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Poincaré Type Inequalities for Linear Differential Operators

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ABSTRACT

Various L_p form Poincaré type inequalities [1], forward and reverse, are given for a Linear Differential Operator L, involving its related initial value problem solution y, Ly, the associated Green's function H and initial conditions at point $x_0 \in \mathbb{R}$.

RESUMEN

Varias L_p desigualdes de tipo Poincaré [1], hacia adelante o atrás, son dadas para un operador diferencial linear L, envolviendo la solución y de un problema de valor inicial asociado, Ly, la función Green asociada H y las condiciones iniciales en un punto $x_0 \in \mathbb{R}$.

Key words and phrases: *Poincaré inequality, linear differential operator.*

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1. Background

Here we follow [2], pp. 145-154.

Let $[a,b] \subset \mathbb{R}$, $a_i(x)$, $i = 0,1,\ldots,n-1$ $(n \in \mathbb{N})$, $h(x)$ be continuous functions on $[a,b]$ and let $L = D^n + a_{n-1}(x) D^{n-1} + \ldots + a_0(x)$ be a fixed linear differential operator on $C^n([a,b])$. Let $y_1(x),\ldots,y_n(x)$ be a set of linear independent solutions to $Ly = 0$. Here the associated Green's functions for L is

$$H(x,t) := \frac{\begin{vmatrix} y_1(t) & \ldots & y_n(t) \\ y_1'(t) & \ldots & y_n'(t) \\ \ldots & \ldots & \ldots \\ y_1^{(n-2)}(t) & \ldots & y_n^{(n-2)}(t) \\ y_1(x) & \ldots & y_n(x) \end{vmatrix}}{\begin{vmatrix} y_1(t) & \ldots & y_n(t) \\ y_1'(t) & \ldots & y_n'(t) \\ \ldots & \ldots & \ldots \\ y_1^{(n-2)}(t) & \ldots & y_n^{(n-2)}(t) \\ y_1^{(n-1)}(t) & \ldots & y_n^{(n-1)}(t) \end{vmatrix}}, \tag{1}$$

which is a continuous function on $[a,b]^2$.

Consider a fixed $x_0 \in [a,b]$, then

$$y(x) = \int_{x_0}^{x} H(x,t)\, h(t)\, dt, \quad \forall x \in [a,b], \tag{2}$$

is the unique solution to the initial value problem

$$Ly = h; \quad y^{(i)}(x_0) = 0, \quad i = 0,1,\ldots,n-1. \tag{3}$$

Next we assume all of the above.

2. Results

We present the following Poincaré type inequalities.

Theorem 1. *Let $x_0 < b$ and $x \in [x_0,b]$, and p, $q > 1 : \frac{1}{p} + \frac{1}{q} = 1$; $\nu > 0$.*

Then

$$1)\ \|y\|_{L_\nu(x_0,b)} \le \left(\int_{x_0}^{b}\left(\int_{x_0}^{x}|H(x,t)|^p\,dt\right)^{\nu/p}dx\right)^{1/\nu}\|Ly\|_{L_q(x_0,b)}. \tag{4}$$

When $\nu = q$ *we have*

$$2)\ \|y\|_{L_q(x_0,b)} \le \left(\int_{x_0}^{b}\left(\int_{x_0}^{x}|H(x,t)|^p\,dt\right)^{q/p}dx\right)^{1/q}\|Ly\|_{L_q(x_0,b)}. \tag{5}$$

When $\nu = p = q = 2$ *we get*

$$3)\ \|y\|_{L_2(x_0,b)} \le \left(\int_{x_0}^{b}\left(\int_{x_0}^{x}H^2(x,t)\,dt\right)dx\right)^{1/2}\|Ly\|_{L_2(x_0,b)}. \tag{6}$$

Proof. From (2) we have

$$|y(x)| \le \int_{x_0}^{x}|H(x,t)|\,|h(t)|\,dt \le$$

$$\left(\int_{x_0}^{x}|H(x,t)|^p\,dt\right)^{1/p}\left(\int_{x_0}^{x}|h(t)|^q\,dt\right)^{1/q} \le$$

$$\left(\int_{x_0}^{x}|H(x,t)|^p\,dt\right)^{1/p}\left(\int_{x_0}^{b}|h(t)|^q\,dt\right)^{1/q}. \tag{7}$$

That is

$$|y(x)|^\nu \le \left(\int_{x_0}^{x}|H(x,t)|^p\,dt\right)^{\nu/p}\|Ly\|_{L_q(x_0,b)}^{\nu}, \tag{8}$$

Therefore

$$\int_{x_0}^{b}|y(x)|^\nu\,dx \le \left(\int_{x_0}^{b}\left(\int_{x_0}^{x}|H(x,t)|^p\,dt\right)^{\nu/p}dx\right)\|Ly\|_{L_q(x_0,b)}^{\nu}, \tag{9}$$

proving the claim. □

We continue with

Theorem 2. *Let* $x_0 > a$ *and* $x \in [a, x_0]$, *and* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1;\ \nu > 0$.

Then

$$1)\ \|y\|_{L_\nu(a,x_0)} \le \left(\int_{a}^{x_0}\left(\int_{x}^{x_0}|H(x,t)|^p\,dt\right)^{\nu/p}dx\right)^{1/\nu}\|Ly\|_{L_q(a,x_0)}. \tag{10}$$

When $\nu = q$ *we have*

$$2)\ \|y\|_{L_q(a,x_0)} \le \left(\int_{a}^{x_0}\left(\int_{x}^{x_0}|H(x,t)|^p\,dt\right)^{q/p}dx\right)^{1/q}\|Ly\|_{L_q(a,x_0)}. \tag{11}$$

When $\nu = p = q = 2$ *we get*

$$3)\ \|y\|_{L_2(a,x_0)} \leq \left(\int_a^{x_0} \left(\int_x^{x_0} H^2(x,t)\,dt\right) dx\right)^{1/2} \|Ly\|_{L_2(a,x_0)}. \tag{12}$$

Proof. From (2) we have

$$|y(x)| \leq \int_x^{x_0} |H(x,t)|\,|h(t)|\,dt \leq$$

$$\left(\int_x^{x_0} |H(x,t)|^p\,dt\right)^{1/p} \left(\int_x^{x_0} |h(t)|^q\,dt\right)^{1/q} \leq$$

$$\left(\int_x^{x_0} |H(x,t)|^p\,dt\right)^{1/p} \left(\int_a^{x_0} |h(t)|^q\,dt\right)^{1/q}. \tag{13}$$

That is

$$|y(x)|^\nu \leq \left(\int_x^{x_0} |H(x,t)|^p\,dt\right)^{\nu/p} \|Ly\|^\nu_{L_q(a,x_0)}, \tag{14}$$

Therefore

$$\int_a^{x_0} |y(x)|^\nu\,dx \leq \left(\int_a^{x_0} \left(\int_x^{x_0} |H(x,t)|^p\,dt\right)^{\nu/p} dx\right) \|Ly\|^\nu_{L_q(a,x_0)}, \tag{15}$$

proving the claim. □

Extreme cases follow

Proposition 3. *Here* $x_0 < b$, $x \in [x_0, b]$, *and* $p = 1$, $q = \infty$.

Then

$$1)\ \|y\|_{L_\nu(x_0,b)} \leq \left(\int_{x_0}^{b} \left(\int_{x_0}^{x} |H(x,t)|\,dt\right)^\nu dx\right)^{1/\nu} \|Ly\|_{L_\infty(x_0,b)}. \tag{16}$$

When $\nu = 1$ *we have*

$$2)\ \|y\|_{L_1(x_0,b)} \leq \left(\int_{x_0}^{b} \left(\int_{x_0}^{x} |H(x,t)|\,dt\right) dx\right) \|Ly\|_{L_\infty(x_0,b)}. \tag{17}$$

Proof. From (2) we have

$$|y(x)| \leq \int_{x_0}^{x} |H(x,t)|\,|h(t)|\,dt \leq$$

$$\left(\int_{x_0}^{x} |H(x,t)|\,dt\right) \|h\|_{L_\infty(x_0,b)}. \tag{18}$$

That is

$$|y(x)|^{\nu} \leq \left(\int_{x_0}^{x} |H(x,t)|\, dt \right)^{\nu} \|Ly\|^{\nu}_{L_\infty(x_0,b)}, \tag{19}$$

and

$$\int_{x_0}^{b} |y(x)|^{\nu}\, dx \leq \left(\int_{x_0}^{b} \left(\int_{x_0}^{x} |H(x,t)|\, dt \right)^{\nu} dx \right) \|Ly\|^{\nu}_{L_\infty(x_0,b)}, \tag{20}$$

proving the claim. □

We continue with

Proposition 4. *Here* $x_0 > a$, $x \in [a, x_0]$, *and* $p = 1$, $q = \infty$.

Then

$$1)\ \|y\|_{L_\nu(a,x_0)} \leq \left(\int_{a}^{x_0} \left(\int_{x}^{x_0} |H(x,t)|\, dt \right)^{\nu} dx \right)^{1/\nu} \|Ly\|_{L_\infty(a,x_0)}. \tag{21}$$

When $\nu = 1$ *we get*

$$2)\ \|y\|_{L_1(a,x_0)} \leq \left(\int_{a}^{x_0} \left(\int_{x}^{x_0} |H(x,t)|\, dt \right) dx \right) \|Ly\|_{L_\infty(a,x_0)}. \tag{22}$$

Proof. From (2) we have

$$|y(x)| \leq \int_{x}^{x_0} |H(x,t)|\, |h(t)|\, dt \leq$$

$$\left(\int_{x}^{x_0} |H(x,t)|\, dt \right) \|h\|_{L_\infty(a,x_0)}. \tag{23}$$

That is

$$|y(x)|^{\nu} \leq \left(\int_{x}^{x_0} |H(x,t)|\, dt \right)^{\nu} \|Ly\|^{\nu}_{L_\infty(a,x_0)}, \tag{24}$$

and

$$\int_{a}^{x_0} |y(x)|^{\nu}\, dx \leq \left(\int_{a}^{x_0} \left(\int_{x}^{x_0} |H(x,t)|\, dt \right)^{\nu} dx \right) \|Ly\|^{\nu}_{L_\infty(a,x_0)}, \tag{25}$$

proving the claim. □

Next we give reverse Poincaré type inequalities.

Theorem 5. *Let* $x_0 < b$, $x \in [x_0, b]$, *and* $0 < p < 1$, $q < 0 : \frac{1}{p} + \frac{1}{q} = 1$, $\nu > 0$.

Assume $H(x,t) \geq 0$ *for* $x_0 \leq t \leq x$, *and* $Ly = h$ *is of fixed sign and nowhere zero.*

Then

$$1)\ \|y\|_{L_\nu(x_0,b)} \geq \left(\int_{x_0}^{b} \left(\int_{x_0}^{x} (H(x,t))^{p}\, dt \right)^{\nu/p} dx \right)^{1/\nu} \|Ly\|_{L_q(x_0,b)}. \tag{26}$$

When $\nu = p$ we get

$$2)\ \|y\|_{L_p(x_0,b)} \geq \left(\int_{x_0}^{b}\left(\int_{x_0}^{x}(H(x,t))^p\,dt\right)dx\right)^{1/p}\|Ly\|_{L_q(x_0,b)}. \tag{27}$$

When $\nu = 1$ we obtain

$$3)\ \|y\|_{L_1(x_0,b)} \geq \left(\int_{x_0}^{b}\left(\int_{x_0}^{x}(H(x,t))^p\,dt\right)^{1/p}dx\right)\|Ly\|_{L_q(x_0,b)}. \tag{28}$$

Proof. By (2) we have

$$|y(x)| = \int_{x_0}^{x} H(x,t)\,|h(t)|\,dt,\ \text{for all } x_0 \leq x \leq b. \tag{29}$$

From (29) by reverse Hölder's inequality we obtain

$$|y(x)| \geq \left(\int_{x_0}^{x}(H(x,t))^p\,dt\right)^{1/p}\left(\int_{x_0}^{x}|h(t)|^q\,dt\right)^{1/q}$$

$$\geq \left(\int_{x_0}^{x}(H(x,t))^p\,dt\right)^{1/p}\left(\int_{x_0}^{b}|h(t)|^q\,dt\right)^{1/q}, \tag{30}$$

for all $x_0 < x \leq b$.

I.e. it holds

$$|y(x)|^{\nu} \geq \left(\int_{x_0}^{x}(H(x,t))^p\,dt\right)^{\nu/p}\|h\|^{\nu}_{L_q(x_0,b)}, \tag{31}$$

for all $x_0 \leq x \leq b$, and

$$\int_{x_0}^{b}|y(x)|^{\nu}\,dx \geq \left(\int_{x_0}^{b}\left(\int_{x_0}^{x}(H(x,t))^p\,dt\right)^{\nu/p}dx\right)\|h\|^{\nu}_{L_q(x_0,b)}, \tag{32}$$

proving the claim. □

We continue with

Theorem 6. *Let $x_0 > a$, $x \in [a, x_0]$, and $0 < p < 1$, $q < 0 : \frac{1}{p} + \frac{1}{q} = 1$, $\nu > 0$.*

Assume $H(x,t) \leq 0$ for $x \leq t \leq x_0$, and $Ly = h$ is of fixed sign and nowhere zero. Then

$$1)\ \|y\|_{L_\nu(a,x_0)} \geq \left(\int_{a}^{x_0}\left(\int_{x}^{x_0}(-H(x,t))^p\,dt\right)^{\nu/p}dx\right)^{1/\nu}\|Ly\|_{L_q(a,x_0)}. \tag{33}$$

When $\nu = p$ *we get*

$$2)\ \ \|y\|_{L_p(a,x_0)} \geq \left(\int_a^{x_0} \left(\int_x^{x_0} (-H(x,t))^p \, dt \right) dx \right)^{1/p} \|Ly\|_{L_q(a,x_0)} . \tag{34}$$

When $\nu = 1$ *we have*

$$3)\ \ \|y\|_{L_1(a,x_0)} \geq \left(\int_a^{x_0} \left(\int_x^{x_0} (-H(x,t))^p \, dt \right)^{1/p} dx \right) \|Ly\|_{L_q(a,x_0)} . \tag{35}$$

Proof. From (2) we have

$$|y(x)| = \left| \int_{x_0}^{x} H(x,t)\, h(t)\, dt \right| = \left| \int_x^{x_0} H(x,t)\, h(t)\, dt \right| = \left| \int_x^{x_0} (-H(x,t))\, h(t)\, dt \right| = \int_x^{x_0} (-H(x,t))\, |h(t)|\, dt. \tag{36}$$

From (36) by reverse Hölder's inequality we obtain

$$|y(x)| \geq \left(\int_x^{x_0} (-H(x,t))^p \, dt \right)^{1/p} \left(\int_x^{x_0} |h(t)|^q \, dt \right)^{1/q} \geq \left(\int_x^{x_0} (-H(x,t))^p \, dt \right)^{1/p} \left(\int_a^{x_0} |h(t)|^q \, dt \right)^{1/q} . \tag{37}$$

for all $a \leq x < x_0$.

I.e. it holds

$$|y(x)|^{\nu} \geq \left(\int_x^{x_0} (-H(x,t))^p \, dt \right)^{\nu/p} \|Ly\|^{\nu}_{L_q(a,x_0)} , \tag{38}$$

for all $a \leq x \leq x_0$, and

$$\int_a^{x_0} |y(x)|^{\nu} \, dx \geq \left(\int_a^{x_0} \left(\int_x^{x_0} (-H(x,t))^p \, dt \right)^{\nu/p} dx \right) \|Ly\|^{\nu}_{L_q(a,x_0)} , \tag{39}$$

proving the claim. □

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