CUBO A Mathematical Journal
Vol.10, Nº04, (101–108). December 2008

Browder Convergence and Mosco Convergence for Families of Nonexpansive Mappings

Tomonari Suzuki*
Department of Mathematics, Kyushu Institute of Technology,
Tobata, Kitakyushu 804-8550, Japan
email: `suzuki-t@mns.kyutech.ac.jp`

ABSTRACT

We study the relationship between Browder's strong convergence and Mosco convergence of fixed-point set for families of nonexpansive mappings.

RESUMEN

Estudiamos la relación entre la convergencia fuerte de Browder y la convergencia de Mosco del conjunto de puntos fijos para familias de aplicaciones no espansivas.

Key words and phrases: *Nonexpansive mapping, nonexpansive semigroup, fixed point, Browder convergence, Mosco convergence.*

Math. Subj. Class.: *47H10, 47H09, 47H20.*

*The author is supported in part by Grants-in-Aid for Scientific Research from the Japanese Ministry of Education, Culture, Sports, Science and Technology.

1. Introduction

Let C be a subset of a Banach space E. A mapping T on C is called a *nonexpansive mapping* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. Using the results in Gossez and Lami Dozo [6] and Kirk [8], we can prove that $F(T)$ is nonempty in the case where C is weakly compact, convex and has the Opial property. See also [1, 5, 7] and others. In 1967, Browder [2] proved the following strong convergence theorem,

Theorem 1 (Browder [2]). *Let C be a bounded closed convex subset of a Hilbert space E and let T be a nonexpansive mapping on C. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ converging to 0. Fix $u \in C$ and define a sequence $\{x_n\}$ in C by $x_n = (1 - \alpha_n) T x_n + \alpha_n u$ for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to Pu, where P is the metric projection from C onto $F(T)$.*

A family of mappings $\{T(t) : t \geq 0\}$ is called a *one-parameter strongly continuous semigroup of nonexpansive mappings* (*nonexpansive semigroup*, for short) on C if the following are satisfied:

(NS1) For each $t \geq 0$, $T(t)$ is a nonexpansive mapping on C.

(NS2) $T(s + t) = T(s) \circ T(t)$ for all $s, t \geq 0$.

(NS3) For each $x \in C$, the mapping $t \mapsto T(t)x$ from $[0, \infty)$ into C is strongly continuous.

Suzuki [15] proved that $\bigcap_t F(T(t))$ is nonempty provided C is bounded closed convex and every nonexpansive mapping on C has a fixed point. He also proved a semigroup version of Browder's convergence theorem in [11, 17].

Theorem 2 ([11, 17]). *Let E be a smooth Banach space with the Opial property such that the normalized duality mapping J of E is weakly sequentially continuous at zero. Let C be a weakly compact convex subset of E. Let $\{T(t) : t \geq 0\}$ be a nonexpansive semigroup on C. Let τ be a nonnegative real number. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences in $\mathbb{R}$ satisfying $0 < \alpha_n < 1$, $0 \leq \tau + t_n$ and $t_n \neq 0$ for $n \in \mathbb{N}$, and $\lim_n t_n = \lim_n \alpha_n / t_n = 0$. Fix $u \in C$ and define a sequence $\{x_n\}$ in C by $x_n = (1 - \alpha_n) T(\tau + t_n) x_n + \alpha_n u$ for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to Pu, where P is the unique sunny nonexpansive retraction from C onto $\bigcap_t F(T(t))$.*

Motivated by Theorem 2, Suzuki [19] considered the Mosco convergence of $\{F(T(\tau + t_n))\}$. The following theorem is a corollary of the main result in [19].

Theorem 3 ([19]). *Let E, C and $\{T(t) : t \geq 0\}$ be as in Theorem 2. Let τ be a nonnegative real number and let $\{t_n\}$ be a sequence in $\mathbb{R}$ satisfying $0 \leq \tau + t_n$ and $t_n \neq 0$ for $n \in \mathbb{N}$, and $\lim_n t_n = 0$. Then $\{F(T(\tau + t_n))\}$ converges to $\bigcap_t F(T(t))$ in the sense of Mosco.*

Therefore we can guess that Browder convergence is strongly connected with Mosco convergence. In this paper, we study the relationship between Browder convergence and Mosco convergence for families of nonexpansive mappings.

2. Preliminaries

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers.

Let E be a Banach space and let $\{A_n\}$ be a sequence of subsets of E. Define two sets

$$\operatorname*{s\text{-}liminf}_{n\to\infty} A_n \quad \text{and} \quad \operatorname*{w\text{-}limsup}_{n\to\infty} A_n$$

as follows: $x \in \text{s-liminf}_n A_n$ if and only if there exist a sequence $\{x_n\}$ in E and $n_0 \in \mathbb{N}$ such that $\{x_n\}$ converges strongly to x and $x_n \in A_n$ for $n \in \mathbb{N}$ with $n \geq n_0$. $x \in \text{w-limsup}_n A_n$ if and only if there exists a sequence $\{x_n\}$ in E such that $\{x_n\}$ converges weakly to x and $\{n \in \mathbb{N} : x_n \in A_n\}$ is an infinite subset of $\mathbb{N}$. It is obvious that $\text{s-liminf}_n A_n \subset \text{w-limsup}_n A_n$ holds. We say $\{A_n\}$ converges to a subset A of E in the *sense of Mosco* [9] if $A = \text{s-liminf}_n A_n = \text{w-limsup}_n A_n$. And we write

$$A = \operatorname*{M\text{-}lim}_{n\to\infty} A_n.$$

Let E be a Banach space. The *normalized duality mapping* J of E is defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}.$$

E is said to be *smooth* if and only if $J(x)$ consists of one element for every $x \in E$. If E is smooth, then we can consider that J is a mapping from E into E^*. J is said to be *weakly sequentially continuous at zero* if for every sequence $\{x_n\}$ in E which converges weakly to $0 \in E$, $\{J(x_n)\}$ converges weakly* to $0 \in E^*$.

A nonempty subset C of a Banach space E is said to have the *Opial property* [10] if for each weakly convergent sequence $\{x_n\}$ in C with weak limit $z_0 \in C$,

$$\liminf_{n\to\infty} \|x_n - z_0\| < \liminf_{n\to\infty} \|x_n - z\|$$

holds for $z \in C$ with $z \neq z_0$. All nonempty compact subsets have the Opial property. Also, all Hilbert spaces, $\ell^p (1 \leq p < \infty)$ and finite dimensional Banach spaces have the Opial property. A Banach space with a duality mapping which is weakly sequentially continuous also has the Opial property [6]. We know that every separable Banach space can be equivalently renormed so that it has the Opial property [4].

Let C and K be subsets of a Banach space E. A mapping P from C into K is called sunny [3] if

$$P\big(Px + t\,(x - Px)\big) = Px$$

for $x \in C$ and $t \geq 0$ with $Px + t\,(x - Px) \in C$.

Let $\{S_n\}$ be a sequence of nonexpansive mappings on a closed convex subset C of a Banach space E and let $\{\alpha_n\}$ be a sequence in $(0, 1]$ with $\lim_n \alpha_n = 0$. $(E, C, \{S_n\}, \{\alpha_n\})$ is said to have *Browder's property* [16] if for each $u \in C$, a sequence $\{x_n\}$ defined by

$$x_n = (1 - \alpha_n)\, S_n x_n + \alpha_n\, u \tag{1}$$

for $n \in \mathbb{N}$ converges strongly. We note that $\{x_n\}$ is well defined because $x \mapsto (1-\alpha_n) S_n x + \alpha_n u$ is contractive. We know the following.

Lemma 1 ([16]). *Let $(E, C, \{S_n\}, \{\alpha_n\})$ have Browder's property. For each $u \in C$, put*

$$Pu = \lim_{n\to\infty} x_n, \tag{2}$$

where $\{x_n\}$ is a sequence in C defined by (1). Then P is a nonexpansive mapping on C.

Using P, we can rewrite Theorem 2 as follows.

Theorem 4. *Let E, C, $\{T(t) : t \geq 0\}$, τ, $\{\alpha_n\}$ and $\{t_n\}$ be as in Theorem 2. Then $\big(E, C, \{T(\tau + t_n)\}, \{\alpha_n\}\big)$ has Browder's property. Moreover a mapping P defined by (2) is the unique sunny nonexpansive retraction from C onto $\bigcap_t F(T(t))$.*

3. Main results

In this section, we prove our main results.

Theorem 5. *Let $(E, C, \{S_n\}, \{\alpha_n\})$ satisfy Browder's property. Assume that C has the Opial property. Define a mapping P on C by (2). Then* w-limsup$_n$ $F(S_n) \subset F(P)$ *holds.*

Proof. Fix $x \in$ w-limsup$_n$ $F(S_n)$. Then there exist a subsequence $\{n_k\}$ of $\{n\}$ and a sequence $\{u_k\}$ in C such that $u_k \in F(S_{n_k})$ and $\{u_k\}$ converges weakly to x. We note that $\{u_k\}$ is bounded. Define a sequence $\{v_n\}$ in C by

$$v_n = (1-\alpha_n) S_n v_n + \alpha_n x.$$

Then from the assumption, $\{v_n\}$ converges strongly to Px. We have

$$\begin{aligned} \|u_k - v_{n_k}\| &\leq (1-\alpha_{n_k}) \|u_k - S_{n_k} v_{n_k}\| + \alpha_{n_k} \|u_k - x\| \\ &\leq (1-\alpha_{n_k}) \|u_k - v_{n_k}\| + \alpha_{n_k} \|u_k - x\| \end{aligned}$$

and hence $\|u_k - v_{n_k}\| \leq \|u_k - x\|$. So

$$\liminf_{k\to\infty} \|u_k - Px\| \leq \liminf_{k\to\infty} \big(\|u_k - v_{n_k}\| + \|v_{n_k} - Px\|\big) \leq \liminf_{k\to\infty} \|u_k - x\|.$$

From the Opial property, we obtain $Px = x$. □

As a direct consequence of Theorem 5, we obtain the following.

Theorem 6. *Let $(E, C, \{S_n\}, \{\alpha_n\})$ satisfy Browder's property. Define a mapping P on C by (2). Assume that C has the Opial property and $F(P) \subset F(S_n)$ for $n \in \mathbb{N}$. Then* M-lim$_n$ $F(S_n) = F(P)$ *holds.*

Proof. From the assumption, $F(P) \subset \text{s-liminf}_n F(S_n)$. So by Theorem 5, we obtain the desired result. $\square$

Remark. Using Theorems 2 and 6, we can prove Theorem 3.

We next apply Theorem 6 to infinite families of nonexpansive mappings. The following convergence theorem was proved in [12, 14].

Theorem 7 ([12, 14])**.** *Let E and C be as in Theorem 2. Let $\{T_n : n \in \mathbb{N}\}$ be an infinite family of commuting nonexpansive mappings on C. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences in $(0, 1/2)$ satisfying $\lim_n t_n = \lim_n \alpha_n / {t_n}^\ell = 0$ for $\ell \in \mathbb{N}$. Let $\{I_n\}$ be a sequence of nonempty subsets of $\mathbb{N}$ such that $I_n \subset I_{n+1}$ for $n \in \mathbb{N}$, and $\bigcup_n I_n = \mathbb{N}$. Define a sequence $\{S_n\}$ of nonexpansive mappings on C by*

$$S_n x = \left(\left(1 - \sum_{k \in I_n} {t_n}^k \right) T_1 x + \sum_{k \in I_n} {t_n}^k \, T_{k+1} x \right).$$

Then $\big(E, C, \{S_n\}, \{\alpha_n\}\big)$ has Browder's property. Moreover a mapping P defined by (2) is the unique sunny nonexpansive retraction from C onto $\bigcap_n F(T_n)$.

By Theorem 6, we obtain the following.

Theorem 8. *Let E and C be as in Theorem 2. Let $\{T_n : n \in \mathbb{N}\}$ be an infinite family of commuting nonexpansive mappings on C. Let $\{t_n\}$ be a sequence in $(0, 1/2)$ converging to 0. Let $\{I_n\}$ and $\{S_n\}$ be as in Theorem 7. Then $\{F(S_n)\}$ converges to $\bigcap_n F(T_n)$ in the sense of Mosco.*

Proof. Put $\alpha_n = {t_n}^n$. Then it is obvious that $\lim_n \alpha_n / {t_n}^\ell = 0$ holds for $\ell \in \mathbb{N}$. By Theorem 7, $\big(E, C, \{S_n\}, \{\alpha_n\}\big)$ has Browder's property and a mapping P defined by (2) is the unique sunny nonexpansive retraction from C onto $\bigcap_n F(T_n)$. Since $F(P) = \bigcap_n F(T_n)$, we have $F(P) \subset F(T_n)$. So by Theorem 6, we obtain the desired result. $\square$

We recall that a family of mappings $\{T(p) : p \in [0, \infty)^\ell\}$ is said to be an *ℓ-parameter nonexpansive semigroup* on a subset C of a Banach space E if the following are satisfied:

(ℓNS1) For each $p \in [0, \infty)^\ell$, $T(p)$ is a nonexpansive mapping on C.

(ℓNS2) $T(p + q) = T(p) \circ T(q)$ for all $p, q \in [0, \infty)^\ell$.

(ℓNS3) For each $x \in C$, the mapping $p \mapsto T(p)x$ from $[0, \infty)^\ell$ into C is continuous.

We denote by $\mathbb{Q}$ the set of all rational numbers. Using the result in [13], we obtain the following.

Theorem 9. *Let E and C be as in Theorem 2. Let $\{T(p) : p \in [0, \infty)^\ell\}$ be an ℓ-parameter nonexpansive semigroup on C. Let $p_1, p_2, \cdots, p_\ell \in [0, \infty)^\ell$ such that $\{p_1, p_2, \cdots, p_\ell\}$ is linearly independent in the usual sense. Let $\beta_1, \beta_2, \cdots, \beta_\ell \in \mathbb{R}$ such that $\{1, \beta_1, \beta_2, \cdots, \beta_\ell\}$ is linearly* *independent over $\mathbb{Q}$. Suppose $p_0 := \beta_1 p_1 + \beta_2 p_2 + \cdots + \beta_\ell p_\ell \in [0,\infty)^\ell$. Let $\{t_n\}$ be a sequence in $(0,1/2)$ converging to 0. Define a sequence $\{S_n\}$ of nonexpansive mappings on C by*

$$S_n x = \left(1 - \sum_{k=1}^{\ell} {t_n}^k\right) T(p_0)x + \sum_{k=1}^{\ell} {t_n}^k \, T(p_k)x.$$

Then $\{F(S_n)\}$ converges to $\bigcap_p F(T(p))$ in the sense of Mosco.

4. Additional results

In this section, we observe Browder's property.

Proposition 1. *Let $(E, C, \{T\}, \{\alpha_n\})$ satisfy Browder's property. Define a mapping P on C by (2). Then P is a nonexpansive retraction from C onto $F(T)$.*

Proof. We first fix $x \in C$ and define a sequence $\{u_n\}$ in C by $u_n = (1-\alpha_n)\,Tu_n + \alpha_n\, x$. Then since $\{u_n\}$ converges strongly to Px, we obtain $Px = TPx$, which implies $Px \in F(T)$. We next fix $y \in F(T)$ and define a sequence $\{v_n\}$ in C by $v_n = (1-\alpha_n)\,Tv_n + \alpha_n\, y$. Then since $y = (1-\alpha_n)\,Ty + \alpha_n\, y$, we have $v_n = y$ and hence $Py = y$. This completes the proof. $\square$

Remark. Though it is not interesting, we have confirmed that $\text{M-}\lim_n F(T) = F(P)$ holds.

There is an example such that P is not a retraction. See also [18].

Example 1. Let E be the two dimensional real Hilbert space and put $C = E$. For $t \geq 0$, define a 2×2 matrices $T(t)$ by

$$T(t) = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}.$$

We can consider that $\{T(t) : t \geq 0\}$ is a linear nonexpansive semigroup on C. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences in $\mathbb{R}$ satisfying $0 < \alpha_n < 1$ and $0 < t_n$ for $n \in \mathbb{N}$, $\lim_n \alpha_n = \lim_n t_n = 0$ and $\eta := \lim_n t_n/\alpha_n \in (0,\infty)$. Then $(E, C, \{T(t_n)\}, \{\alpha_n\})$ satisfies Browder's property. However, a mapping P defined by (2) is not a retraction.

Proof. For $\alpha \in (0,1)$ and $t \in (0,\infty)$, we put a 2×2 matrix $P(\alpha, t)$ by

$$P(\alpha,t) = \frac{\alpha}{4\,(1-\alpha)\,\sin^2(t/2) + \alpha^2} \begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

where $a = \alpha + 2\,(1-\alpha)\,\sin^2(t/2)$ and $b = (1-\alpha)\,\sin(t)$. It is easy to verify that for $u \in C$, $P(\alpha,t)u$ is the unique point satisfying $x = (1-\alpha)\,T(t)x + \alpha\, u$. We have

$$P := \lim_{n\to\infty} P(\alpha_n, t_n) = \frac{1}{\eta^2+1} \begin{bmatrix} 1 & -\eta \\ \eta & 1 \end{bmatrix} = \frac{1}{\sqrt{\eta^2+1}}\, T(\theta),$$

where $\theta := \arctan(\eta) \in (0, \pi/2)$. Hence $(E, C, \{T(t_n)\}, \{\alpha_n\})$ satisfies Browder's property. However, P does not satisfy $P^2 = P$. $\square$

We finally give an example such that $\text{M-lim}_n F(S_n) \subsetneqq F(P)$.

Example 2. Let T be a nonexpansive mapping on a bounded closed convex subset C of a Banach space E. Assume that T is not the identity mapping on C. Define a sequence $\{S_n\}$ of nonexpansive mappings on C by

$$S_n x = (1 - t_n)\, x + t_n\, Tx,$$

where $\{t_n\}$ is a sequence in $(0, 1)$ converging to 0. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$ such that $\lim_n \alpha_n = 0$ and $\lim_n \alpha_n / t_n = \infty$. Then $(E, C, \{S_n\}, \{\alpha_n\})$ satisfies Browder's property, a mapping P defined by (2) is the identity mapping on C and $\text{M-lim}_n F(S_n) \subsetneqq F(P)$ holds.

Proof. Fix $x \in C$ and define a sequence $\{u_n\}$ in C by $u_n = (1 - \alpha_n)\, S_n u_n + \alpha_n\, x$. We have

$$\begin{aligned} \|u_n - x\| &= (1 - \alpha_n)\, \|S_n u_n - x\| \\ &\leq (1 - \alpha_n)\,(1 - t_n)\, \|u_n - x\| + (1 - \alpha_n)\, t_n\, \|Tu_n - x\| \end{aligned}$$

and hence

$$\lim_{n \to \infty} \|u_n - x\| \leq \lim_{n \to \infty} \frac{(1 - \alpha_n)\, t_n}{\alpha_n + t_n - \alpha_n\, t_n}\, \|Tu_n - x\| = 0.$$

Thus, $\{u_n\}$ converges strongly to x. Therefore $(E, C, \{S_n\}, \{\alpha_n\})$ satisfies Browder's property and $Px = x$ holds. From the assumption, $F(T) \subsetneqq C = F(P)$. Since $F(S_n) = F(T)$, we have $\text{M-lim}_n F(S_n) = F(T)$ and hence $\text{M-lim}_n F(S_n) \subsetneqq F(P)$. $\square$

Received: April 2008. Revised: April 2008.

References

[1] F.E. Browder, *Fixed-point theorems for noncompact mappings in Hilbert space*, Proc. Nat. Acad. Sci. USA, **53** (1965), 1272–1276.

[2] ———, *Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces*, Arch. Ration. Mech. Anal., **24** (1967), 82–90.

[3] R.E. Bruck, *Nonexpansive retracts of Banach spaces*, Bull. Amer. Math. Soc., **76** (1970), 384–386.

[4] D. van Dulst, *Equivalent norms and the fixed point property for nonexpansive mappings*, J. London Math. Soc., **25** (1982), 139–144.

[5] J. García Falset, E. Llorens Fuster and E.M. Mazcuñán Navarro, *Uniformly nonsquare Banach spaces have the fixed point property for nonexpansive mappings*, J. Funct. Anal., **233** (2006), 494–514.

[6] J.-P. Gossez and E. Lami Dozo, *Some geometric properties related to the fixed point theory for nonexpansive mappings*, Pacific J. Math., **40** (1972), 565–573.

[7] M. Kato and T. Tamura, *Weak nearly uniform smoothness and worth property of ψ-direct sums of Banach spaces*, Comment. Math. Prace Mat., **46** (2006), 113–129.

[8] W.A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly, **72** (1965), 1004–1006.

[9] U. Mosco, *Convergence of convex sets and of solutions of variational inequalities*, Adv. Math., **3** (1969), 510–585.

[10] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc., **73** (1967), 591–597.

[11] T. Suzuki, *On strong convergence to common fixed points of nonexpansive semigroups in Hilbert spaces*, Proc. Amer. Math. Soc., **131** (2003), 2133–2136.

[12] ______, *Strong convergence theorems of Browder's type sequences for infinite families of nonexpansive mappings in Hilbert spaces*, Bull. Kyushu Inst. Technol., **52** (2005), 21–28.

[13] ______, *The set of common fixed points of an n-parameter continuous semigroup of mappings*, Nonlinear Anal., **63** (2005), 1180–1190.

[14] ______, *Browder's type strong convergence theorems for infinite families of nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl., **2006** (2006), Article ID 59692, 1–16.

[15] ______, *Common fixed points of one-parameter nonexpansive semigroups*, Bull. London Math. Soc., **38** (2006), 1009–1018.

[16] ______, *Moudafi's viscosity approximations with Meir-Keeler contractions*, J. Math. Anal. Appl., **325** (2007), 342–352.

[17] ______, *Browder's type convergence theorems for one-parameter semigroups of nonexpansive mappings in Banach spaces*, Israel J. Math., **157** (2007), 239–257.

[18] ______, *Some comments about recent results on one-parameter nonexpansive semigroups*, Bull. Kyushu Inst. Technol., **54** (2007), 13–26.

[19] ______, *Mosco convergence of the sets of fixed points for one-parameter nonexpansive semigroups*, Nonlinear Anal. (2007), doi:10.1016/j.na.2007.04.026.