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A New Version of Fan's Theorem and its Applications

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ABSTRACT

In this article, using a generalized version of Ky Fan's Theorem, we deduce new proofs for some fixed point theorems and new existence theorems for equilibrium problems.

RESUMEN

Usando una versión generalizada del Teorema de Ky Fan, deducimos nuevas demostraciones para algunos teoremas de punto fijo y nuevos teorema de existencia para problemas de equilibrio.

Key words and phrases: *Fan's Theorem, fixed point theorem, equilibrium problem, Variational inequalities.*

Math. Subj. Class.: *47H10; 49J53; 54H25.*

1 Introduction

Many problems in nonlinear analysis can be solved by showing the nonemptyness of the intersection of certain family of subsets of an underlying set. Each point of the intersection can be a fixed point, a coincidence point, an equilibrium point, a saddle point or an optimal. The first remarkable result on nonempty intersection was the celebrated Knaster, Kuratowski and Mazurkiewicz in 1929 [15], which concerns with certain type of multimaps called the KKM maps later. Fan [11] proved that the assertion of the KKM theorem for infinite dimensional topological vector space. Brézis, Nirenberg and Stampacchia [4] improved Fan's KKM lemma [11] by assuming the closedness condition only on finite dimensional subspaces, with some topological pseudomonotone condition. Chowdhury and Tan [5], replacing finite dimensional subspaces by polytopes, restated the Brézis, Nirenberg and Stampacchia result under weaker assumptions. Ding and Tarafdar [7] obtained the result of Chowdhury and Tan under weaker compactness condition. The Chowdhury and Tan's result was also proved by Kalmoun [14] for transfer closed-valued multi-valued mappings. Our aim here is to derive a new version of Brézis, Nirenberg and Stampacchia's result and then apply it to obtain some fixed point theorems and established the existence solution of equilibrium problems and generalized variational inequalities.

For the reader's convenience, we review a few basic definitions and notations from the fixed point theory. Let X be a Hausdorff topological vector space and K be a nonempty subset of X, then we denote by $< K >$ the family of all nonempty finite subsets of K. Let K_0 be a nonempty subset of K. A set-valued map $\Gamma : K_0 \rightrightarrows K$ is called a KKM map if for each $A \in < K_0 >$, $\text{conv}(A) \subseteq \bigcup_{x\in A} \Gamma(x)$. Let Y be a nonempty set. Then, $\Gamma : Y \rightrightarrows K$ is said to be transfer closed-valued if for any $(y, x) \in Y \times K$ with $x \notin \Gamma(y)$ there exists $y' \in Y$ such that $x \notin \text{cl}_K\Gamma(y')$. If $Y = K$, then we will call Γ transfer closed-valued on K. If $K_0 \subseteq K$, then a map $\Gamma : K \rightrightarrows K$ is called transfer closed-valued on K_0 if the map $y \mapsto \Gamma(y) \bigcap K_0$, $y \in K_0$, is transfer closed-valued. A set-valued map $\Gamma : K \rightrightarrows K$ is called transfer open-valued on K if the set-valued map $\hat{\Gamma} : K \rightrightarrows K$ defined as follows: $\hat{\Gamma}(x) := K \setminus \Gamma(x)$ is transfer closed-valued on K. Let us recall that a set-valued map $\Gamma : K \rightrightarrows K$ has a maximal element, if there exists a point $\bar{x} \in K$ such that $\Gamma(\bar{x}) = \emptyset$.

Suppose that f is a real-valued bifunction on $Y \times K$. Then, we say that f is transfer lower semicontinuous(l.s.c.) in the second variable if for each $(y, x) \in Y \times K$ with $f(y, x) > 0$ there exist $y' \in Y$ and a neighborhood $U(x)$ of x in K such that $f(y', z) > 0$ for all $z \in U(x)$. If $Y = K$ and $A \subseteq K$, then we call f transfer l.s.c. in the second variable on A, if $f\,|_{A\times A}$ is transfer l.s.c. in the second variable.

Definition 1.1 Let $f : K \times K \to \mathbb{R}$. We recall that:

(i) f is pseudomonotone if, for all $(x, y) \in K \times K$, $\quad f(x, y) \geq 0$ implies $f(y, x) \leq 0$;

(ii) f is called 0-segmentary closed if $\forall x, y \in K$, when (y_α) be a net on K converging to y, then the following implication holds, if $f(u, y_\alpha) \leq 0$ for all $u \in [x, y]$, then $f(x, y) \leq 0$.

(iii) $f(.,y)$ is upper sign continuous if the following implication holds for every $x \in K$,

$$f(u,y) \geq 0, \quad \forall u \in \ ]x,y[\Rightarrow f(x,y) \geq 0,$$

We note that if f is hemicontinuous function, then f and $-f$ both are upper sign continuous.

2 Brézis, Nirenberg and Stampacchia type theorem

In [8, 9], the authors refined the Ding and Tarafdar's result [7] and the Kalmoun's result [14]. Based on the Remark 2 in [4], recently the authors obtain a short and direct proof of the following Brézis, Nirenberg and Stampacchia version of Fan's KKM Theorem [10]

Lemma 2.1 *Let K be a nonempty and convex subset of a Hausdorff t.v.s. X. Suppose that $\Gamma : K \rightrightarrows K$ is a set-valued mapping such that the following conditions are satisfied:*

(i) Γ is a KKM map;

(ii) for all $A \in \, < K >$, Γ is transfer closed-valued on conv(A);

(iii) for all $x, y \in K$, $cl_K(\bigcap_{u\in[x,y]} \Gamma(u)) \cap [x,y] = (\bigcap_{u\in[x,y]} \Gamma(u)) \cap [x,y]$;

(iv) there is a nonempty compact convex set $B \subseteq K$, such that $cl_K(\bigcap_{x\in B} \Gamma(x))$ is compact.

Then, $\bigcap_{x\in K} \Gamma(x) \neq \emptyset$.

Based on the above Lemma, here we obtain another new version of the above result.

Theorem 2.1. *Let K be a nonempty convex subset of a Hausdorff t.v.s. X. Suppose that $\Gamma : K \rightrightarrows K$ is a set-valued mapping such that the following conditions are satisfied:*

(H1) Γ is a KKM map;

(H2) $\forall A \in \langle K \rangle$, Γ is transfer closed-valued on conv(A);

(H3) $\forall x, y \in K$, $cl_K(\bigcap_{u\in[x,y]} \Gamma(u)) \cap [x,y] = (\bigcap_{u\in[x,y]} \Gamma(u)) \cap [x,y]$;

(H4) there exist a nonempty compact convex subset B of K and a nonempty compact subset D of K such that, for each $y \in K \setminus D$ there exists $x \in conv(B \cup \{y\})$ such that $y \notin \Gamma(x)$.

Then, $\bigcap_{x\in K} \Gamma(x) \neq \emptyset$.

Proof. Suppose that $A \in < K >$ and $L_A = \text{conv}(A \bigcup B)$, then L_A is compact. Let $\Gamma_A : L_A \rightrightarrows L_A$ be defined as $\Gamma_A(x) = \Gamma(x) \cap L_A$. Then, from Lemma 2.1, we have

$$\bigcap_{x\in L_A} \Gamma_A(x) \neq \emptyset.$$

Now, we show that

$$\bigcap_{x\in L_A} \Gamma_A(x) \subseteq D.$$

Suppose that this claim is not true, then there exists $y \in \bigcap_{x\in L_A} \Gamma_A(x)$ such that $y \in K \setminus D$. But by assumption (H4) there exists $x \in \text{conv}(B \cup \{y\})$ such that $y \notin \Gamma(x)$. Therefore, $x \notin L_A$. But since $y \in L_A$, then $\text{conv}(B \cup \{y\}) \subseteq L_A$ which contradicts (H4).

Assume that

$$M_A = \bigcap_{x\in L_A} \Gamma(x) \text{ for any } A \in < K >, \tag{1}$$

then

$$M_A \subseteq D \text{ for all } A \in < K > . \tag{2}$$

If $\mathcal{M} = \{M_A : A \in < K >\}$, then by (1) one can see that the class $\mathcal{M}$ has the finite intersection property. Therefore, from (2), we have

$$\bigcap_{A\in<K>} \text{cl}_K M_A \neq \emptyset.$$

If $\bar{x} \in \bigcap_{A\in<K>} \text{cl}_K M_A$, $x \in X$ and $A_0 = \{\bar{x}, x\}$, then $\text{conv}(A_0) = [\bar{x}, x]$ and

$$\bar{x} \in \text{cl}_K M_{A_0} = \text{cl}_K \left(\bigcap_{u\in L_{A_0}} \Gamma(u) \right) \subseteq \text{cl}_K \left(\bigcap_{u\in[\bar{x},x]} \Gamma(u) \right)$$

Hence, by condition (H3)

$$\bar{x} \in \text{cl}_K \left(\bigcap_{u\in[\bar{x},x]} \Gamma(u) \right) \cap [\bar{x}, x] = \left(\bigcap_{u\in[\bar{x},x]} \Gamma(u) \right) \cap [\bar{x}, x].$$

Therefore, $\bar{x} \in \Gamma(x)$ for all $x \in X$ and the proof is complete.

□

Remark 2.2. *(a) By a similar proof as that of the above theorem, we can obtain some other versions of Fan's KKM Theorem.*
Let K_0 be a nonempty subset of K and $\Gamma : K_0 \rightrightarrows K$ satisfying the following conditions:

(i) Γ is a KKM map,

(ii) for each $A \in \mathcal{F}(K_0)$, $\Gamma : A \rightrightarrows conv(A)$ is transfer closed valued,

(iii) for each $x, y \in K_0$,

$$cl_K \left(\bigcap_{u\in[x,y]\cap K_0} \Gamma(u) \right) \cap [x, y] = \left(\bigcap_{u\in[x,y]\cap K_0} \Gamma(u) \right) \cap [x, y],$$

(iv) there exists a nonempty convex compact subset B of K such that for each $y \in K \setminus B$ there exists $x \in conv(B \cup \{y\}) \cap K_0$ such that $y \notin \Gamma(x)$.

Then, $\bigcap_{x\in K_0}\Gamma(x)\neq\emptyset$.

(b) Instead of assumptions (ii) and (iii) in part (a) we can assume that Γ *is transfer closed valued. Furthermore, in this case, condition (iv) of part (a) can replaced by the following condition:*

(iv)′ there exist a nonempty compact convex subset B *of* K *and a nonempty compact subset* D *of* K *such that, for each* $y\in K\setminus D$ *there exists* $x\in conv(B\cup\{y\})\cap K_0$ *such that* $y\notin\Gamma(x)$.

3 Fixed point Theorems

In this section, we deduce slight generalizations of known fixed point theorems from Theorem 2.1.

Theorem 3.1. *Let* K *be a nonempty convex subset of a (t.v.s.)* X *and* $S: K\rightrightarrows K$ *a set-valued map such that:*

(i) for each $A\in<K>$, S^- *is transfer open valued on* $conv(A)$;

(ii) for each $x,y\in K$;

$$int\left(\bigcup_{z\in[x,y]}S^-(z)\right)\cap[x,y]=\left(\bigcup_{z\in[x,y]}S^-(z)\right)\cap[x,y];$$

(iii) there exist a nonempty convex compact subset B *of* K *and a nonempty compact subset* D *of* K *such that, for each* $y\in K\setminus D$ *there exists* $x\in conv(B\cup\{y\})$ *such that* $x\in S(y)$.

Then, either S *has a maximal element or* $convS$ *has a fixed point.*

Proof. Suppose that S has no maximal element, then $\bigcup_{y\in X}S^-(y)=K$. If $\Gamma(x)=K\setminus S^-(x)$ for all $x\in X$, then

$$\bigcap_{x\in K}\Gamma(x)=\emptyset.$$

Therefore, one of the assumptions of Theorem 2.1 does not hold for Γ. By condition (i), Γ is transfer closed-valued on conv$A$ for any $A\in<K>$. Now suppose that $x,y\in K$, $z\in[x,y]$ and $z\notin\cap_{u\in[x,y]}\Gamma(u)$. Then $z\in(\bigcup_{u\in[x,y]}S^-(u))\cap[x,y]$ and so $z\in int(\bigcup_{u\in[x,y]}S^-(u))\cap[x,y]$. Therefore, there exists an open neighborhood U of z in K such that $U\subseteq\bigcup_{u\in[x,y]}S^-(u)$. Hence,

$$U\bigcap(\cap_{u\in[x,y]}\Gamma(u))=\emptyset.$$

That is $z\notin \mathrm{cl}_K\left(\bigcap_{u\in[x,y]}\Gamma(u)\right)\cap[x,y]$ and so condition (H3) is satisfied. Also condition (iii) implies condition (H4). Therefore, Γ is not KKM map. Thus, there exists a finite subset $A=\{x_1,...,x_n\}$ of K such that

$$\text{conv(A)}\nsubseteq\bigcup_{i=1}^{n}\Gamma(x_i).$$

This implies that there is a point $x \in \text{conv}(A)$ such that $x \in S^-(x_i)$ for all $i = 1, ..., n$. Therefore, for all $i = 1, ..., n$, $x_i \in S(x)$ and $x \in \text{conv}S(x)$.

□

Remark 3.2. *When K is compact, then trivially condition (iii) holds. Furthermore, if S^- is transfer open valued on K, then we can replace $S^-(z)$ in condition (ii) by $intS^-(z)$. Hence, in this case conditions (ii) and (iii) are fulfilled.*

Now we deduce the following version of Theorem 1.2 in Ansari and Yao [1] as a corollary of our Theorem 3.1.

Corollary 3.3. *Let K be a nonempty convex subset of a Hausdorff topological vector space X. Suppose that $S, T : K \rightrightarrows K$ are two set-valued maps with nonempty values such that*

(a) for each $x \in K$, $A \in \langle S(x) \rangle$, $conv(A) \subset T(x)$;

(b) for each $A \in < K >$, S transfer open valued on $conv(A)$;

(c) for each $x, y \in K$,

$$int \left(\bigcup_{z \in [x,y]} S^-(z) \right) \cap [x, y] = \left(\bigcup_{z \in [x,y]} S^-(z) \right) \cap [x, y];$$

(e) there exist a nonempty convex compact subset B of K and a nonempty compact subset D of K such that, for each $y \in K \setminus D$ there exists $x \in conv(B \cup \{y\})$ such that $x \in S(y)$.
Then T has a fixed point.

From Theorem 3.1, we also deduce Theorem 1.1 of Djafari-Rouhani, Tarafdar and Watson [6].

Corollary 3.4. *Let K be a nonempty convex subset of a Hausdorff topological vector space X. Suppose that $S, T : K \rightrightarrows K$ are two set-valued maps with nonempty values that*
(a) for each $x \in K$, $A \in \langle S(x) \rangle$, $conv(A) \subset T(x)$.
(b) for each $y \in K$, $S^-(y)$ contains an open set O_y, which may be empty such that $K = \cup\{O_y : y \in K\}$
(c) there exist a nonempty convex compact subset B of K and a points $\{\hat{x}_1, \hat{x}_2, ..., \hat{x}_n\}$ in K such that

$$\bigcap_{x \in B} O_x^c \subseteq \bigcup_{i=1}^{n} O_{\hat{x}_i},$$

where O_x^c is the complement of O_x in K. Then T has a fixed point.

Proof. From (b-c), we obtain that

$$\bigcap_{x \in B} \{K \setminus S^-(x)\} \subseteq \bigcap_{x \in B} O_x^c \subseteq \bigcup_{i=1}^{n} O_{\hat{x}_i} \subseteq \bigcup_{i=1}^{n} S^-(\hat{x}_i).$$

Let $C = B \cup \{\hat{x}_1, \hat{x}_2, ..., \hat{x}_n\}$. Then

$$K = \bigcup_{x \in C} S^-(x).$$

Let $H = \text{conv}(C)$, then H is compact and convex and moreover,

$$H = \bigcup_{x \in C} S^-(x) \cap H \subseteq \bigcup_{x \in H} S^-(x).$$

Now we define the set-valued mapping $\Gamma : H \rightrightarrows H$ as

$$\Gamma(x) = H \setminus S^-(x).$$

Then from Remark 3.2, we conclude the proof.

□

As another consequence of Theorem 3.1, we obtain the following fixed point theorem of Ansari and Lin [2].

Corollary 3.5. *Let K be a nonempty convex subset of a Hausdorff topological vector space X. Suppose that $S, T : K \rightrightarrows K$ are two mutivalued maps such that*

(a) for each $x \in K$, $A \in \langle S(x) \rangle$, $conv(A) \subset T(x)$;

(b) there exist a nonempty convex compact subset B of K and a points $\{\hat{x}_1, \hat{x}_2, ..., \hat{x}_n\}$ in K such that

$$\bigcap_{x \in B} (K \setminus int_K S^-(x)) \subseteq \bigcup_{i=1}^{n} int_K S^-(\hat{x}_i).$$

Then T has a fixed point.

Proof. It is enough in the above corollary to set $O_x = \text{int}_K S^-(x)$ for all $x \in K$. □

Remark 3.6. *By the same argument as in the above corollary,one can also obtain a proof for Theorem 2.1 in [2].*

4 Equilibrium Problems

We now give some new applications of Theorem 2.1 in obtaining existence results of equilibrium problem.

Theorem 4.1. *Let K be a nonempty convex subset of a Hausdorff (t.v.s.) X. Suppose f is a pseudomomtone real-valued on $K \times K$ such that:*

(A1) $f(x,x) = 0$ for any $x \in X$;

(A2) for each $x, y, z \in X$ if $f(x,y) < 0$ and $f(x,z) \leq 0$, then $f(x,u) < 0$ for all $u \in]y,z[$;

(A3) for each $A \in < K >$, f is transfer l.s.c. in the second variable on conv(A);

(A4) f is 0-segmentary closed;

(A5) there exist a nonempty compact subset $D \subseteq K$ and a nonempty convex compact subset B of K such that for each $x \in K \setminus D$, there exists $y \in conv(B \cup \{x\})$ such that $f(x,y) > 0$.

Then, there exists $\bar{x} \in X$ such that $f(y,\bar{x}) \leq 0$ for all $y \in X$.

Proof. Assume that $\hat{\Gamma} : K \rightrightarrows K, \Gamma : K \rightrightarrows K$ are defined by:

$$\hat{\Gamma}(y) = \{x \in X : f(x,y) \geq 0\},$$
$$\Gamma(y) = \{x \in X : f(y,x) \leq 0\}.$$

Then, as f is pseudomonotone, $\hat{\Gamma}(y) \subseteq \Gamma(y)$ for all $y \in K$. By (A1) and (A2) $\hat{\Gamma}$ is a KKM map, so Γ is a KKM map. Condition (A5) implies condition (H4). Therefore, the conditions (H1) and (H4) are fulfilled by the set-valued map Γ. The condition (A3) implies that the condition (H2) holds for Γ. For condition (H3), suppose

$$z \in \mathrm{cl}_K \left(\bigcap_{u \in [x,y]} \Gamma(u) \right) \cap [x,y].$$

Then, there exists a net (z_α) converging to z such that $f(u, z_\alpha) \leq 0$ for all $u \in [x,y]$. Since $z \in [x,y]$, for each $v \in [u,z]$, we have also $f(v, z_\alpha) \leq 0$, hence by (A4), we obtain $f(u,z) \leq 0$ for each $u \in [x,y]$. Thus, we have

$$z \in \left(\bigcap_{u \in [x,y]} \Gamma(u) \right) \cap [x,y].$$

Hence Γ satisfies also the condition (H3). Therefore, from Theorem 2.1, we have

$$\bigcap_{y \in X} \Gamma(y) \neq \emptyset.$$

Thus, any point $\hat{x}$ in this intersection is a solution for our problem.

□

Corollary 4.2. *In Theorem 4.1, if $f(.,y)$ is upper sign continuous for every $y \in K$, then there exists $\bar{x} \in K$ such that $f(x,\bar{x}) \geq 0$ for all $x \in K$.*

Proof. By Theorem 4.1, suppose that $\bar{x} \in K$ such that $f(y,\bar{x}) \leq 0$ for all $y \in X$. Assume that there exists $\bar{y} \in K$ such that $f(\bar{x},\bar{y}) < 0$. By our assumption on $\bar{x}$ we have also $f(\bar{y},\bar{x}) \leq 0$. We will show that $f(u,\bar{y}) \geq 0$ for all $u \in]\bar{x},\bar{y}[$. Indeed, if $f(u,\bar{y}) < 0$ for some $u \in]\bar{x},\bar{y}[$, then as

$f(u, \bar{x}) \leq 0$, we obtain from (A2) that $f(u, u) < 0$, which contradicts (A1). Now by upper sign continuity of f, we have $f(\bar{x}, \bar{y}) \geq 0$, which is a contradiction. □

The following example shows that Corollary 4.2 improves the corresponding results in [3] and [13].

Example 4.3. Assume that $K = \mathbb{R}$ and $f : K \times K \to \mathbb{R}$ is defined as follows:

$$f(x,y) := \begin{cases} -y & \text{if } x = 0, \\ 1-y & \text{if } x = 1, \\ 1 \quad \text{if} & x = \frac{3}{2}, 5 > |y| \geq 3, \\ -2+y & \text{if } x = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Let $D = [-1, 2]$ and $B = [1, 2]$. For $y < -1$, we have $f(1, y) > 0$. If $y > 2$, then $f(2, y) > 0$. Therefore, f satisfies all of the condition of Corollary 4.2. But for $x = 3/2$, the set

$$\{y \in \mathbb{R} : f(3/2, y) \leq 0\} =]-3, 3[\cup] - \infty, -5[\cup]5, \infty[,$$

which is not convex and K is not compact.

As a consequence of our results, we conclude a new version of Theorem 15 in [12] and its corollary for existence of variational inequalities problem.

Corollary 4.3. *Let K be a nonempty convex subset of a Hausdorff (t.v.s.) X and $T : K \rightrightarrows X^*$. Suppose that*

(i) T is upper semicontinuous from conv(A) of any $A \in < K >$ to X^* endowed with w-topology and for each $x \in K$, $T(x)$ is convex w*-compact;*

(ii) $f(x, y) := \inf_{y^* \in T(y)} \langle y^*, y - x\rangle$ is 0-segmentary closed;

(iii) there exist a nonempty compact subset D and a nonempty convex compact subset B of K such that, for each $y \in K \backslash D$, there exists $x \in$ conv$(B \cup \{y\})$ such that

$$\inf_{y^* \in T(y)} \langle y^*, y - x\rangle > 0.$$

Then, there exist $\bar{y} \in K$ and $y_0^* \in T(\bar{y})$ such that $\langle y_0^*, x - \bar{y}\rangle \geq 0, \forall x \in K$.

Proof. Let

$$f(x, y) = \inf_{y^* \in T(y)} \langle y^*, y - x\rangle.$$

We will show that all of the conditions of Theorem 4.1 and its corollary are fulfilled by f. Lemma 2.2 in [7] implies that for each fixed $x \in K$, the function $y \to \inf_{y^* \in T(y)} \langle y^*, y - x\rangle$ is l.s.c. on conv(A) of any $A \in < K >$, hence we trivially have condition(A3) of Theorem 4.1. Since $f(x,x) = 0$, for every $x \in K$and f is affine in the first argument, hence f satisfies conditions (A1) and (A2). Trivially $f(.,y)$ is upper sign continuous, thus from Corollary 4.2, there exists $\bar{y} \in K$ such that

$$\inf_{y^* \in T(\bar{y})} \langle y^*, \bar{y} - x\rangle \leq 0, \forall x \in K.$$

Now, let $g : K \times T(\bar{y}) \to \mathbb{R}$ be defined as follows:

$$g(x, y^*) = \langle y^*, \bar{y} - x\rangle,$$

then since $T(\bar{y})$ is convex, by Kneser's minimax theorem, we have

$$\inf_{y^* \in T(\bar{y})} \sup_{x \in K} \langle y^*, \bar{y} - x\rangle = \sup_{x \in K} \inf_{y^* \in T(\bar{y})} \langle y^*, \bar{y} - x\rangle \leq 0.$$

Therefore, there exists a point $y_0^* \in T(\bar{y})$ such that

$$\sup_{x \in K} \langle y_0^*, \bar{y} - x\rangle \leq 0.$$

□

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