wedge M_k]_8 = \pi_8(T(m) \wedge M_k)$. Indeed, $\pi_9(T(m) \wedge M_k) = BP_9/(k) = 0$. We further extend it to a self-map $A = \widetilde{v}_1^4 \wedge M_k = T(m) \wedge \alpha : T(m) \wedge M_k \to T(m) \wedge M_k$ by the ring structure of $T(m)$. Now the cofiber of $A^a$ is $T(m)/(v_1^{4a}) \wedge M_k = T(m) \wedge K_k^a$.

□

This lemma implies

$$V_m(1)_{4a} = T(m) \wedge K_2^a \tag{2.5}$$

for the spectrum $V_m(1)_{4a}$ in (1.2).

**Lemma 2.6.** *Let $F$ denote one of the spectra $M_k$ and $K_k^a$ for $k = 2, 4$ and $a > 0$. Then, there is a pairing $\nu_F : F \wedge F \to T(m) \wedge F$ such that $\nu_F \circ (F \wedge i_F) = \iota \wedge F\colon F \to T(m) \wedge F$ for $m > 0$. Here $i_F : S^0 \to F$ denotes the inclusion to the bottom cell.*

**Proof.** The pairing for $F = M_4$ or $K_4^a$ is the composite $(\iota\wedge F\wedge F)(T(m)\wedge\mu_F)$ for the multiplication $\mu_F$ of the ring spectrum of $F$ (see [5]).

For $F = M_2$, we see that $\pi_0(T(m)\wedge M_2) = BP_0/(2) = \mathbb{Z}/2$ and $\pi_1(T(m)\wedge M_2) = BP_1/(2) = 0$ by Lemma 2.2, and so $[M_2, T(m) \wedge M_2]_0 = \mathbb{Z}/2$.

Note that $M_2 \wedge M_4 = M_2 \vee \Sigma M_2$. Then, by Lemma 2.4,

$$\begin{aligned} T(m) \wedge M_2 \wedge K_4^a &= T(m)/(v_1^{4a}) \wedge M_2 \wedge M_4 \;=\; T(m)/(v_1^{4a}) \wedge (M_2 \vee \Sigma M_2) \\ &= T(m)/(v_1^{4a}) \wedge M_2 \vee \Sigma T(m)/(v_1^{4a}) \wedge M_2 \;=\; T(m) \wedge K_2^a \vee \Sigma T(m) \wedge K_2^a. \end{aligned}$$

We also see that $T(m) \wedge K_2^a \wedge K_4^a = T(m)/(v_1^{4a}) \wedge K_2^a \wedge M_4 = T(m)/(v_1^{4a}) \wedge (K_2^a \vee \Sigma K_2^a)$, and so $T(m) \wedge K_2^a \wedge K_4^a \wedge M_2 = T(m) \wedge K_2^a \wedge K_2^a \vee \Sigma T(m) \wedge K_2^a \wedge K_2^a$. Then,

$$\begin{aligned} T(m) \wedge M_2 \wedge K_4^a \wedge K_4^a \wedge M_2 &= T(m) \wedge K_2^a \wedge K_4^a \wedge M_2 \vee \Sigma T(m) \wedge K_2^a \wedge K_4^a \wedge M_2 \\ &= T(m) \wedge K_2^a \wedge K_2^a \vee \Sigma T(m) \wedge K_2^a \wedge K_2^a \vee \Sigma T(m) \wedge K_2^a \wedge K_2^a \wedge M_2. \end{aligned}$$

Let $\mu_K : K_4^a \wedge K_4^a \to K_4^a$ denote the multiplication of the ring spectrum $K_4^a$, and $\widetilde{\nu}$ be the composite $T(m) \wedge M_2 \wedge M_2 \xrightarrow{T(m) \wedge \nu_{M_2}} T(m) \wedge T(m) \wedge M_2 \xrightarrow{\mu \wedge M_2} T(m) \wedge M_2$. Then the desired pairing is a composite

$$K_2^a \wedge K_2^a \xrightarrow{\iota \wedge K \wedge K} T(m) \wedge K_2^a \wedge K_2^a \xrightarrow{inc \wedge K_2^a} T(m) \wedge M_2 \wedge K_4^a \wedge K_4^a \wedge M_2 \xrightarrow{switch}$$

$$T(m) \wedge M_2 \wedge M_2 \wedge K_4^a \wedge K_4^a \xrightarrow{\widetilde{\nu}} T(m) \wedge M_2 \wedge K_4^a \wedge K_4^a \xrightarrow{T(m) \wedge M_2 \wedge \mu_K} T(m) \wedge M_2 \wedge K_4^a \xrightarrow{prj} T(m) \wedge K_2^a.$$

$\square$

**Corollary 2.7.** *The spectra $V_m(0)$ and $V_m(1)_{4a}$ for $a > 0$ are ring spectra.*

We say that a spectrum $X$ has $M_2$-action, if there is a pairing $\varphi_X : X \wedge M_2 \to X$ such that $\varphi_X(X \wedge i) = id_X$. Here $i : S^0 \to M_2$ is the inclusion of (1.1) and $id_X : X \to X$ denotes the identity map.

**Lemma 2.8.** *$V_m(1)_t$ has $M_2$-action.*

**Proof.** Since $T(m)$ is an associative ring spectrum, $T(m)/(v_1^t)$ is a $T(m)$-module spectrum. The action $\varphi_{V_m(1)_t}$ is defined by the composite $V_m(1)_t \wedge M_2 = T(m)/(v_1^t) \wedge M_2 \wedge M_2 \xrightarrow{T(m)/(v_1^t) \wedge \nu_{M_2}}$ $T(m)/(v_1^t) \wedge T(m) \wedge M_2 \longrightarrow T(m)/(v_1^t) \wedge M_2 = V_m(1)_t$. $\square$

Since $V_m(1)_t$ is a $T(m)$-module spectrum, it implies the following

**Corollary 2.9.** *$V_m(1)_t$ is a $V_m(0)$-module spectrum.*

# 3 The homotopy groups of $L_2V_m(1)$

Note that if $BP_*(X)$ is $(2, v_1)$-nil, then $BP_*(L_2X) = v_2^{-1}BP_*(X)$, since $L_2$ is smashing (*cf.* [8], [9]). Therefore, the Adams-Novikov $E_2$-term $E_2^*(L_2V_m(1)_t)$ is $\mathrm{Ext}_\Gamma^*(A, v_2^{-1}BP_*/(2, v_1^t)[t_1, \ldots, t_m])$, which is isomorphic to

$$E_2^*(L_2V_m(1)_t) = \mathrm{Ext}_{\Gamma_m}^*(A, v_2^{-1}BP_*/(2, v_1^t))$$

by (2.1). Consider a spectrum

$$E_m(2) = v_2^{-1}BP\langle m+2\rangle$$

for the Johnson-Wilson spectrum $BP\langle m+2\rangle$. Then we obtain a Hopf algebroid

$$(E_m(2)_*, \Sigma_m(2)) = (v_2^{-1}\mathbb{Z}_{(2)}[v_1, v_2, \dots, v_{m+2}], E_m(2)_* \otimes_A \Gamma_m \otimes_A E_m(2)_*).$$

Since

$$v_2^{-1}BP_*/J \xrightarrow{1\otimes\eta_R} E_m(2)_*/J \otimes_A \Gamma_m$$

for an invariant regular ideal $J = (2^b, v_1^a)$ is a faithfully flat extension, we have an isomorphism

$$\mathrm{Ext}^*_{\Gamma_m}(A, BP_*/J) \cong \mathrm{Ext}^*_{\Sigma_m(2)}(E_m(2)_*, E_m(2)_*/J)$$

by a theorem of Hopkins' (*cf.* [1, Th. 3.3]). Note that $m+2$ is the smallest number $n$, for which $v_2^{-1}BP_*/J \xrightarrow{1\otimes\eta_R} v_2^{-1}BP\langle n\rangle_*/J \otimes_A \Gamma_m$ is a faithfully flat extension. We use the abbreviation

$$H^*M = \mathrm{Ext}^*_{\Sigma_m(2)}(E_m(2)_*, M) \tag{3.1}$$

for a $\Sigma_m(2)$-comodule $M$. We compute the Ext group $H^*M$ by the reduced cobar complex $\widetilde{\Omega}^*_{\Sigma_m(2)}M$ (*cf.* [10]). Since the differentials of the cobar complex are defined by the right unit $\eta_R : E_m(2)_* \to \Sigma_m(2)$ and the diagonal $\Delta : \Sigma_m(2) \to \Sigma_m(2) \otimes_{E_m(2)_*} \Sigma_m(2)$, we write down here some formulas on them based on the Hazewinkel and the Quillen formulas:

$$\begin{array}{l} v_n = 2\ell_n - \sum_{k=1}^{n-1} \ell_k v_{n-k}^{2^k} \in \mathbb{Q}\otimes A = \mathbb{Q}[\ell_1, \ell_2, \dots], \\ \eta_R(\ell_n) = \sum_{k=0}^{n} \ell_k t_{n-k}^{2^k} \in \mathbb{Q}\otimes\Gamma = \mathbb{Q}\otimes A[t_1, t_2, \dots] \quad \text{and} \\ \sum_{i+j=n} \ell_i \Delta(t_j^{2^i}) = \sum_{i+j+k=n} \ell_i t_j^{2^i} \otimes t_k^{2^{i+j}} \in \mathbb{Q}\otimes\Gamma\otimes_A\Gamma. \end{array} \tag{3.2}$$

Hereafter, we put $v_2 = 1$ and use the following notation:

$$u_i = v_{m+i} \quad \text{and} \quad s_i = t_{m+i}.$$

Since the structure maps preserve degrees, we may recover $v_2$'s from its degrees. Then, we obtain the following two lemmas immediately from (3.2) by a routine computation:

**Lemma 3.3.** *The right unit $\eta_R : A \to \Gamma_m/(2)$ acts as follows:*

$$\begin{array}{rcl} \eta_R(v_n) & = & v_n \quad \textit{for } n \le m+1, \\ \eta_R(u_2) & = & u_2 + v_1 s_1^2 + v_1^{2^{m+1}} s_1, \\ \eta_R(u_3) & \equiv & u_3 + s_1^4 + s_1 + v_1 r_1 \mod (2, v_1^{2^{m+2}}), \\ \eta_R(u_4) & \equiv & u_4 + s_2^4 + s_2 + v_3 s_1^8 + v_3^{2^{m+1}} s_1 \mod (2, v_1) \end{array}$$

*for $r_1 = s_2^2 + v_1 u_2 s_1^2$.*

This yields the relations in $\Sigma_m(2)$:

$$s_1^4 + s_1 \equiv v_1 r_1 \mod (2, v_1^{2^{m+2}}) \quad \text{and} \quad s_2^4 + s_2 + v_3 s_1^8 + v_3^{2^{m+1}} s_1 \equiv 0 \mod (2, v_1). \tag{3.4}$$

**Lemma 3.5.** *The diagonal $\Delta$ behaves on the generators $s_i$ as follows:*

$$\begin{array}{rcl}\Delta(s_1) & = & s_1\otimes 1+1\otimes s_1,\\ \Delta(s_2) & = & s_2\otimes 1+1\otimes s_2+v_1s_1\otimes s_1,\\ \Delta(s_3) & \equiv & s_3\otimes 1+1\otimes s_3+v_2s_1^2\otimes s_1^2 \mod (2,v_1).\end{array}$$

**Lemma 3.6.** *Let $z$ denote an element defined by $r_1^4+r_1+v_3^2s_1^4+v_3^{2^{m+2}}s_1^2=v_1z$. Then the cochains $r_1, z\in\widetilde{\Omega}^1_{\Sigma_m(2)}E_m(2)_*/(2)$ are cocycles. Besides, $z\equiv u_2s_1^2$ modulo $(v_1^2)$.*

**Proof.** Since $v_1\in\widetilde{\Omega}^0_{\Sigma_m(2)}E_m(2)_*/(2)$ and $s_1\in\widetilde{\Omega}^1_{\Sigma_m(2)}E_m(2)_*/(2)$ are both cocycles, so is $r_1$ by the relation $v_1r_1=s_1^4+s_1\in\Sigma_m(2)$ in (3.4). Furthermore, $v_3\in\widetilde{\Omega}^0_{\Sigma_m(2)}E_m(2)_*/(2)$ is a cocycle. It follows similarly from its definition that $z$ is a cocycle. By the definition of $r_1$, $r_1^4+r_1\equiv s_2^8+s_2^2+v_1u_2s_1^2\equiv v_1u_2s_1^2+v_3^2s_1^{16}+v_3^{2^{m+2}}s_1^2$ modulo $(2,v_1^2)$ by (3.4). $\square$

We now work as [6].

**Lemma 3.7.** *$u_2^t\in E_2^0(V_m(1))$ and $u_2^th_{2,0}\in E_2^1(V_m(1))$ for each $t>0$ are permanent cycles.*

**Proof.** For $t=1$, the lemma is seen by Lemma 2.2. Consider the cofiber sequence $\Sigma^2V_m(0)\xrightarrow{v_1}V_m(0)\xrightarrow{i_1}V_m(1)\xrightarrow{j_1}\Sigma^3V_m(0)$. Put $d(u_2^t)=v_1k_t'\in\widetilde{\Omega}^1_{\Sigma_m(2)}E_m(2)_*/(2)$ by virtue of Lemma 3.3, and let $k_t\in E_2^1(V_m(0))$ be the homology class of the cocycle $k_t'$. Then, $k_1=h_{1,1}$, $v_1k_t=0$ and $k_{t+1}=\langle k_1,v_1,k_t\rangle$. Indeed, $\langle k_1,v_1,k_t\rangle$ is the class of $k_1'\eta_R(u_2^t)+u_2k_t'=d(u_2^{t+1})/v_1=k_{t+1}'$. Besides, $\delta(u_2^t)=k_t$ for the connecting homomorphism associated to the cofiber sequence. Let $\xi_1\in\pi_{q_m-1}(V_m(0))$ denote the homotopy element detected by $k_1$. Then, $v_1\xi_1=\xi_1v_1=0$.

Suppose now that $u_2^t\in E_2^0(V_m(1))$ is a permanent cycle. Then, $k_t$ is a permanent cycle that detects the element $\xi_t=j_1u_2^t$ by the Geometric Boundary Theorem. Since $v_1\xi_t=0$, the Toda bracket $\{\xi_1,v_1,\xi_k\}$ is defined, which is detected by the Massey product $\langle k_1,v_1,k_t\rangle$. Note here that the Toda bracket is defined since $V_m(0)$ is a ring spectrum. It follows that $k_{t+1}$ is a permanent cycle and detects a homotopy element, which we denote by $\xi_{t+1}$. Since the Massey product $\langle v_1,k_1,v_1\rangle$ is zero in the $E_2$-term $E_2^{0,q_m+4}(V_m(0))$, we see that $\{v_1,\xi_1,v_1\}=0$ by Lemma 2.2. Now we compute $v_1\{\xi_1,v_1,\xi_k\}=\{v_1,\xi_1,v_1\}\xi_k=0$, and $\xi_{t+1}$ is pulled back to $u_2^{t+1}$ under the map $j_1$.

Turn to $u_2^th_{2,0}$. In this case a similar argument works. For the connecting homomorphism $\delta$, $\delta(u_2^th_{2,0})=\langle h_{1,0}^2,v_1,k_t\rangle$, which detects a homotopy element $\{\eta_0^2,v_1,\xi_t\}$, where $\eta_0$ denotes an element detected by $h_{1,0}$. Applying $v_1$ shows $\{v_1,\eta_0^2,v_1\}\xi_t=0$. Indeed, $\{v_1,\eta_0^2,v_1\}$ is detected by $E_2^{s,2q_m+4+s}(V_m(0))$ for $s>2$. $\square$

**Lemma 3.8.** *The elements $h_{1,0},h_{1,1}\in E_2^1(V_m(0))$ and $h_{2,1}\in E_2^1(L_2V_m(0))$ are permanent cycles.*

**Proof.** $h_{1,0},h_{1,1}$ are seen immediately by Lemma 2.2.

The cobar module $\widetilde{\Omega}^{4,4q_m+6}_{\Gamma_m} BP_*/(2)$ is generated by $v_1^3 s_1^{\otimes 4}$ and $v_2 s_1^{\otimes 4}$ by degree reason. The first generator cobounds $v_1^2 s_2 \otimes s_1 \otimes s_1$, and we obtain $E_2^{4,4q_m+6}(V_m(0)) = \mathbb{Z}/2\{v_2 h_{1,0}^4\}$. Put $d_3(h_{2,1}) = av_2 h_{1,0}^4 \in E_2^{4,4q_m+6}(V_m(0))$ for $a \in \mathbb{Z}/2$. Let $w$ be an element fit in $d(s_3) = v_2 s_1^2 \otimes s_1^2 + v_1 w$ by virtue of Lemma 3.5. Then, $d(w) = 0$ in the cobar complex $\widetilde{\Omega}^3_{\Sigma_m(2)} E_m(2)_*/(2)$, and we see that $s_1^{\otimes 4}$ cobounds $s_3^2 \otimes s_1 \otimes s_1 + v_1 w^2 \otimes s_2 + (r_1 \otimes s_1 + s_1 \otimes r_1 + v_1 r_1 \otimes r_1) \otimes s_2$ (in which we set $v_2 = 1$). It follows that $d_3(h_{2,1}) = av_2 h_{1,0}^4 = 0 \in E_2^4(L_2 V_m(0))$ as desired. Indeed, $v_2 h_{1,0}^4 = v_1 g h_{1,0}^2 = 0$, since $v_2 h_{1,0}^2 = v_1 g$ for an element $g$ and $v_1 h_{1,0}^2 = 0$ by $d(s_2) = v_1 s_1 \otimes s_1$.

□

**Proof of Theorem 1.4.** Every element $x \in E_2^s(L_2 V_m(1))$ is decomposed as $x = x'x''$ for $x' \in \mathbb{Z}/2[u_2] \otimes \wedge(h_{2,0})$ and $x'' \in K_{m-1}(2)_* \otimes \wedge(h_{1,0}, h_{1,1}, h_{2,1})$. Note that $K_{m-1}(2)_* \otimes \wedge(h_{1,0}, h_{1,1}, h_{2,1}) \subset E_2^*(L_2 V_m(0))$. Since $x'$ (resp. $x''$) is a permanent cycle of the Adams-Novikov spectral sequence for computing $\pi_*(L_2 V_m(1))$ (resp. $\pi_*(L_2 V_m(0))$) by Lemma 3.7 (resp. 3.8), we obtain that the element $x$ is a permanent cycle from Corollary 2.9. We see that the extension problem is trivial by Lemma 2.8. Indeed, $\mathbb{Z}/2 = \pi_0(M_2)$ acts on $\pi_*(L_2 V_m(1))$. □

## 4 The elements $x_n$

We introduce the integer $b_n$ for $n \geq 0$ by

$$b_n = \begin{cases} a_n - 8 & n \equiv 1\ (3) \\ a_n - 3 & n \equiv 2\ (3) \\ 0 & n \equiv 0\ (3), \end{cases}$$

and the elements $x_n \in E_m(2)_*$ defined by

$$x_n = x_{n-1}^2 + v_1^{b_n} y_{n-1}, \quad \text{where} \quad y_n = \begin{cases} 0 & n \leq 0 \text{ or } n \equiv 2\ (3) \\ x_0 & n = 1 \\ x_2 + v_1^2 v_3^4 x_1^2 + v_1^4 v_3^{2^{m+3}} x_1 & n = 3 \\ x_{n-2} y_{n-3} & n \equiv 0, 1\ (3) \text{ and } n \geq 4. \end{cases} \tag{4.1}$$

We also consider cocycles $z_n \in \Sigma_m(2)$:

$$z_n = \begin{cases} s_1^{2^{n+1}} & n = 0, 1 \\ r_1^{2^{n-1}} & n = 2, 3 \\ x_{n-3} z_{n-3} & n > 3. \end{cases} \tag{4.2}$$

**Proposition 4.3.** *For the differential $d : \Omega^0_{\Sigma_m(2)} E_m(2)_*/(2) \to \Omega^1_{\Sigma_m(2)} E_m(2)_*/(2)$ of the cobar complex,*

$$d(x_n) = v_1^{a_n} z_n.$$

**Proof.** For $n = 0$ and $1$, it is immediate from Lemma 3.3, and the cases for $n = 2$ and $3$ follow from the computation $d(x_2) = d(u_2^4 + v_1^3 u_2) = v_1^4 s_1^8 + v_1^4 s_1^2 = v_1^6 r_1^2$ by (3.4). For $n = 4$,

$$\begin{array}{rcl} d(x_4) & \equiv & d(x_2^4 + v_1^{18} x_2 + v_1^{20} v_3^4 x_1^2 + v_1^{22} v_3^{2^{m+3}} x_1) \\ & \equiv & v_1^{24} r_1^8 + v_1^{24} r_1^2 + v_1^{24} v_3^4 s_1^8 + v_1^{24} v_3^{2^{m+3}} s_1^4 \ \equiv \ v_1^{26} z^2 \ \equiv \ v_1^{26} x_1 z_1 \mod (2, v_1^{28}) \end{array}$$

by the definition of $z$.

Suppose inductively that $d(x_{3k+1}) = v_1^{a_{3k+1}} x_{3k-2} z_{3k-2} \mod (2, v_1^{a_{3k+1}+2})$ for $k > 0$.

$$\begin{array}{rcl} d(x_{3k+1}^2) & \equiv & v_1^{2a_{3k+1}} x_{3k-2}^2 z_{3k-2}^2 \mod (2, v_1^{2a_{3k+1}+4}) \\ d(v_1^{a_{3k+2}-3} y_{3k+1}) & \equiv & d(v_1^{a_{3k+2}-3} x_{3k-1} y_{3k-2}) \\ & \equiv & v_1^{a_{3k+2}-3} x_{3k-1}(v_1 z_{3k-2}^2 + v_1^3 z_{3k-1}) \mod (2, v_1^{a_{3k+2}-3+a_{3k-1}}) \end{array}$$

and the sum shows $d(x_{3k+2}) \equiv v_1^{a_{3k+2}} x_{3k-1} z_{3k-1} \mod (2, v_1^{a_{3k+2}+2})$. Similarly,

$$\begin{array}{rcl} d(x_{3k+2}^4) & \equiv & v_1^{4a_{3k+2}} x_{3k-1}^4 z_{3k-1}^4 \mod (2, v_1^{4a_{3k+2}+8}) \\ d(v_1^{a_{3k+4}-8} y_{3k+3}) & \equiv & d(v_1^{a_{3k+4}-8} x_{3k+1} y_{3k}) \\ & \equiv & v_1^{a_{3k+4}-8} x_{3k+1}(v_1^6 z_{3k}^2 + v_1^8 z_{3k+1}) \mod (2, v_1^{a_{3k+4}-8+a_{3k+1}}) \end{array}$$

and we have $d(x_{3k+4}) = v_1^{a_{3k+4}} x_{3k+1} z_{3k+1} \mod (2, v_1^{a_{3k+4}+2})$, which completes the induction. □

**Proof of Lemma 1.10.** It suffices to show that $h_{1,0}/v_1^j \in E_2^1(L_2 V_m(1)_\infty)$ equals $\zeta/v_1^{j-2}$. The element $h_{1,0}/v_1^j$ is represented by $s_1/v_1^j$. We make a computation in the cobar complex

$$\begin{array}{rcl} d(u_2^2/v_1^{j+2}) & = & s_1^4/v_1^j \ = \ s_1/v_1^j + r_1/v_1^{j-1} \\ d(v_3^2 u_2^2/v_1^{j+1}) & = & v_3^2 s_1^4/v_1^{j-1} \\ d(v_3^{2^{m+2}} u_2/v_1^j) & = & v_3^{2^{m+2}} s_1^2/v_1^{j-1} \\ d(x_2^2/v_1^{j+11}) & = & r_1^4/v_1^{j-1} \end{array}$$

by Lemma 3.3 and Proposition 4.3. The sum yields the homologous relation $s_1/v_1^j \sim z/v_1^{j-2}$ by Lemma 3.6, and so $h_{1,0}/v_1^j = \zeta/v_1^{j-2}$ in $E_2^1(L_2 V_m(1)_\infty)$. □

**Proof of Proposition 1.6.** We consider the $v_1$-Bockstein spectral sequence given by the short exact sequence $0 \to E_m(2)_*(V_m(1)) \xrightarrow{\varphi} E_m(2)_*(V_m(1)_\infty) \xrightarrow{v_1} E_m(2)_*(V_m(1)_\infty) \to 0$ for $\varphi$ given by $\varphi(x) = x/v_1$. Let $B^*$ denote the $\mathbb{Z}/2[v_1]$-module of the proposition. Then, it is easy to see that $B^s$ contains the image of $\varphi_*\colon E_2^s(L_2 V_m(1)) \to E_2^s(L_2 V_m(1)_\infty)$ and that Proposition 4.3 defines a homomorphism $f\colon B^s \to E_2^s(L_2 V_m(1)_\infty)$. We also consider the composite $\partial = \delta \circ f\colon B^s \to E_2^{s+1}(L_2 V_m(1))$, where $\delta\colon E_2^s(L_2 V_m(1)_\infty) \to E_2^{s+1}(L_2 V_m(1))$ denotes the connecting homomorphism associated to the short exact sequence. By [7, Remark 3.11], it suffices to show the sequence

$$0 \longrightarrow \mathrm{Coker}\ \partial \xrightarrow{\varphi_*} B^* \xrightarrow{v_1} B^* \xrightarrow{\partial} \mathrm{Im}\ \partial \longrightarrow 0 \qquad (4.4)$$

is exact.

We decompose $E_2^*(L_2V_m(1))$ into the direct sum of $M_C = K_{m-1}(2)_*[u_2^2]\{u_2\}\otimes\wedge(h_{10}, h_{20}, h_{21})$, $M_I = K_{m-1}(2)_*[u_2^2]\{h_{11}\} \otimes \wedge(h_{10}, h_{20}, h_{21})$ and $N \otimes \wedge(\zeta) = K_{m-1}(2)_*[u_2^2] \otimes \wedge(h_{10}, h_{20}, h_{21}, \zeta)$. We notice that for non-negative integers $n$ and $r$ with $r < 8$, there exist uniquely non-negative integers $t$ and $q$ such that $n = 8^q t + re_q$. By this fact, we decompose summands of $N$ as follows:

$$\begin{aligned}
&K_{m-1}(2)_*[u_2^2]\\
&\quad= \quad K_{m-1}(2)_* \oplus \textstyle\bigoplus_{k\geq 1} \underline{x_k K_{m-1}(2)_*[x_{k+1}]}_A,\\
&K_{m-1}(2)_*[u_2^2]h_{10}\\
&\quad= \quad \textstyle\bigoplus_{q\geq 0} \left(\left(\underline{x_{3q+2}K_{m-1}(2)_*[x_{3q+3}]}_a \oplus \underline{x_{3q+3}K_{m-1}(2)_*[x_{3q+4}]}_b\right)\zeta_{3q+4} \oplus \underline{\underline{K_{m-1}(2)_*[x_{3q+2}]\zeta_{3q+1}}}_A\right),\\
&K_{m-1}(2)_*[u_2^2]h_{20}\\
&\quad= \quad \textstyle\bigoplus_{q\geq 0} \left(\underline{x_{3q+3}K_{m-1}(2)_*[x_{3q+4}]\zeta_{3q+5}}_c \oplus \left(\underline{x_{3q+1}K_{m-1}(2)_*[x_{3q+2}]}_d \oplus \underline{\underline{K_{m-1}(2)_*[x_{3q+3}]}}_A\right)\zeta_{3q+2}\right),\\
&K_{m-1}(2)_*[u_2^2]h_{21}\\
&\quad= \quad \textstyle\bigoplus_{q\geq 0} \left(\underline{x_{3q+1}K_{m-1}(2)_*[x_{3q+2}]}_e \oplus \underline{x_{3q+2}K_{m-1}(2)_*[x_{3q+3}]}_f \oplus \underline{\underline{K_{m-1}(2)_*[x_{3q+4}]}}_A\right)\zeta_{3q+3},
\end{aligned}$$

$$\begin{aligned}
&K_{m-1}(2)_*[u_2^2]h_{10}h_{20}\\
&\quad= \quad \textstyle\bigoplus_{q\geq 0} \Big(\underline{\underline{K_{m-1}(2)_*[x_{3q+3}]\zeta_{3q+4}\zeta_{3q+2}}}_a \oplus \underline{x_{3q+3}K_{m-1}(2)_*[x_{3q+4}]\zeta_{3q+4}\zeta_{3q+5}}_B\\
&\qquad\qquad \oplus \underline{\underline{K_{m-1}(2)_*[x_{3q+2}]\zeta_{3q+1}\zeta_{3q+2}}}_d\Big),\\
&K_{m-1}(2)_*[u_2^2]h_{20}h_{21}\\
&\quad= \quad \textstyle\bigoplus_{q\geq 0}\left(\underline{\underline{K_{m-1}(2)_*[x_{3q+4}]\zeta_{3q+3}\zeta_{3q+5}}}_c \oplus \left(\underline{x_{3q+1}K_{m-1}(2)_*[x_{3q+2}]}_B \oplus \underline{\underline{K_{m-1}(2)_*[x_{3q+3}]}}_f\right)\zeta_{3q+2}\zeta_{3q+3}\right),\\
&K_{m-1}(2)_*[u_2^2]h_{10}h_{21}\\
&\quad= \quad \textstyle\bigoplus_{q\geq 0}\left(\left(\underline{K_{m-1}(2)_*[x_{3q+3}]x_{3q+2}}_B \oplus \underline{\underline{K_{m-1}(2)_*[x_{3q+4}]}}_b\right)\zeta_{3q+4}\zeta_{3q+3} \oplus \underline{\underline{K_{m-1}(2)_*[x_{3q+2}]\zeta_{3q+1}\zeta_{3q+3}}}_e\right),\\
&K_{m-1}(2)_*[u_2^2]h_{10}h_{20}h_{21}\\
&\quad= \quad \textstyle\bigoplus_{k\geq 1} \underline{\underline{K_{m-1}(2)_*[x_{k+1}]\zeta_k\zeta_{k+1}\zeta_{k+2}}}_B.
\end{aligned}$$

Here, $\underline{M}_X$ and $\underline{\underline{M'}}_X$ for modules $M$ and $M'$ mean that $M$ and $M'$ are isomorphic under a Bockstein differential $d_r$ for some $r$ so that $d_r(M) = M'$, which is seen by Proposition 4.3. Let $N_C$ (resp. $N_I$) be the direct sum of single (resp. double) underlined submodules of $N$, and put $\widetilde{M} = Q(0) \otimes \wedge(h_{1,0}, h_{2,0}, h_{2,1})$, $\widetilde{N} = \bigoplus_{k>0} Q(k) \otimes \wedge(\zeta_{k+1}, \zeta_{k+2})$. Then we have the three exact sequences

$$\begin{gathered}
0 \to M_C \xrightarrow{\varphi_*} \widetilde{M} \xrightarrow{v_1} \widetilde{M} \to M_I \to 0, \quad 0 \to N_C \xrightarrow{\varphi_*} \widetilde{N} \xrightarrow{v_1} \widetilde{N} \to N_I \to 0 \quad \text{and}\\
0 \to K_{m-1}(2)_* \to E_{m-1}(2)_*/(2, v_1^\infty) \to E_{m-1}(2)_*/(2, v_1^\infty) \to 0,
\end{gathered}$$

the direct sum of which yields the sequence (4.4). □

# 5 The Adams-Novikov $E_\infty$-term for $\pi_*(L_2T(m) \wedge M_2)$

We first show that all elements of the Adams-Novikov $E_2$-term for $\pi_*(L_2V_m(1)_\infty)$ are permanent cycles. Take an element $x/v_1^t \in E_2^0(L_2V_m(1)_\infty)$. Then $x \in E_2^0(L_2V_m(1)_t)$. Thus, if $x = y^2/v_1^t$ for some $y \in E_2^0(L_2V_m(1)_{4t})$, then $x$ is a permanent cycle. So it is sufficient to show that $d_3(x_n) = 0 \in E_2^3(L_2V_m(1)_{a_n})$ for each $n \geq 0$. We consider the integer

$$\varepsilon_n = \begin{cases} 2 & n \not\equiv 0 \ (3) \\ 0 & n \equiv 0 \ (3) \end{cases}$$

so that $V_m(1)_{a_n+\varepsilon_n}$ is a ring spectrum by Corollary 2.7.

**Lemma 5.1.** *$d_3(x_n) = 0 \in E_2^3(L_2V_m(1)_{a_n})$ for $n \geq 0$.*

**Proof.** For $n = 0$, it is shown in Lemma 3.7.

Suppose that $d_3(x_n) = \xi \in E_2^3(L_2V_m(1)_{a_n})$ for $n > 0$. Send this to $E_2^3(L_2V_m(1)_{a_{n-1}})$, and we see that $\xi = d_3(x_n) = d_3(x_{n-1}^2) \in E_2^3(L_2V_m(1)_{a_{n-1}})$. Then, the map $v_1^{\varepsilon_{n-1}} : E_2^3(L_2V_m(1)_{a_{n-1}}) \to E_2^3(L_2V_m(1)_{a_{n-1}+\varepsilon_{n-1}})$ assigns $v_1^{\varepsilon_{n-1}}\xi$ to $v_1^{2\varepsilon_{n-1}}\xi = d_3((v_1^{\varepsilon_{n-1}}x_{n-1})^2)$, which is zero, since $v_1^{\varepsilon_{n-1}}x_{n-1} \in E_2^0(L_2V_m(1)_{a_{n-1}+\varepsilon_{n-1}})$ and $V_m(1)_{a_{n-1}+\varepsilon_{n-1}}$ is a ring spectrum. It follows that $\xi = v_1^{a_{n-1}-\varepsilon_{n-1}}\xi'$ for some $\xi' \in E_2^3(L_2V_m(1)_{a_n-a_{n-1}+\varepsilon_{n-1}})$. Note that this works even if $n = 1$, though $V_m(1)$ is not a ring spectrum. Consider the commutative diagram

$$\begin{tikzcd}
V_m(1) \arrow[r, equal] \arrow[d, "v_1^{a_n-a}"'] & V_m(1) \arrow[r] \arrow[d, "v_1^{a_n}"'] & * \arrow[r] \arrow[d] & V_m(1) \arrow[d, "v_1^{a_n-a}"] \\
V_m(1)_{a_n-a+1} \arrow[r, "v_1^a"] \arrow[d] & V_m(1)_{a_n+1} \arrow[r, "i_v"] \arrow[d] & V_m(1)_a \arrow[r, "j_v"] \arrow[d, equal] & V_m(1)_{a_n-a+1} \arrow[d, "p"] \\
V_m(1)_{a_n-a} \arrow[r, "v_1^a"] & V_m(1)_{a_n} \arrow[r, "i'_v"] & V_m(1)_a \arrow[r, "j'_v"] & V_m(1)_{a_n-a},
\end{tikzcd}$$

($a = a_{n-1} - \varepsilon_{n-1}$) in which rows and columns are cofiber sequences. Let $\langle x \rangle \in \pi_*(X)$ denote a homotopy element detected by $x \in E_2^*(X)$. Noticing that $x_n \in E_2^0(L_2V_m(1)_{a_{n-1}-\varepsilon_{n-1}})$ is a permanent cycle, we see that $j_{v*}(\langle x_n \rangle) = \langle v_1^{a_n-a_{n-1}+\varepsilon_{n-1}}\zeta_n \rangle$ and $j'_{v*}(\langle x_n \rangle) = \langle \xi' \rangle$, and so $p_*(\langle v_1^{a_n-a_{n-1}+\varepsilon_{n-1}}\zeta_n \rangle) = \langle \xi' \rangle$. Since $\langle \zeta_n \rangle \in \pi_*(L_2V_m(1))$ by Theorem 1.4, we obtain $\langle \xi' \rangle = 0$, and $\langle x_n \rangle$ is in the image under the map $i'_{v*}$. It follows that there is a permanent cycle $x'_n \in E_2^0(L_2V_m(1)_{a_n})$, whose leading term is $x_n$, such that $i_{v*}(\langle x'_n \rangle) = \langle x_n \rangle \in \pi_*(L_2V_m(1)_{a_{n-1}-\varepsilon_{n-1}})$. The lemma now follows by replacing $x_n$ by $x'_n$. □

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