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Fixed Point Results for Set-Valued and Single-Valued Mappings in Ordered Spaces

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ABSTRACT

In this article we use a recursion principle and generalized iteration methods to prove existence and comparison results for fixed points of set- and single-valued mappings in ordered spaces.

RESUMEN

En este artículo usamos el principio de recurrencia y métodos de iteración generalizados para provar resultados de exitencia y comparación para puntos fijos de aplicaciones conjunto (uni-)valoradas en espacios ordenados.

Key words and phrases: *Poset, set-valued mapping, fixed point, solution, maximal, minimal, sup-center, inf-center, recursion principle, generalized iteration methods, order compact, chain complete.*

Math. Subj. Class.: *47H04, 47H07, 47H10.*

1 Introduction

Let P be a nonempty partially ordered set (poset). As an introductory result we show that a set-valued mapping $\mathcal{F}$ from P to the set $2^P \setminus \emptyset$ of nonempty subsets of P has minimal and maximal fixed points, that is, the set Fix $\mathcal{F} = \{x \in P \mid x \in \mathcal{F}(x)\}$ has minimal and maximal elements, if the following conditions hold.

(c1) $\sup\{c, y\} \in P$ for some $c \in P$ and for every $y \in P$.

(c2) If $x \leq y$ in P, then for every $z \in \mathcal{F}(x)$ there exists a $w \in \mathcal{F}(y)$ such that $z \leq w$, and for every $w \in \mathcal{F}(y)$ there exists a $z \in \mathcal{F}(x)$ such that $z \leq w$.

(c3) Strictly monotone sequences of $\mathcal{F}[P] = \bigcup\{\mathcal{F}(x) : x \in P\}$ are finite.

As for the proof, denote $x_0 = c$, and choose y_0 from $\mathcal{F}(x_0)$. If $y_0 \not\leq x_0$, then $x_0 < x_1 := \sup\{c, y_0\}$. Apply then condition (c2) to choose y_1 from $\mathcal{F}(x_1)$ such that $y_0 \leq y_1$. If $y_0 = y_1$, then stop. Otherwise, $y_0 < y_1$, whence $x_1 = \sup\{c, y_0\} \leq x_2 := \sup\{c, y_1\}$, and apply again condition (c2) to choose y_2 from $\mathcal{F}(x_2)$ such that $y_1 \leq y_2$. Continuing in a similar way, condition (c3) ensures that after a finite number of choices we get the situation, where $y_{n-1} = y_n \in \mathcal{F}(x_n)$. In view of the above construction we then have $x_n := \sup\{c, y_{n-1}\} = \sup\{c, y_n\}$.

Denoting $z_0 := x_n$ and $w_0 := y_n$ then $w_0 \in \mathcal{F}(z_0)$ and $w_0 \leq \sup\{c, w_0\} = z_0$. If $w_0 = z_0$, then z_0 is a fixed point of $\mathcal{F}$. Otherwise, denoting $z_1 := w_0$, we have $z_1 < z_0$. In view of condition (c2) there exists a $w_1 \in \mathcal{F}(z_1)$ such that $w_1 \leq w_0$. If equality holds, then $z_1 = w_0 = w_1 \in \mathcal{F}(z_1)$, so that z_1 is a fixed point of $\mathcal{F}$. Otherwise, $w_1 < w_0$, denote $z_2 := w_1$, and choose by (c2) such a $w_2 \in \mathcal{F}(z_2)$ that $w_2 \leq w_1$, and so on. Condition (c3) implies that a finite number of steps yields the situation $z_m := w_{m-1} = w_m \in \mathcal{F}(z_m)$. Thus z_m belongs to Fix $\mathcal{F}$. Being a subset of $\mathcal{F}[P]$, strictly monotone sequences of Fix $\mathcal{F}$ are finite by condition (c3). This property implies in turn that Fix $\mathcal{F}$ has minimal and maximal elements.

The above described result will be generalized in Section 3. For instance, we show that $\mathcal{F}$ has minimal and maximal fixed points when the above condition (c1) holds, condition (c2) is replaced by a stronger monotonicity condition, and (c3) is replaced by a chain completeness of the order closure of the range $\mathcal{F}[P]$. Applications to single-valued mappings are also given. Fixed points of a concrete mapping are approximated by using an algorithmic method developed from the above described reasoning.

The obtained results are used in Section 4 to derive fixed point results in ordered normed spaces and in ordered topological spaces. Existence proofs require several consecutive applications of a recursion principle and generalized iteration methods introduced in [4, 6] and presented in Section 2.

2 Recursions and iterations in posets

Given a nonempty set P, a relation $x < y$ in $P \times P$ is called a *partial ordering*, if $x < y$ implies $y \not< x$, and if $x < y$ and $y < z$ imply $x < z$. Defining $x \leq y$ if and only if $x < y$ or $x = y$, we say that $P = (P, \leq)$ is a partially ordered set (poset).

An element b of a poset P is called an *upper bound* of a subset A of P if $x \leq b$ for each $x \in A$. If $b \in A$, we say that b is the *greatest element* of A, and denote $b = \max A$. A lower bound of A and the least element, $\min A$, of A are defined similarly, replacing $x \leq b$ above by $b \leq x$. If the set of all upper bounds of A has the least element, we call it a *supremum of* A and denote it by $\sup A$. We say that y is a *maximal element* of A if $y \in A$, and if $z \in A$ and $y \leq z$ imply that $y = z$. An infimum of A, $\inf A$, and a minimal element of A are defined similarly. We say that a poset P is a *lattice* if $\inf\{x, y\}$ and $\sup\{x, y\}$ exist for all $x, y \in P$. W is called a *chain* if $x \leq y$ or $y \leq x$ for all $x, y \in W$. We say that W is *well-ordered* if nonempty subsets of W have least elements, and *inversely well-ordered* if nonempty subsets of W have greatest elements. In both cases W is a chain.

A basis to our considerations is the following recursion principle (cf. [6], Lemma 1.1.1).

Lemma 2.1. *Given a nonempty poset P, a subset $\mathcal{D}$ of $2^P = \{A : A \subseteq P\}$ with $\emptyset \in \mathcal{D}$ and a mapping $f : \mathcal{D} \to P$, there is a unique well-ordered chain C in P such that*

$$x \in C \quad \textit{if and only if } x = f(C^{<x}), \textit{ where } \; C^{<x} = \{y \in C : y < x\}. \tag{2.1}$$

If $C \in \mathcal{D}$, then $f(C)$ is not a strict upper bound of C.

As an application of Lemma 2.1 we get the following result (cf. [4], Lemma 2).

Lemma 2.2. *Given $G : P \to P$ and $c \in P$, there exists a unique well-ordered chain $C = C(G)$ in P, called* a w-o chain of cG-iterations, *satisfying*

$$x \in C \quad \textit{if and only if} \quad x = \sup\{c, G[C^{<x}]\}. \tag{2.2}$$

Proof. Denote $\mathcal{D} = \{W \subseteq P : W$ is well-ordered and $\sup\{c, G[W]\}$ exists$\}$. Defining $f(W) = \sup\{c, G[W]\}$, $W \in \mathcal{D}$, we get a mapping $f : \mathcal{D} \to P$, and (2.1) is reduced to (2.2). Thus the assertion follows from Lemma 2.1. $\square$

A subset W of a chain C is called an *initial segment* of C if $x \in W$ and $y < x$ imply $y \in W$. The following result is also used in the sequel.

Lemma 2.3. *Given $\mathcal{F} : P \to 2^P \setminus \emptyset$, denote by $\mathcal{G}$ the set of all selections from $\mathcal{F}$, i.e.,*

$$\mathcal{G} := \{G : P \to P : G(x) \in \mathcal{F}(x) \textit{ for all } x \in P\}. \tag{2.3}$$

For every $G : P \to P$ denote by C_G the longest such an initial segment of the w-o chain $C(G)$ of cG-iterations that the restriction $G|C_G$ of G to C_G is increasing (i.e., $G(x) \leq G(y)$ whenever $x \leq y$ in C_G). Define a partial ordering $\prec$ on $\mathcal{G}$ as follows.

(O) $F \prec G$ if and only if C_F is a proper initial segment of C_G and $G|C_F = F|C_F$.

Then $(\mathcal{G}, \preceq)$ has a maximal element.

Proof. Let $\mathcal{C}$ be a chain in $\mathcal{G}$. The definition (O) of $\prec$ implies that the sets C_F, $F \in \mathcal{C}$, form a nested family of well-ordered sets of P. Thus the set $C := \cup\{C_F : F \in \mathcal{C}\}$ is well-ordered. Moreover, it follows from (O) that the functions $F|C_F$, $F \in \mathcal{C}$, considered as relations in $P \times P$, are nested. This ensures that $g := \cup\{F|C_F : F \in \mathcal{C}\}$ is a function from C to P. Since each $F \in \mathcal{C}$ is increasing in C_F, then g is increasing, and $g(x) \in \mathcal{F}(x)$ for each $x \in C$. Let G be such a selection from $\mathcal{F}$ that $G|C = g$. Then $G \in \mathcal{G}$, and G is increasing on C. If $x \in C$, then $x \in C_F$ for some $F \in \mathcal{C}$. The definitions of C and the partial ordering $\prec$ imply that C_F is C or its initial segment, whence $C_F^{<x} = C^{<x}$. Because $F|C_F = g|C_F = G|C_F$, then

$$x = \sup\{c, F[C_F^{<x}]\} = \sup\{c, G[C^{<x}]\}. \tag{2.4}$$

This result implies (cf. the proof of Lemma 2.1) that C is $C(G)$ or its proper initial segment. Since G is increasing on C, then C is C_G or its proper initial segment. Consequently, G is an upper bound of $\mathcal{C}$ in $\mathcal{G}$. This result implies by Zorn's Lemma that $\mathcal{G}$ has a maximal element. □

Let $X = (X, \leq)$ be a poset. When z, $w \in X$, denote

$$[z) = \{x \in X : z \leq x\}, \ (w] = \{x \in X : x \leq w\} \text{ and } [z, w] = [z) \cap (w].$$

A subset A of X is called a *causal set* (causet) if $[z, w] \cap A$ is finite for all z, $w \in A$. We say that A is *bounded from above* if $A \subseteq (z]$, *bounded from below* if $A \subseteq [z)$, and *order bounded* if $A \subseteq [z, w]$ for some z, $w \in X$.

We say that X, equipped with a topology is an *ordered topological space* if the order intervals $[z)$ and $(z]$ are closed for each $z \in X$. If the topology of X is induced by a metric, we say that X is an *ordered metric space.*

Next we define some concepts for sequences and set-valued functions.

Definition 2.1. A sequence $(z_n)_{n=0}^\infty$ of a poset is called *increasing* if $z_n \leq z_m$ whenever $n \leq m$, *decreasing* if $z_m \leq z_n$ whenever $n \leq m$, and *monotone* if it is increasing or decreasing. If the above inequalities are strict, the sequence $(z_n)_{n=0}^\infty$ is called strictly increasing, strictly decreasing or strictly monotone, respectively.

Definition 2.2. Given posets X and P, we say $\mathcal{F} : X \to 2^P \setminus \emptyset$ is *increasing upward* if $x \leq y$ in X and $z \in \mathcal{F}(x)$ imply that $[z) \cap \mathcal{F}(y)$ is nonempty. $\mathcal{F}$ is *increasing downward* if $x \leq y$ in X and $w \in \mathcal{F}(y)$ imply that $(w] \cap \mathcal{F}(x)$ is nonempty. If $\mathcal{F}$ is increasing upward and downward, we say that $\mathcal{F}$ is *increasing.*

Definition 2.3. Given posets P and X and a set-valued function $\mathcal{F} : X \to 2^P \setminus \emptyset$, consider chains of the form $G[C]$, where C is a nonempty chain in X and G is an increasing selection from $\mathcal{F}|C$. $\mathcal{F}$

is called *chain complete upward* if such chains $G[C]$ have supremums whenever C is well-ordered, *chain complete downward* if such chains $G[C]$ have infimums whenever C is inversely well-ordered, and *chain complete* if both these conditions hold. If every such a chain $G[C]$ has an upper bound in $\mathcal{F}(x)$ whenever x is an upper bound of C in X, we say that $\mathcal{F}$ is called *strongly increasing upward*. If this condition holds with upper bounds replaced by lower bounds, we say that $\mathcal{F}$ is *strongly increasing downward*. If both these conditions hold, then $\mathcal{F}$ is said to be *strongly increasing*.

The following Corollary is an easy consequence of Definitions 2.2 and 2.3.

Corollary 2.1. *(a) $\mathcal{F} : X \to 2^P \setminus \emptyset$ is chain complete upward (respectively downward) if every nonempty chain of $\mathcal{F}[X]$ has a supremum (respectively an infimum).*
(b) If $\mathcal{F}$ is strongly increasing upward (respectively downward), then it is increasing upward (respectively downward).
(c) If $\mathcal{F}$ is increasing upward (respectively downward), and if $\max \mathcal{F}(x)$ (respectively $\min \mathcal{F}(x)$) exists for every $x \in X$, then $\mathcal{F}$ is strongly increasing upward (respectively downward).
(d) An increasing mapping $\mathcal{F}$ is strongly increasing, if chains of X are causets, or if chains of $\mathcal{F}[X]$ are finite, or if the values of $\mathcal{F}$ are finite sets.

In the case when P is an ordered topological space we have the following results.

Lemma 2.4. *Let X be a poset, P an ordered topological space, and $\mathcal{F} : X \to 2^P \setminus \emptyset$.*
(a) If $\mathcal{F}$ is increasing upward (respectively downward), and if its values are compact, then $\mathcal{F}$ is strongly increasing upward (respectively downward).
(b) Assume that (y_n) converges whenever $y_n \in \mathcal{F}(x_n)$ for every n and both (x_n) and (y_n) are increasing (respectively decreasing). If P is second countable or metrizable, then $\mathcal{F}$ is chain complete upward (respectively downward).

Proof. Consider a chain $W = G[C]$, where C is a chain in X and G is an increasing selection from $\mathcal{F}|C$.

(a) Assume that $x \in X$ is an upper bound of C. If $\mathcal{F}$ is increasing upward, then to every $y \in W$ there corresponds a $z \in [y) \cap \mathcal{F}(x)$. Because W is a chain, then the sets $[y) \cap \mathcal{F}(x)$, $y \in W$, satisfy the finite intersection property. Thus their intersection is nonempty if $\mathcal{F}(x)$ is compact, and every element from that intersection is an upper bound of W in $\mathcal{F}(x)$. Similarly one can prove that if $x \in X$ is a lower bound of C, if $\mathcal{F}$ is increasing downward, and if $\mathcal{F}(x)$ is compact, then $\mathcal{F}(x)$ contains a lower bound of W.

(b) If C is well-ordered, and (y_n) is an increasing sequence of $W = G[C]$, then $y_n = G(x_n)$, where $x_n = \min\{x \in C : G(x) = y_n\}$ for every n, and (x_n) is increasing. Thus (y_n) converges by a hypothesis of (b). If P is second countable, then every subset of W is separable. It then follows from [6], Lemma 1.1.7 that W contains an increasing sequence which converges to $\sup W$. This result follows from [6], Proposition 1.1.5 if P is an ordered metric space. These results and their duals imply the conclusions of (b). $\square$

3 Fixed point results in posets

In this section we prove existence and comparison results for fixed points of a set-valued and single-valued functions in a poset P.

3.1 Fixed point results for set-valued functions

As an application of Lemma 2.1 we obtain the following result.

Proposition 3.1. *Assume that $\mathcal{F} : P \to 2^P \setminus \emptyset$ is strongly increasing upward and chain complete upward, and that the set $S_+ = \{x \in P : [x) \cap \mathcal{F}(x) \neq \emptyset\}$ is nonempty. Then $\mathcal{F}$ has a maximal fixed point, which is also a maximal element of S_+.*

Proof. Denote $\mathcal{D} = \{W \subset S_+ : W$ is well-ordered and has a strict upper bound in $S_+\}$. Because S_+ is nonempty, then $\emptyset \in \mathcal{D}$. Let $f : \mathcal{D} \to P$ be a function which assigns to each $W \in \mathcal{D}$ an element $y = f(W) \in [x) \cap \mathcal{F}(x)$, where x is a fixed strict upper bound of W in S_+. By Lemma 2.1 there exists exactly one well ordered chain W in P satisfying (2.1). By the above construction and (2.1) each element y of W belongs to $[x) \cap \mathcal{F}(x)$, where x is a fixed strict upper bound of $W^{<y}$ in S_+. Because $\mathcal{F}$ is increasing upward and $x \leq y \in \mathcal{F}(x)$, then $[y) \cap \mathcal{F}(y) \neq \emptyset$, so that $y \in S_+$. It is easy to verify that the set C of these elements x form a well ordered chain in S_+, that the correspondence $x \mapsto y$ defines an increasing selection $G : C \to S_+$ from $\mathcal{F}|C$, and that $W = G[C]$. Because $\mathcal{F}$ is chain complete upward, then $x = \sup W$ exists in P. The above construction implies that x is also an upper bound of C. Since $\mathcal{F}$ is strongly increasing upward, then W has an upper bound y in $\mathcal{F}(x)$. Because $x = \sup W$, then $x \leq y$, so that $y \in [x) \cap \mathcal{F}(x)$, and thus $x = \sup W$ belongs to S_+. $x = \max W$, for otherwise $f(W)$ would exist, and being a strict upper bound of W, would contradict the last conclusion of Lemma 2.1. By the same reason x is a maximal element of S_+.

Because $x \leq y \in F(x)$, then $[y) \cap \mathcal{F}(y) \neq \emptyset$, or equivalently, $y \in S_+$, since $\mathcal{F}$ is increasing upward. Because x is a maximal element of S_+, then $x = y \in \mathcal{F}(x)$, so that x is a fixed point of $\mathcal{F}$. If z is a fixed point of $\mathcal{F}$ and $x \leq z$, then $z \in S_+$, whence $x = z$. Thus x is a maximal fixed point of $\mathcal{F}$. $\square$

The next result is the dual of Proposition 3.1.

Proposition 3.2. *Assume that $\mathcal{F} : P \to 2^P \setminus \emptyset$ is strongly increasing downward and chain complete downward, and that the set $S_- = \{x \in P : (x] \cap \mathcal{F}(x) \neq \emptyset\}$ is nonempty. Then $\mathcal{F}$ has a minimal fixed point, which is also a minimal element of S_-.*

If $\mathcal{F}[P]$ has an upper bound (respectively a lower bound) in P, it belongs to S_- (respectively to S_+).

Next we derive other conditions under which the set S_- or the set S_+ is nonempty.

Definition 3.1. Let A be a subset of a poset P. The set $ocl(A)$ of all possible supremums and infimums of chains of A is called an *order closure* of A. If $A = ocl(A)$, then A is *order closed.* We say that a subset A of poset P has a *sup-center* c in P if $c \in P$ and $\sup\{c, x\}$ exists in P for each $x \in A$. If $\inf\{c, x\}$ exists in P for each $x \in A$, we say that c is an *inf-center* of A in P. If c has both these properties it is called an *order center* of A in P. Phrase "in P" is omitted if $A = P$.

If P is an ordered topological space, then the order closure $ocl(A)$ of A is contained in the topological closure of A. If c is the greatest element (respectively the least element) of P, then c is an inf-center, (respectively a sup-center) of P. If P is a lattice, then its every point is an order center of P. If P is a subset of $\mathbb{R}^2$, ordered coordinatewise, a necessary and sufficient condition for a point $c = (c_1, c_2)$ of P to be a sup-center of a subset A of P in P is that whenever a point $y = (y_1, y_2)$ of A and c are unordered, then $(y_1, c_2) \in P$ if $y_2 < c_2$ and $(c_1, y_2) \in P$ if $y_1 < c_1$. No conditions are imposed on other points of A.

The following result is an application of Lemma 2.3.

Proposition 3.3. *Let $\mathcal{F} : P \to 2^P \setminus \emptyset$ be chain complete upward and strongly increasing upward. If $ocl(\mathcal{F}[P])$ has a sup-center in P, then the set $S_- = \{x \in P : (x] \cap \mathcal{F}(x) \neq \emptyset\}$ is nonempty.*

Proof. Let c be a sup-center of $ocl(\mathcal{F}[P])$ in P, let $\mathcal{G}$ be defined by (2.3), and let the partial ordering $\prec$ be defined by (O). By Lemma 2.3 $(\mathcal{G}, \preceq)$ has a maximal element G. Let $C(G)$ be the w-o chain of cG-iterations, and let $C = C_G$ be the longest initial segment of $C(G)$ on which G is increasing. Thus C is well-ordered and G is an increasing selection from $\mathcal{F}|C$. Since $\mathcal{F}$ is chain complete upward, then $w = \sup G[C]$ exists. Moreover, $\overline{x} = \sup\{c, w\}$ exists in P by the choice of c, and it is easy to see that $\overline{x} = \sup\{c, G[C]\}$. This result and (2.4) imply that for each $x \in C$,

$$x = \sup\{c, G[C^{<x}]\} \leq \sup\{c, G[C]\} = \overline{x}.$$

This proves that $\overline{x}$ is an upper bound of C. Since $\mathcal{F}$ is strongly increasing upward, then $W = G[C]$ has an upper bound z in $\mathcal{F}(\overline{x})$, and $w = \sup G[C] \leq z$. To show that $\overline{x} = \max C$, assume on the contrary that $\overline{x}$ is a strict upper bound of C. Let F be a selection from $\mathcal{F}$ whose restriction to $C \cup \{\overline{x}\}$ is $G|C \cup \{(\overline{x}, z)\}$. Since G is increasing on C and $F(x) = G(x) \leq w \leq z = F(\overline{x})$ for each $x \in C$, then F is increasing on $C \cup \{\overline{x}\}$. Moreover,

$$\overline{x} = \sup\{c, G[C]\} = \sup\{c, F[C]\} = \sup\{c, F[\{y \in C \cup \{\overline{x}\} : y < \overline{x}\}]\},$$

whence $C \cup \{\overline{x}\}$ is a subset of the longest initial segment C_F of the w-o chain of cF-iterations where F is increasing. Thus $C = C_G$ is a proper subset of C_F, and $F|C_G = F|C_F$. This means by (O) that $G \prec F$. But this is impossible because G is a maximal element of $(\mathcal{G}, \preceq)$. Consequently, $\overline{x} = \max C$. Since G is increasing on C, then $\overline{x} = \sup\{c, G[C]\} = \sup\{c, G(\overline{x})\}$. In particular, $\mathcal{F}(\overline{x}) \ni G(\overline{x}) \leq \overline{x}$, whence $G(\overline{x})$ belongs to the set $(\overline{x}] \cap \mathcal{F}(\overline{x})$. □

As a consequence of Propositions 3.1, 3.2 and 3.3 we obtain the following result.

Theorem 3.1. *Assume that $\mathcal{F} : P \to 2^P \setminus \emptyset$ is strongly increasing and chain complete. If $ocl(\mathcal{F}[P])$ has a sup-center or an inf-center in P, then $\mathcal{F}$ has minimal and maximal fixed points.*

Proof. We shall give the proof in the case when $ocl(\mathcal{F}[P])$ has a sup-center in P, the proof in the case of an inf-center being similar. The hypotheses of Proposition 3.3 are then valid, whence there exists a $\overline{x} \in P$ such that $(\overline{x}] \cap \mathcal{F}(\overline{x}) \neq \emptyset$. Thus the hypotheses of Proposition 3.2 hold, whence $\mathcal{F}$ has by Proposition 3.2 a minimal fixed point x_-. In particular $[x_-) \cap \mathcal{F}(x_-) \neq \emptyset$. The hypotheses of Proposition 3.1 are then valid, whence $\mathcal{F}$ has also a maximal fixed point. $\square$

Example 3.1. Assume that $\mathbb{R}^m$ is ordered as follows. For all $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$,

$$x \leq y \text{ if and only if } x_i \leq y_i,\ i = 1, \ldots, j, \text{ and } x_i \geq y_i,\ i = j+1, \ldots, m, \tag{3.1}$$

where $j \in \{0, \ldots, m\}$. Show that if $\mathcal{F} : \mathbb{R}^m \to 2^{\mathbb{R}^m} \setminus \emptyset$ is increasing, and its values are closed subsets of $\mathbb{R}^m$, and if $\mathcal{F}[\mathbb{R}^m]$ is contained in

$$B^p_R(c) = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : \sum_{i=1}^m |x_i - c_i|^p \leq R^p\}$$

for some p, $R \in (0, \infty)$ and $c = (c_1, \ldots, c_m) \in \mathbb{R}^m$, then $\mathcal{F}$ has minimal and maximal fixed points.

Solution. Let $x = (x_1, \ldots, x_m) \in B^p_R(c)$ be given. Since $|\max\{c_i, x_i\} - c_i| \leq |x_i - c_i|$ and $|\min\{c_i, x_i\} - c_i| \leq |x_i - c_i|$ for each $i = 1, \ldots, m$, it follows that $\sup\{c, x\}$ and $\inf\{c, x\}$ belong to $B^p_R(c)$ for all $x \in B^p_R(c)$. Moreover, every $B^p_R(c)$ is a closed and bounded subset of $\mathbb{R}^m$, whence its monotone sequences converge in $B^p_R(c)$ with respect to the Euclidean metric of $\mathbb{R}^m$. These results, Lemma 2.4 and the given hypotheses imply that $\mathcal{F}$ is chain complete, and strongly increasing, and that c is an order center of $ocl(\mathcal{F}[\mathbb{R}^m])$. Thus the hypotheses of Theorem 3.1 hold, whence $\mathcal{F}$ has minimal and maximal fixed points. $\square$

3.2 Fixed point results for single-valued functions

Next we present existence and comparison results for fixed points of single-valued functions. In the proofs we use the following consequence of Proposition 3.3.

Proposition 3.4. *Assume that $G : P \to P$ is increasing, that $ocl(G[P])$ has a sup-center c in P, and that $\sup G[C]$ exists whenever C is a nonempty chain in P. If C is the w-o chain of cG-iterations, then $\overline{x} = \max C$ exists, $\overline{x} = \sup\{c, G(\overline{x})\} = \sup\{c, G[C]\}$ and*

$$\overline{x} = \min\{z \in P : \sup\{c, G(z)\} \leq z\}. \tag{3.2}$$

Moreover, $\overline{x}$ is the least solution of the equation $x = \sup\{c, G(x)\}$ and is increasing in G.

Proof. When $\mathcal{F} : P \to 2^P \setminus \emptyset$ is single-valued, it coincides with its unique selection function $G : P \to P$. Moreover, $\mathcal{F}$ is strongly increasing upward if and only if G is increasing, in which case C in Lemma 2.3 is the w-o chain of cG-iterations. The hypotheses given for G imply also that $G = \mathcal{F}$ is chain complete upward, and that c is a sup-center of $ocl(\mathcal{F}[P])$ in P. As a single valued mapping it is also strongly increasing. Thus the proof of Proposition 3.3 implies that $\overline{x} = \max C$ exists and $\overline{x} = \sup\{c, G(\overline{x})\} = \sup\{c, G[C]\}$. To prove (3.2), let $z \in P$ satisfy $\sup\{c, G(z)\} \leq z$. Then $c = \min C \leq z$. If $x \in C$ and $\sup\{c, G(y)\} \leq z$ for each $y \in C^{<x}$, then $x = \sup\{c, G[C^{<x}]\} \leq z$. This implies by transfinite induction that $x \leq z$ for each $x \in C$. In particular $\overline{x} = \max C \leq z$. This result and the fact that $\overline{x} = \sup\{c, G(\overline{x})\}$ imply that $\overline{x} = x$ is the least solution of the equation $x = \sup\{c, G(x)\}$, and that (3.2) holds. The last assertion is an immediate consequence of (3.2). $\square$

The results presented in the next proposition are dual to those of Lemma 2.2 and Proposition 3.4.

Proposition 3.5. *Given $G : P \to P$ and $c \in P$ there exists exactly one inversely well-ordered chain D in P, called an* inversely well-ordered (i.w-o) chain of cG- iterations*, satisfying*

$$x \in D \quad \text{if and only if } x = \inf\{c, G[\{y \in C : x < y\}]\}. \tag{3.3}$$

Assume that G is increasing, that $ocl(G[P])$ has an inf-center c in P, and that $\inf G[C]$ exists whenever C is a nonempty chain in P. If D is the i.w-o chain of cG-iterations, then $\underline{x} = \min D$ exists, $\underline{x} = \inf\{c, G(\underline{x})\} = \inf\{c, G[D]\}$ and

$$\underline{x} = \max\{z \in P : z \leq \inf\{c, G(z)\}\}. \tag{3.4}$$

Moreover, $\underline{x}$ is the greatest solution of the equation $x = \inf\{c, G(x)\}$ and is increasing in G.

Our first fixed point result is a consequence of Propositions 3.4 and 3.5.

Lemma 3.1. *Let P be a poset and $G : P \to P$ an increasing mapping.*
(a) If $P \ni \underline{x} \leq G(\underline{x})$, and if $\sup G[C]$ exists whenever C is a chain in $[\underline{x})$, then the w-o chain C of $\underline{x}G$-iterations has a maximum x_* and

$$x_* = \max C = \sup G[C] = \min\{y \in [\underline{x}) : G(y) \leq y\}. \tag{3.5}$$

Moreover, x_* is the least fixed point of G in $[\underline{x})$ and is increasing in G.
(b) If $G(\overline{x}) \leq \overline{x} \in P$, and if $\inf G[C]$ exists whenever C is a chain $(\overline{x}]$, then the i.w-o chain D of $\overline{x}G$-iterations has a minimum x^* and

$$x^* = \min D = \inf G[D] = \max\{y \in (\overline{x}] : y \leq G(y)\}. \tag{3.6}$$

Moreover, x^* is the greatest fixed point of G in $(\overline{x}]$ and is increasing in G.

Proof. (a) Since G is increasing and $\underline{x} \leq G(\underline{x})$, then $G[[\underline{x})] \subset [\underline{x})$. Thus the conclusions of (a) are immediate consequences of the conclusion of Proposition 3.4 when $c = \underline{x}$ and G is replaced by its restriction to $[\underline{x})$.

The proof of (b) is dual to that of (a). □

As an application of Propositions 3.4 and 3.5 and Lemma 3.1 we get the following fixed point results.

Theorem 3.2. *Assume that $G : P \to P$ is increasing and that $c \in P$.*
(a) If c is a sup-center of $ocl(G[P])$ in P, and if $\sup G[C]$ and $\inf G[C]$ exist whenever C is a chain in P, then G has minimal and maximal fixed points. Moreover, G has the greatest fixed point x^* in $(\overline{x}]$, where $\overline{x}$ is the least solution of the equation $x = \sup\{c, G(x)\}$. Both $\overline{x}$ and x^* are increasing with respect to G.
(b) If c is an inf-center of $ocl(G[P])$ in P, and if $\sup G[C]$ and $\inf G[C]$ exist whenever C is a chain in P, then G has minimal and maximal fixed points. Moreover, G has the least fixed point x_* in $[\underline{x})$, where $\underline{x}$ is the greatest solution of the equation $x = \inf\{c, G(x)\}$. Both $\underline{x}$ and x_* are increasing with respect to G.

Lemma 3.1, Proposition 3.1 and Proposition 3.2 imply the following results.

Proposition 3.6. *Assume that $G : P \to P$ is increasing.*
(a) If $\sup G[C]$ exists whenever C is a well-ordered chain in P, and if $G[P]$ has a lower bound in P, then G has the least fixed point and a maximal fixed point.
(b) If $\sup G[D]$ exists whenever D is an inversely well-ordered chain in P, and if $G[P]$ has an upper bound in P, then G has the greatest and a minimal fixed point.

Example 3.2. Let $\mathbb{R}^m$ be ordered coordinatewise, and assume that $G : \mathbb{R}^m \to \mathbb{R}^m$ maps increasing sequences of $\mathbb{R}^m$ to bounded and increasing sequences of $\mathbb{R}^m_+$, where $\mathbb{R}_+$ is the set of nonnegative reals. Show that G has the least fixed point and a maximal fixed point.

Solution. Let C be a well-ordered chain in $\mathbb{R}^m$. Since G is increasing, by definition, then $G[C]$ is a well-ordered chain in $\mathbb{R}^m_+$. If (y_n) is an increasing sequence in $G[C]$, and $x_n = \min\{x \in C : G(x) = y_n\}$, then the sequence (x_n) is increasing and $y_n = G(x_n)$ for every n. Thus (y_n) is bounded, by definition of G, and hence converges with respect to the Euclidean metric of $\mathbb{R}^m$. This result implies by Lemma 2.4 that $\sup G[C]$ exists. Moreover, the origin is a lower bound of $G[\mathbb{R}^m]$. Thus the assertions follow from Proposition 3.6. □

3.3 Algorithmic methods

It can be shown that the first elements of the w-o chain C of cG-iterations are: $x_0 = c$, $x_{n+1} = \sup\{c, Gx_n\}$, $n = 0, 1, \ldots$, as long as x_{n+1} exists and $x_n < x_{n+1}$. Assuming that strictly monotone sequences of $G[P]$ are finite, then C is a finite strictly increasing sequence $(x_n)_{n=0}^m$. If $\sup\{c, x\}$ exists for every $x \in G[P]$, then $\overline{x} = \sup\{c, G[C]\} = \max C = x_m$ is the least solution of the equation $x = \sup\{c, G(x)\}$ by the proof of Proposition 3.4. In particular, $G\overline{x} \le \overline{x}$. If $G(\overline{x}) < \overline{x}$, then first elements of the i.w-o chain D of $\overline{x}G$-iterations of $\overline{x}$ are $y_0 = \overline{x} = x_m$, $y_{j+1} = Gy_j$, as long as $y_{j+1} < y_j$. Since strictly monotone sequences of $G[P]$ are finite, D is a finite strictly decreasing sequence $(y_j)_{j=0}^k$, and $x^* = \inf G[D] = y_k$ is the greatest fixed point of G in $(\overline{x}]$ by the proof of Lemma 3.1. This reasoning and its dual imply the following results.

Corollary 3.1. *Conclusions of Theorem 3.2 hold if $G : P \to P$ is increasing and strictly monotone sequences of $G[P]$ are finite, and if $\sup\{c, x\}$ and $\inf\{c, x\}$ exist for every $x \in G[P]$. Moreover, x^* is the last element of the finite sequence determined by the following algorithm:*

(i) $x_0 = c$. For n from 0 while $x_n \neq Gx_n$ do: $x_{n+1} = Gx_n$ if $Gx_n < x_n$ else $x_{n+1} = \sup\{c, Gx_n\}$,

and x_* is the last element of the finite sequence determined by the following algorithm:

(ii) $x_0 = c$. For n from 0 while $x_n \neq Gx_n$ do: $x_{n+1} = Gx_n$ if $Gx_n > x_n$ else $x_{n+1} = \inf\{c, Gx_n\}$.

Let $G : P \to P$ satisfy the hypotheses of Theorem 3.2. The result Corollary 3.1 can be applied to approximate the fixed points x^* and x_* of G introduced in Theorem 3.2 in the following manner. Assume that $\underline{G}$, $\overline{G} : P \to P$ satisfy the hypotheses given for G in Corollary 3.1, and that

$$\underline{G}(x) \leq G(x) \leq \overline{G}(x) \quad \text{for all} \quad x \in P. \tag{3.7}$$

Since x^* is increasing with respect to G, it follows from (3.7) that $\underline{x}^* \leq x^* \leq \overline{x}^*$, where $\underline{x}^*$ and $\overline{x}^*$ are obtained by algorithm (i) of Corollary 3.1 with G replaced by $\underline{G}$ and $\overline{G}$, respectively.

Since partial ordering is the only structure needed in the proofs, the above results can be applied to problems where only ordinal scales are available. On the other hand, these results have some practical value also in real analysis. We shall demonstrate this by an example where the above described method is applied to a system of the form

$$x_i = G_i(x_1, \ldots, x_m), \quad i = 1, \ldots, m, \tag{3.8}$$

where the functions G_i are real valued functions of m real variables.

Example 3.3. In this example we approximate a solution $x^* = (x_1, y_1)$ of the system

$$x = G_1(x, y) := \frac{N_1(x, y)}{2 - |N_1(x, y)|}, \; y = G_2(x, y) := \frac{N_2(x, y)}{3 - |N_2(x, y)|}, \tag{3.9}$$

where

$$N_1(x, y) = \frac{11}{12}x + \frac{12}{13}y + \frac{1}{234} \quad \text{and} \quad N_2(x, y) = \frac{15}{16}x + \frac{14}{15}y - \frac{7}{345}, \tag{3.10}$$

by calculating such upper and lower estimates of (x_1, y_1) whose corresponding coordinates differ less than 10^{-100}.

The mapping $G = (G_1, G_2)$, defined by (3.9), (3.10) maps the set $P = \{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq \frac{1}{2}\}$ into P, and is increasing on P. It follows from Example 2.1 that $c = (0, 0)$ is an order center of P, and that P is chain complete. Thus the results of Theorem 3.2 are valid.

Upper and lower estimates to the fixed point $x^* = (x_1, y_1)$ of G, and hence to a solution (x_1, y_1) of system (3.9), (3.10), can be obtained by applying the algorithm (i) given in Corollary 3.1 to operators $\overline{G}$ and $\underline{G}$, defined by

$$\begin{cases} \overline{G}(x,y) = (10^{-101}\mathrm{ceil}(10^{101}G_1(x,y)), 10^{-101}\mathrm{ceil}(10^{101}G_2(x,y)), \\ \underline{G}(x,y) = (10^{-101}\mathrm{floor}(10^{101}G_1(x,y)), 10^{-101}\mathrm{floor}(10^{101}G_2(x,y)), \end{cases} \tag{3.11}$$

where ceil(x) is the least integer $\geq x$ and floor(x) is the greatest integer $\leq x$. The so defined operators $\underline{G}$, $\overline{G}$ are increasing and map the set $P = \{(x,y) \in \mathbb{R}^2 : |x| + |y| \leq \frac{1}{2}\}$ into finite subsets of P, and (3.7) holds. We are going to show that the required upper and lower estimates are obtained by algorithm (i) of Corollary 3.1 with G replaced by $\underline{G}$ and $\overline{G}$, respectively. The following Maple program is used in calculations of the upper estimate $\overline{x}^* = (x1, y1)$.

```
(N1, N2):=(11/12*x+12/13*y+1/234,15/16*x+14/15*y-7/345):
(z,w):=(N1/(2-abs(N1)),N2/(3-abs(N2))):(G1,G2):=(ceil(10^101 z)/10^101 ,ceil(10^101 w)/10^101):
(x0,y0):=(0,0);x:=x0:y:=y0:u:=G1:v:=G2:b[0]:=[x,y]:
for k from 1 while abs(u-x)+abs(v-y)> 0 do:
if u<=x and v<=y then (x,y):=(u,v) else (x,y):=(max(x,u),max(y,v)):fi:
u:=G1:v:=G2:b[k]:=[x,y]:od:n:=k-1: x1:=x;y1=y;
```

The above program yields the following results (n=1246).

$$\begin{cases} x1 = -0.007753186849780811654910693041037019619471431387747172549504569995356264082732785848367182252372500 43, \\ y1 : -0.013599615424610901489836719913129280024524254401289927375880599161783854868392762013556944139785572 1 \end{cases}$$

In particular, $(x1, y1)$ is the fixed point $\overline{x}^*$ of $\overline{G}$.

Replacing 'ceil' by 'floor' in the above program we obtain components of the fixed point $\underline{x}^* = (x2, y2)$ of $\underline{G}$ (n:=1248).

$$\begin{cases} x2 = -0.007753186849780811654910693041037019619471431387747172549504569995356264082732785848367182252372500 5, \\ y2 = -0.013599615424610901489836719913129280024524254401289927375880599161783854868392762013556944139785572 15 \end{cases}$$

The above calculated components of $\overline{x}^*$ and $\underline{x}^*$ are exact, and their differences are $< 10^{-100}$. According to the above reasoning the exact fixed point x^* of G belongs order interval $[\underline{x}^*, \overline{x}^*]$. In particular, both $(x1, y1)$ and $(x2, y2)$ approximate an exact solution (x_1, y_1) of system (3.9), (3.10) with the required precision. Moreover, $x1 \leq x_1 \leq y1$ and $x2 \leq y_1 \leq y2$. $\square$

Remarks 3.1. The results of Lemma 3.1 and its dual and Corollary 3.1 could be combined to obtain upper and lower estimates also to fixed points of set-valued functions.

4 Special cases

In this section we shall first present existence and comparison results for equations and inclusions in ordered normed spaces. Next we formulate in ordered topological spaces some existence and comparison results derived in section 3.

4.1 Equations and inclusions in ordered normed spaces

Definition 4.1. A closed subset E_+ of a normed space E is called an *order cone* if $E_+ + E_+ \subseteq E_+$, $E_+ \cap (-E_+) = \{0\}$ and $cE_+ \subseteq E_+$ for each $c \geq 0$. The space E, equipped with an order relation '$\leq$', defined by

$$x \leq y \ \text{ if and only if } \ y - x \in E_+$$

is called an *ordered normed space.*

It is easy to see that the above defined order relation $\leq$ is a partial ordering in E.

Lemma 4.1. *Let C be a chain in an ordered normed space E, and assume that each monotone sequence of C has a weak limit in E. Then C contains an increasing sequence which converges weakly to* $\sup C$ *and a decreasing sequence which converges weakly to* $\inf C$*. This result holds also when weak convergence is replaced by strong convergence.*

Proof. C has by [6], Lemma 1.1.2 a well-ordered cofinal subchain W. Since all increasing sequences of W have weak limits, there is by [2], Lemma A.3.1 an increasing sequence (x_n) in W which converges weakly to $x = \sup W = \sup C$. Noticing that $-C$ is a chain whose increasing sequences have weak limits, there exists an increasing sequence (x_n) of $-C$ which converges weakly to $\sup(-C) = -\inf C$. Denoting $y_n = -x_n$, we obtain a decreasing sequence (y_n) of C which converges weakly to $\inf C$. In the case of strong convergence the conclusion follows from [6], Proposition 1.1.5.

In what follows, E is an ordered normed space having some of the following properties.

(E0) Bounded and monotone sequences of E have weak limits.

(E1) $x^+ = \sup\{0, x\}$ exists, and $\|x^+\| \leq \|x\|$ for every $x \in E$.

When $c \in E$ and $R \in [0, \infty)$, denote $B_R(c) := \{x \in E : \|x - c\| \leq R\}$. Recall (cf. e.g., [11]) that if a sequence (x_n) of a normed space E converges weakly to x, then (x_n) is bounded, i.e. $\sup_n \|x_n\| < \infty$, and

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|. \tag{4.1}$$

The next auxiliary result is needed in the sequel.

Lemma 4.2. *Let E be an ordered normed space with properties (E0) and (E1). If $c \in E$ and $R \in (0, \infty)$, then c is an order center of $B_R(c)$, and for every chain C of $B_R(c)$ both $\sup C$ and $\inf C$ exist and belong to $B_R(c)$.*

Proof. Since

$$\sup\{c, x\} = (x - c)^+ - c \quad \text{and} \quad \inf\{c, x\} = c - (c - x)^+, \quad \text{for all} \quad x \in E, \tag{4.2}$$

then (E1) and (4.2) imply that

$$\|\sup\{c, x\} - c\| = \|\inf\{c, x\} - c\| = \|(x - c)^+\| \le \|x - c\| \le R \quad \text{for every} \quad x \in B_R(c).$$

Thus both $\sup\{c, x\}$ and $\inf\{c, x\}$ belong to $B_R(c)$.

Let C be a chain in $B_R(c)$. Since C is bounded there is by (E0) and Lemma 4.1 an increasing sequence (x_n) in C which converges weakly to $x = \sup C$. Since $\|x_n - c\| \le R$ for each n, it follows from (4.1) that

$$\|x - c\| \le \liminf_{n\to\infty} \|x_n - c\| \le R.$$

Thus $x = \sup C$ exists and belongs to $B_R(c)$. Similarly one can show that $\inf G[C]$ exists in E and belongs to $B_R(c)$. □

Applying Theorem 3.2 and Lemmas 4.1 and 4.2 we obtain the following fixed point results.

Theorem 4.1. *Given a subset P of E, assume that $G : P \to P$ is increasing, and that $G[P] \subseteq B_R(c) \subseteq P$ for some $c \in E$ and $R \in (0, \infty)$. Then G has*
(a) minimal and maximal fixed points;
(b) least and greatest fixed points x_* and x^* in the order interval $[\underline{x}, \overline{x}]$ of P, where $\underline{x}$ is the greatest solution of $x = \inf\{c, G(x)\}$, and $\overline{x}$ is the least solution of $x = \sup\{c, G(x)\}$.
Moreover, x^*, x_*, $\underline{x}$ and $\overline{x}$ are all increasing with respect to G.

Proof. Let C be a chain in P. Since $G[C]$ is a chain in $B_R(c)$, then both $\sup G[C]$ and $\inf G[C]$ exist in E and belongs to $B_R(c) \subseteq P$ by Lemma 4.2. Because c is an order center of $B_R(c)$ and $ocl(G[P]) \subseteq \overline{G[P]} \subseteq B_R(c) \subseteq P$, then c is an order center of $ocl(G[P])$ in P.

The above proof shows that the hypotheses of Theorem 3.2 are valid. □

In the set-valued case we have the following consequence of Theorem 3.1.

Theorem 4.2. *Assume that P is a subset of E which contains $B_R(c)$ for some $c \in E$ and $R \in (0, \infty)$. Let $\mathcal{F} : P \to 2^P \setminus \emptyset$ be an increasing mapping whose values are weakly compact in E, and whose range $\mathcal{F}[P]$ is contained in $B_R(c)$. Then $\mathcal{F}$ has minimal and maximal fixed points.*

Remarks 4.1. Each of the following spaces has properties (E0) and (E1) (as for the proofs, see, e.g. [1, 2, 3, 5, 6, 7, 8, 10]):

(a) A Sobolev space $W^{1,p}(\Omega)$ or $W_0^{1,p}(\Omega)$, $1 < p < \infty$, ordered a.e. pointwise, where Ω is a bounded domain in $\mathbb{R}^N$.
(b) A finite-dimensional normed space ordered by a cone generated by a basis.
(c) l^p, $1 \leq p \leq \infty$, normed by p-norm and ordered coordinatewise.
(d) $L^p(\Omega)$, $1 \leq p \leq \infty$, normed by p-norm and ordered a.e. pointwise, where Ω is a σ-finite measure space.
(e) A separable Hilbert space whose order cone is generated by an orthonormal basis.
(f) A weakly complete Banach lattice or a UMB-lattice (cf.[1]).
(g) $L^p(\Omega, Y)$, $1 \leq p \leq \infty$, normed by p-norm and ordered a.e. pointwise, where Ω is a σ-finite measure space and Y is any of the spaces (b)–(f).
(h) Newtonian spaces $N^{1,p}(Y)$, $1 < p < \infty$, ordered a.e. pointwise, where Y is a metric measure space.

Thus the results of Theorems 4.1–4.2 hold if E is any of the spaces listed in (a)–(h).

4.2 Fixed point results in ordered topological spaces

Let $P = (P, \leq)$ be an ordered topological space, i.e., for each $a \in P$ the order intervals $[a) = \{x \in P : a \leq x\}$ and $(a] = \{x \in P : x \leq a\}$ are closed in the topology of P. In what follows, we assume that P has the following property:

(C) Each well-ordered chain C of P whose increasing sequences converge in P contains an increasing sequence which converges to $\sup C$, and each inversely well-ordered chain C of P whose decreasing sequences converge in P contains a decreasing sequence which converges to $\inf C$.

Corollary 4.1. *The following ordered topological spaces have property (C).*
(a) Ordered metric spaces.
(b) Order closed subsets of ordered normed spaces equipped with a norm topology.
(c) Order closed subsets of ordered normed spaces equipped with a weak topology.
(d) Ordered topological spaces which satisfy the second countability axiom.

Proof. (a) and (b) follow from the result of [6], Proposition 1.1.5 and from its dual.

(c) is a consequence of [2], Appendix, Lemma A.3.1 and its dual.

(d) If P is an ordered topological spaces which satisfies the second countability axiom, then each chain of P is separable, whence P has property (C) by the result of [6], Lemma 1.1.7 and its dual. □

The following result is a consequence of Proposition 3.6.

Proposition 4.1. *Given an ordered topological space P with property (C), assume that $G : P \to P$ is an increasing function.*

(a) If $G[P]$ has an upper bound in P, and if G maps decreasing sequences of P to convergent sequences, then G has greatest and minimal fixed points.
(b) If $G[P]$ has a lower bound in P, and if G maps increasing sequences of P to convergent sequences, then G has least and maximal fixed points.

Proof. (a) Let D be an inversely well-ordered chain in P. Since G is increasing, then $G[D]$ is inversely well-ordered. Every decreasing sequence of $G[D]$ is of the form $(G(x_n))$, where (x_n) is a decreasing sequence in D. Thus the hypotheses of (a) and property (C) imply that $x^* = \inf G[D]$ exists and belongs to P. It then follows from Proposition 3.6(b) that G has the greatest fixed point and a minimal fixed point.

The conclusions of (b) is a similar consequence of Proposition 3.6(a). □

The next fixed point result is a consequence of Proposition 4.1 and Lemma 4.1.

Corollary 4.2. *Let P be an order closed subset of an ordered normed space E whose (order) bounded and monotone sequences have weak or strong limits, and let $G : P \to P$ be increasing.*
(a) If $G[P]$ has an upper bound in P, and if G maps decreasing sequences of P to (order) bounded sequences, then G has the greatest and a minimal fixed point.
(b) If $G[P]$ has a lower bound in P, and G maps increasing sequences of P to (order) bounded sequences, then G has the least and a maximal fixed point.

The next result is a consequence of Theorem 3.2.

Theorem 4.3. *Given an ordered topological space P with property (C), assume that $G : P \to P$ is increasing and maps monotone sequences of P to convergent sequences.*
(a) If c is a sup-center of $ocl(G[P])$ in P, then G has minimal and maximal fixed points. Moreover, G has the greatest fixed point x^* in $(\overline{x}]$, where $\overline{x}$ is the least solution of the equation $x = \sup\{c, G(x)\}$. Both $\overline{x}$ and x^* are increasing with respect to G.
(b) If c is an inf-center of $ocl(G[P])$ in P, then G has minimal and maximal fixed points. Moreover, G has the least fixed point x_* in $[\underline{x})$, where $\underline{x}$ is the greatest solution of the equation $x = \inf\{c, G(x)\}$. Both $\underline{x}$ and x_* are increasing with respect to G.

As a consequence of Propositions 3.1 and 3.2 and Theorem 3.1 we get the following result.

Proposition 4.2. *Let P be an ordered topological space with property (C), and let the values of $\mathcal{F} : P \to 2^P \setminus \emptyset$ be compact.*
(a) Assume that following hypothesis holds.

$(\mathcal{F}_+)$ If (x_n) and (y_n) are increasing and $y_n \in \mathcal{F}(x_n)$ for every n, then (y_n) converges.

If the set $S_+ = \{x \in P : [x) \cap \mathcal{F}(x) \neq \emptyset\}$ is nonempty, then $\mathcal{F}$ has a maximal fixed point.
(b) Assume that $\mathcal{F}$ satisfies the following hypothesis.

($\mathcal{F}_-$) If (x_n) and (y_n) are decreasing and $y_n \in \mathcal{F}(x_n)$ for every n, then (y_n) converges.

If the set $S_- = \{x \in P : (x] \cap \mathcal{F}(x) \neq \emptyset\}$ is nonempty, then $\mathcal{F}$ has a minimal fixed point.
(b) Assume that the hypotheses ($\mathcal{F}_\pm$) hold. If $ocl(\mathcal{F}[P])$ has a sup-center or an inf-center in P, then $\mathcal{F}$ has minimal and maximal fixed points.

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