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A Disc-Cutting Theorem and Two-Dimensional Bifurcation of a Reaction-Diffusion System with Inclusions

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ABSTRACT

We provide a topological disc-cutting theorem which allows to prove that unilateral inclusions in a reaction-diffusion system of prey-predator type with a two-dimensional bifurcation parameter necessarily have a certain global branch of (global) bifurcation points.

RESUMEN

Presentamos un teorema "Disc-Cutting" topológico el cual permite probar que inclusiones unilaterales en un sistema de reación-difusión de tipo predador-presa con parametro de bifurcación 2-dimencional, necessariamente tiene una cierta rama global de puntos de bifucarción (global).

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1 Introduction

Although we provide in Section 2 a general topological theorem about the existence of a global branch which is applicable to a large class of bifurcation problems with a parameter from a space of dimension at least 2, our main motivation for the result comes from the following particular problem.

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with a Lipschitz boundary, and let measurable (possibly empty) subsets $\Omega_0 \subseteq \Omega$ and $\Gamma_0, \Gamma \subseteq \partial\Omega$ be fixed with $\mathrm{mes}(\Gamma_0 \cap \Gamma) = 0$. We consider the reaction-diffusion system

$$\begin{aligned} u_t &= d_1\Delta u + b_{11}u + b_{12}v + f_1(d,x,u,v,\nabla u,\nabla v) = 0 \qquad \text{on } \Omega, \\ v_t &\in d_2\Delta v + b_{21}u + b_{22}v + f_2(d,x,u,v,\nabla u,\nabla v) + \begin{cases} \{0\} & \text{on } \Omega \setminus \Omega_0, \\ m_0(d,x,u,v,\nabla u,\nabla v) & \text{on } \Omega_0, \end{cases} \end{aligned} \tag{1.1}$$

with the boundary conditions

$$\begin{cases} u = v = 0 & \text{on } \Gamma_0, \\ \dfrac{\partial u}{\partial n} = g_1(d,x,u,v) & \text{on } \partial\Omega \setminus \Gamma_0, \\ \dfrac{\partial v}{\partial n} = g_2(d,x,u,v) & \text{on } \partial\Omega \setminus (\Gamma_0 \cup \Gamma), \\ \dfrac{\partial v}{\partial n} \in g_2(d,x,u,v) + m_1(d,x,u,v) & \text{on } \Gamma. \end{cases} \tag{1.2}$$

Here, $d = (d_1, d_2) \in \mathbb{R}_+^2$ is a bifurcation parameter, the nonlinearities f_i and g_i are small at $(u,v) = 0$, and m_i are nonnegative interval functions specified later. The scalar parameters b_{ij} are assumed to satisfy

$$\begin{aligned} &b_{11} > 0,\ b_{12} < 0,\ b_{21} > 0,\ b_{22} < 0, \\ &b_{11} + b_{22} < 0,\ b_{11}b_{22} - b_{12}b_{21} > 0, \end{aligned} \tag{1.3}$$

which means that system (1.1) is a special system of activator-inhibitor or prey-predator type such that in case $d_1 = d_2 = 0$ (i.e. without diffusion) the solution $(0,0)$ is stable. However, it is known (see e.g. [11] or [2, Appendix] or [1]) that the stability of (1.1)/(1.2) with classical data $m_0 = m_1 = 0$ depends on $d = (d_1, d_2)$. In fact, the domain D_S of those $d \in \mathbb{R}_+^2$ where this system is exponentially stable is the right-hand side of the "envelope" of the sequence of hyperbolas

$$C_n := \left\{ (d_1, d_2) \in \mathbb{R}_+^2 : d_2 = \frac{b_{12}b_{21}/\kappa_n^2}{d_1 - b_{11}/\kappa_n} + \frac{b_{22}}{\kappa_n} \right\}. \tag{1.4}$$

where $\kappa_1 \leq \kappa_2 \leq \cdots \to \infty$ denotes the sequence of eigenvalues of $-\Delta$, i.e. for which a (weak) nontrivial solution of the problem

$$\begin{cases} -\Delta u = \kappa_n u & \text{on } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \setminus \Gamma_0 \end{cases} \tag{1.5}$$

exists.

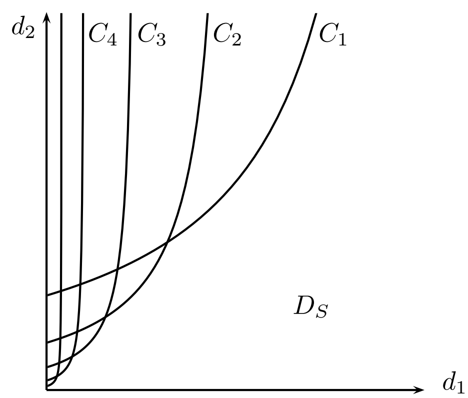

Figure 1: *Hyperbolas* (1.4) *determining* D_S

Using degree theory for multivalued maps, it was shown in [6] (for similar results and related systems, see also [2]–[5], [7]–[10]) that in the case of natural unilateral (possibly multivalued) functions m_0 and/or m_1 there is a destabilizing effect in the sense that even the stationary points admit a global bifurcation along certain paths in D_S. In this paper, we will show by a purely topological argument that results of such a type actually imply the existence of certain *global branches of bifurcation points* (i.e. not only the bifurcation branch is global but actually even the set of bifurcation points itself). Such phenomena can naturally arise only because our bifurcation parameter d is not only from a one-dimensional space.

We point out that although we concentrate only on the system (1.1)/(1.2), the same results (and proofs) hold for all systems for which corresponding results along paths are available. In particular, this is the case when we consider instead of (1.2) the Signorini type boundary conditions (see e.g. [9], [10])

$$\begin{cases} u = v = 0 & \text{on } \Gamma_0, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \setminus \Gamma_0, \\ \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \setminus (\Gamma_0 \cup \Gamma), \\ \dfrac{\partial v}{\partial n} \geq 0,\ v \geq 0,\ \dfrac{\partial v}{\partial n} \cdot v = 0 & \text{on } \Gamma. \end{cases}$$

2 The Disc-Cutting Theorem

Our main topological tool is based on a generalization of the Whyburn lemma yielding global branches [12] and on the following result.

Theorem 2.1 (Boundaries connect squaresides)**.** *Let X be a topological space into which a square with (compact) "square-sides" A_1, A_2, A_3, A_4 and "square-interior" Q is homeomorphically embedded. Let $V \subseteq X$ be open such that $A_1 \subseteq V$ and $\overline{V} \cap A_3 = \varnothing$. Then there is a connected subset $C \subseteq Q \cap \partial V$ such that $\overline{C} \cap A_i \neq \varnothing$ for $i = 2, 4$.*

The afore mentioned generalization of the Whyburn lemma is the following (this version can be proved using only the [countable] axiom of dependent choices):

Theorem 2.2. *Let X be a regular space, $A \subseteq X$ compact, and $S \subseteq X$. Then for each open set $U \supseteq A$ for which $\overline{U} \cap S$ is compact and metrizable the following statements are equivalent:*

1. *For each open set $\Omega \supseteq A$ with $\overline{\Omega} \subseteq U$ there is some $x \in \partial\Omega$ in S.*
2. *There is a connected set $C \subseteq (S \cap U) \setminus A$ such that $\overline{C} \cap S$ intersects A and ∂U.*

For the proof of Theorem 2.2 we refer to [12]. Let us now use this result and some winding number theory to prove Theorem 2.1.

Proof of Theorem 2.1. We show first that it suffices to show the claim for the case $X = \mathbb{R}^2$ and $Q = (0,1) \times (0,1)$ with $A_1 := [0,1] \times \{0\}$, $A_2 := \{1\} \times [0,1]$, $A_3 := [0,1] \times \{1\}$, and $A_4 := \{0\} \times [0,1]$.

To see that then the general case holds also, assume that we have a homeomorphism f of $\overline{Q}$ onto a subset X_0 of a general space X. Note that X_0 is the union of the sets $\widetilde{Q} := f(Q)$ and $\widetilde{A}_i := f(A_i)$. Note that we do not assume that X is Hausdorff, so X_0 might not be closed, although it is compact. However, if $V \subseteq X$ is open with $\widetilde{A}_1 \subseteq V$ and $\overline{V} \cap \widetilde{A}_3 = \varnothing$, then $V_0 := f^{-1}(V \cap X_0)$ is open in $\overline{Q}$, and so there is some open $V_1 \subseteq \mathbb{R}^2$ with $V_1 \cap \overline{Q} = V_0$. Since $\overline{V}_0 \subseteq f^{-1}(\overline{V} \cap X_0)$ is disjoint from the compact set $f^{-1}(\widetilde{A}_3) = A_3$, and since $\mathbb{R}^2$ is regular, we thus find a closed neighborhood of A_3 which is disjoint from V_0. Eliminating this neighborhood from V_1 if necessary, we may thus assume without loss of generality that $\overline{V}_1 \cap A_3 = \varnothing$. By the special case of the Theorem, we thus find a connected set $C \subseteq Q \cap \partial V_1$ with $\overline{C} \cap A_i \neq \varnothing$ for $i = 2, 4$. Then $\widetilde{C} := f(C)$ is connected, and its closure contains $f(\overline{C})$ and thus intersects $\widetilde{A}_i = f(A_i)$ for $i = 2, 4$. Moreover, $\widetilde{C}$ is contained in $f(Q \cap \partial V_1)$. Note that, since Q is open, $\overline{V}_1 \cap Q = (\overline{V_1 \cap Q}) \cap Q = \overline{V}_0 \cap Q$, and so $\widetilde{C} \subseteq f(\overline{V}_0 \cap Q) \subseteq \widetilde{Q} \cap \overline{V}$. Moreover, $\widetilde{C}$ is disjoint from V, since $\widetilde{C} = f(C)$ is contained in X_0 and disjoint from $V \cap X_0 = f(V_0)$ in view of $C \cap V_0 = \varnothing$. Hence, we have indeed found a connected set $\widetilde{C} \subseteq Q \cap \overline{V} \setminus V = Q \cap \partial V$ whose closure intersects $\widetilde{A}_i$ for $i = 2, 4$. This proves the general case of the claim.

We prove now the claim in the special case $X := \mathbb{R}^2$, $Q := (0,1) \times (0,1)$ and A_i as above. Thus, let $V \subseteq \mathbb{R}^2$ be open with $A_1 \subseteq V$ and $\overline{V} \cap A_3 = \varnothing$. Without loss of generality, we may assume in addition that $\overline{V}$ is contained in $Q_0 := (-1,2) \times (-1,1)$. Indeed, otherwise, we could replace V by the intersection

$$V_0 := V \cap \{(x_1, x_2) \in (-1/2, 3/2) \times (-1/2, 1) : x_2 < 1 - \operatorname{dist}(x_1, [0,1])\}$$

and note that $A_1 \subseteq V_0$, $\overline{V}_0 \cap A_3 = \varnothing$, and $Q \cap \partial V_0 = Q \cap \partial V$.

Put $S := \overline{Q} \cap \partial V$ and assume by contradiction that there is no connected subset $C \subseteq S \cap Q$ with $\overline{C} \cap A_i \neq \varnothing$ $(i = 2, 4)$. Applying Theorem 2.2 in the space $\overline{Q}$ with $A := A_2$ and $U := \overline{Q} \setminus A_4$, we find some open (in $\overline{Q}$) set $\Omega \subseteq \overline{Q}$ with $A_2 \subseteq \Omega$ and $\overline{\Omega} \cap A_4 = \varnothing$ such that the boundary B_0 of Ω with respect to $\overline{Q}$ contains no point from ∂V. By the compactness of these sets, we thus find some closed neighborhood $M \subseteq \mathbb{R}^2$ of ∂V which is disjoint from B_0. Since ∂V is a compact subset of Q_0 and $\mathbb{R}^2$ is regular, we may assume in addition that $M \subseteq Q_0$.

Covering the compact set $\overline{V}$ with sufficiently small open balls, we find an open set $G \subseteq V \cup M$ containing $\overline{V}$ such that the boundary of G consists of finitely many piecewise smooth closed curves. Fix some $a \in A_1$. Since $a \in V \subseteq G$, the argument principle of complex analysis (or, in other words, the well-known connection between the degree of the identity function with the winding number of the boundary) implies that at least one of these curves must have nonzero winding number with respect to a. We think of such a closed curve as a continuous map $\gamma\colon S^1 \to Q_0$ (where S^1 denotes the unit circle). Note that this curve lies completely in $\partial G \subseteq M$. In particular, $\gamma(S^1)$ is disjoint from B_0 and thus contained in the union of the three sets

$$\Omega_2 := \Omega \cap Q, \quad \Omega_4 := (\overline{Q} \setminus \overline{\Omega}) \cap Q = Q \setminus \overline{\Omega}, \quad R := Q_0 \setminus (Q \cup \{a\}).$$

Since Ω is open in $\overline{Q}$, it follows that Ω_2 is open in Q and thus open in $\mathbb{R}^2$. Analogously, also Ω_4 is open in $\mathbb{R}^2$. With the notation $\gamma = (\gamma_1, \gamma_2)$, we define now a homotopy $h\colon [0,1] \times S^1 \to \mathbb{R}^2 \setminus \{a\}$ by

$$h(t,s) := \begin{cases} ((1-t)\gamma_1(s), \gamma_2(s)) & \text{if } s \in \gamma^{-1}(\Omega_4), \\ ((1-t)\gamma_1(s) + t, \gamma_2(s)) & \text{if } s \in \gamma^{-1}(\Omega_2), \\ (\gamma_1(s), \gamma_2(s)) & \text{otherwise.} \end{cases}$$

This map is indeed continuous by the glueing lemma, because γ can cross the boundary of Ω_i only at A_i $(i = 2, 4)$. We thus have shown that γ is homotopic (in $\mathbb{R}^2 \setminus \{a\}$) to a curve which assumes only values in R. Since R is obviously simply connected, γ is actually homotopic to a constant (in $\mathbb{R}^2 \setminus \{a\}$). Hence, the homotopy invariance of the winding number shows that γ actually has winding number 0 around a. This is the required contradiction. □

Using Theorems 2.1 and 2.2, we can now prove the following disc-cutting theorem:

Theorem 2.3 (Disc-Cutting)**.** *Let X be a topological space into which a (compact) disc with "disc-interior" Q is homeomorphically embedded. Let the "disc-boundary" be the union of four nonempty disjoint connected sets A_1, A_2, A_3, A_4, enumerated in order along the boundary. Assume also that A_2 and A_4 both contain at least two points.*

Let $S \subseteq Q$ be closed in Q such that each compact smooth (via the embedding) injective path P in $Q \cup A_2 \cup A_4$ with $P \cap A_i \neq \varnothing$ $(i = 2, 4)$ contains some point from S. Then there is a connected subset $C \subseteq S$ such that $\overline{C} \cap \overline{A}_i \neq \varnothing$ for $i = 1, 3$.

Remark 2.1. One could also replace "smooth path" by "polygonal path" in the statement of Theorem 2.3 with the obvious modification in the following proof.

Proof. We show first that it suffices to show the claim for the case $X = \mathbb{R}^2$ and the unit disc Q. Indeed, if $f\colon \overline{Q} \to X$ is a homeomorphism onto a subset of a general space X, let $\widetilde{Q} := f(Q)$ and $\widetilde{A}_i := f(A_i)$. Let $S \subseteq \widetilde{Q}$ be as in the claim; in particular, S is closed in $\widetilde{Q}$. Then $S_0 := f^{-1}(S)$ is closed in Q. The hypothesis on S means that each smooth path connecting A_2 with A_4 in Q meets S_0. The special case of the result thus implies that there is a connected subset $C_0 \subseteq S_0$ such that $\overline{C}_0 \cap \overline{A}_i \neq \varnothing$ for $i = 1, 3$. Then $C := f(C_0) \subseteq S$ is connected, and $\overline{C} \cap \overline{\widetilde{A}_i} \supseteq f(\overline{C}_0) \cap f(\overline{A}_i) \neq \varnothing$. Hence, the statement holds also in the general case.

Thus, to prove the theorem, we may assume without loss of generality that $X = \mathbb{R}^2$ and that Q is the unit disc. Assume by contradiction that a set C as in the claim does not exist. We apply Theorem 2.2 in the space $\overline{Q}$ with $A := \overline{A}_1$, $U := \overline{Q} \setminus \overline{A}_3$, and $\overline{S}$ instead of S. Observing that $A \subseteq U$, because A_2 and A_4 are nondegenerate, we find some open in $\overline{Q}$ set $\Omega \supseteq \overline{A}_1$ with $\overline{\Omega} \cap \overline{A}_3 = \varnothing$ such that the boundary B_0 of Ω with respect to $\overline{Q}$ contains no element of $\overline{S}$. Note that B_0 is a closed subset of $\overline{Q}$ and thus compact. Note also that B_0 is disjoint from $\overline{S}$ and from $\overline{A}_i$ $(i = 1, 3)$. We thus find an open neighborhood $M \subseteq \mathbb{R}^2$ of B_0 which is disjoint from $\overline{S} \cup \overline{A}_1 \cup \overline{A}_3$. Moreover, if we let $\hat{A}_i$ $(i = 1, 3)$ be compact "intervals" of the circle boundary which contain the corresponding "intervals" $\overline{A}_i$ $(i = 1, 3)$ in their interior (with respect to the circle boundary) but still satisfy $\hat{A}_1 \subseteq \Omega$ and $\hat{A}_3 \subseteq \overline{Q} \setminus \overline{\Omega}$, and if we let $\hat{A}_i$ $(i = 2, 4)$ denote closure of the corresponding remaining intervals (contained in A_i), we can apply Theorem 2.1 with $V := \Omega$ and the four "square-sides" $\hat{A}_i$. We thus find a connected set $C \subseteq B_0$ such that there are points $a_i \in \overline{C} \cap \hat{A}_i$ for $i = 2, 4$, and so $a_i \in \overline{C} \cap A_i$ $(i = 2, 4)$. Since $\overline{C} \subseteq B_0$ is connected, if follows that a_2 and a_4 belong to the same connected component of B_0. Since $M \subseteq \mathbb{R}^2$ is an open neighborhood of B_0, we may thus connect a_2 and a_4 by a smooth injective path in M. Since M is disjoint from $\overline{A}_i$ $(i = 1, 3)$, we thus find a compact smooth injective path P in $M \cap (Q \cup A_2 \cup A_4)$ with $P \cap A_i \neq \varnothing$ $(i = 2, 4)$. Since $M \cap \overline{S} = \varnothing$, this path cannot contain a point from S, contradicting the hypothesis. □

3 The Reaction-Diffusion System with Inclusions

3.1 Detailed Hypotheses

We will consider the weak formulation of the stationary problem corresponding to (1.1)/(1.2), i.e. we will consider the weak formulation of

$$\left.\begin{aligned} &d_1\Delta u + b_{11}u + b_{12}v + f_1(d, u, v, \nabla u, \nabla v) = 0 \\ &d_2\Delta v + b_{21}u + b_{22}v + f_2(d, u, v, \nabla u, \nabla v) \in -m_0(d, u, v, \nabla u, \nabla v) \end{aligned}\right\} \text{ in } \Omega \tag{3.1}$$

with boundary conditions

$$\begin{cases} u = v = 0 & \text{on } \Gamma_0, \\ \dfrac{\partial u}{\partial n} = g_1(d,u,v) & \text{on } \partial\Omega \setminus \Gamma_0, \\ \dfrac{\partial v}{\partial n} \in g_2(d,u,v) + m_1(d,x,u,v) & \text{on } \partial\Omega \setminus \Gamma_0, \end{cases} \tag{3.2}$$

where we will assume that the (possibly multivalued) functions m_i have the form

$$m_0(d,x,u,v,w,z) := [\underline{c}_0(d)\underline{m}_0(x,u,v,w,z), \overline{c}_0(d)\overline{m}_0(x,u,v,w,z)]$$

and

$$m_1(d,x,u,v) := [\underline{c}_1(d)\underline{m}_1(x,u,v), \overline{c}_1(d)\overline{m}_1(x,u,v)],$$

and where we assumed for the simplicity of notation that $\underline{m}_0$, $\overline{m}_0$, $\underline{m}_1$ and $\overline{m}_1$ vanish for $x \notin \Omega_0$ or $x \notin \Gamma$, respectively, where $\Omega_0 \subseteq \Omega$ and $\Gamma \subseteq \partial\Omega \setminus \Gamma_0$ are measurable. In order to require nontrivial situations, we will assume that

$$\mathrm{mes}\Omega_0 > 0 \text{ or } \mathrm{mes}\Gamma > 0 \text{ (or both).} \tag{3.3}$$

For our considerations it will be crucial that

$$\mathrm{mes}\Gamma_0 > 0 \tag{3.4}$$

so that we can equip the space $\mathbb{H}$ of all functions from $W^{1,2}(\Omega, \mathbb{R}^2)$ vanishing on Γ_0 with the scalar product

$$\langle U, V\rangle := \int_\Omega \langle \nabla U(x), \nabla V(x)\rangle \, dx,$$

which under hypothesis (3.4) generates the inherited topology, see e.g. [13, Theorem 4.8.1].

We assume (1.3), and by $D_S \subseteq \mathbb{R}^2_+$, we denote the (open) domain of stability mentioned in the introduction. Note that all points of $\mathbb{R}^2_+ \cap \partial D_S$ belong to some of the hyperbolas C_n defined by (1.4). We will assume that all of the above functions are at least defined for $d \in D_S \cup \{d^*\}$ where the point $d^* \in C_n \cap \partial D_S$ will be specified later on.

For $i = 0, 1$, we fix exponents p_i, q_i, and q_i^* according to the following restrictions.

$$\begin{cases} p_i \in [1,\infty),\ 1 \le q_i^* < q_i < \infty \text{ arbitrary} & \text{if } n \le 2, \\ p_0 := \frac{n}{n-2},\ p_1 := \frac{n-1}{n-2},\ \infty > q_0 > q_0^* := \frac{2n}{n+2},\ \infty > q_1 > q_1^* := \frac{2n-2}{n} & \text{if } n > 2. \end{cases}$$

Moreover, we assume the following hypothesis.

1. $\underline{c}_i, \overline{c}_i$ are continuous on $D_S \cup \{d^*\}$ and without zeros on D_S.

2. For each $d \in D_S \cup \{d^*\}$ the following holds: The functions $f_i(d, \cdot, u, v, w, z)$ and $g_i(d, \cdot, u, v)$ are measurable, and $f_i(d, x, \cdot, \cdot, \cdot, \cdot)$ and $g_i(d, x, \cdot, \cdot)$ are continuous for almost all x. Moreover, f_i and g_i satisfy the growth estimates

$$|f_i(d,x,u,v,w,z)| \le a_d(x) + b_d \cdot ((|u| + |v|)^{p_0} + \|w\| + \|z\|)^{2/q_0},$$

and

$$|g_i(d,x,u,v,w,z)| \le \widetilde{a}_d(x) + \widetilde{b}_d \cdot ((|u|+|v|)^{p_1})^{2/q_1} ,$$

where the quantities $\|a_d\|_{L_{q_0}(\Omega)}$, $\|\widetilde{a}_d\|_{L_{q_1}(\partial\Omega\setminus\Gamma_0)}$, b_d, and $\widetilde{b}_d$ are locally bounded with respect to d.

3. For each $d_0 \in D_S \cup \{d^*\}$ there are estimates of the form

$$|f_i(d,x,u,v,w,z) - f_i(d_0,x,u,v,w,z)| \le$$
$$c_{d_0}(d)\left(a_{d_0,d}(x) + ((|u|+|v|)^{p_0} + \|w\| + \|z\|)^{2/q_0^*}\right)$$

and

$$|g_i(d,x,u,v) - g_i(d_0,x,u,v)| \le$$
$$\widetilde{c}_{d_0}(d)\left(\widetilde{a}_{d_0,d}(x) + (|u|+|v|)^{2p_1/q_1^*}\right)$$

where $\|a_{d_0,d}\|_{L_{q_0^*}(\Omega)}, \|\widetilde{a}_{d_0,d}\|_{L_{q_1^*}(\partial\Omega\setminus\Gamma_0)} \le 1$ and $c_{d_0}(d), \widetilde{c}_{d_0}(d) \to 0$ as $d \to d_0$.

4. f_i and g_i become uniformly small at $(u,v)=0$ in the sense that for each sufficiently small ball B in D_S (and thus for each nonempty compact subset $B \subseteq D_S$) the following holds:

$$\sup_{w,z\in\mathbb{R}^n}\sup_{d\in B}|f_i(d,x,u,v,w,z)| \le c_B \max\left\{(|u|+|v|)^{2p_0/q_0}, |u|+|v|\right\}$$
$$\lim_{(u,v,w,z)\to 0}\sup_{d\in B}\frac{f_i(d,x,u,v,w,z)}{|u|+|v|+\|w\|+\|z\|} = 0$$
$$\sup_{w,z\in\mathbb{R}^n}\sup_{d\in B}|g_i(d,x,u,v)| \le c_B \max\left\{(|u|+|v|)^{2p_1/q_1}, |u|+|v|\right\}$$
$$\lim_{(u,v,w,z)\to 0}\sup_{d\in B}\frac{g_i(d,x,u,v)}{|u|+|v|} = 0$$

5. The functions $\underline{m}_0(\,\cdot\,,u,v,w,z)$ and $\overline{m}_0(\,\cdot\,,u,v,w,z)$ are measurable, $\underline{m}_0(x,\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,)$ is lower semicontinuous, $\overline{m}_0(x,\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,)$ is upper semicontinuous, and the corresponding superposition operators

$$\underline{M}_0(u,v,w,z)(x) := \underline{m}_0(x,u(x),v(x),w(x),z(x))$$

and

$$\overline{M}_0(u,v,w,z)(x) := \overline{m}_0(x,u(x),v(x),w(x),z(x))$$

send continuous (and thus measurable) functions into measurable functions. Moreover, we require for some $a_0 \in L_{q_0}(\Omega)$ and $b_0 < \infty$ the growth estimates

$$\max\left\{|\underline{m}_0(x,u,v,w,z)|, |\overline{m}_0(x,u,v,w,z)|\right\} \le$$
$$a_0(x) + b_0 \cdot ((|u|+|v|)^{p_0} + \|w\| + \|z\|)^{2/q_0} .$$

6. The functions $\underline{m}_1(\,\cdot\,,u,v)$ and $\overline{m}_1(\,\cdot\,,u,v)$ are measurable, $\underline{m}_1(x,\,\cdot\,,\,\cdot\,)$ is lower semicontinuous, $\overline{m}_1(x,\,\cdot\,,\,\cdot\,)$ is upper semicontinuous, and the corresponding superposition operators

$$\underline{M}_1(u,v)(x) := \underline{m}_1(x,u(x),v(x))$$

and

$$\overline{M}_1(u,v)(x) := \overline{m}_1(x,u(x),v(x))$$

send continuous (and thus measurable) functions into measurable functions. Moreover, we require the following growth estimates for some $a_1 \in L_{q_1}(\Gamma)$ and $b_1 < \infty$:

$$\max\left\{\left|\underline{m}_1(x,u,v)\right|, \left|\overline{m}_1(d,x,u,v)\right|\right\} \le a_1(x) + b_1 \cdot (|u|+|v|)^{2p_1/q_1}\,.$$

7. The following unilateral conditions hold:

$$\begin{aligned}
0 = \underline{c}_0(d)\underline{m}_0(x,u,v,w,z) &= \overline{c}_0(d)\overline{m}_0(x,u,v,w,z) \quad \text{if } v>0,\\
0 = \underline{c}_0(d)\underline{m}_0(x,u,0,w,z) &\le \overline{c}_0(d)\overline{m}_0(x,u,0,w,z)\\
0 \le \underline{c}_0(d)\underline{m}_0(x,u,v,w,z) &\le \overline{c}_0(d)\overline{m}_0(x,u,v,w,z) \quad \text{if } v<0,\\
0 = \underline{c}_1(d)\underline{m}_1(x,u,v) &= \overline{c}_1(d)\overline{m}_1(x,u,v) \quad \text{if } v>0,\\
0 = \underline{c}_1(d)\underline{m}_1(x,u,0) &\le \overline{c}_1(d)\overline{m}_1(x,u,0)\\
0 \le \underline{c}_1(d)\underline{m}_1(x,u,v) &\le \overline{c}_1(d)\overline{m}_1(x,u,v) \quad \text{if } v<0.
\end{aligned}$$

$$\lim_{\substack{(u,v,w,z)\to 0\\ v<0}} \frac{|\underline{m}_0(x,u,v,w,z)|}{v} = -\infty \quad \text{for almost all } x \in \Omega_0,$$

$$\lim_{\substack{(u,v)\to 0\\ v<0}} \frac{|\underline{m}_1(x,u,v)|}{v} = -\infty \qquad \text{for almost all } x \in \Gamma.$$

3.2 Definition of weak solutions

We consider the cone

$$K := \{U = (u_1,u_2) \in \mathbb{H} : u_2|_{\Omega_0} \ge 0 \text{ and } u_2|_\Gamma \ge 0\}$$

and define operators $A(d), G(d,\,\cdot\,), M(d,\,\cdot\,)\colon \mathbb{H} \to \mathbb{H}$ by

$$\begin{aligned}
\langle A(d)U,V\rangle &:= \int_\Omega \left\langle \begin{pmatrix} d_1^{-1}b_{11} & d_1^{-1}b_{12}\\ d_2^{-1}b_{21} & d_2^{-1}b_{22}\end{pmatrix} U(x), V(x)\right\rangle dx,\\
\langle G(d,U),V\rangle &:= \int_\Omega \left\langle \begin{pmatrix} d_1^{-1}f_1(d,U(x),\nabla U(x))\\ d_2^{-1}f_2(d,U(x),\nabla U(x))\end{pmatrix}, V(x)\right\rangle dx\\
&\quad + \int_{\partial\Omega\setminus\Gamma_0} \left\langle \begin{pmatrix} g_1(d,U(x))\\ g_2(d,U(x))\end{pmatrix}, V(x)\right\rangle dx,
\end{aligned}$$

and

$$M(d,U) := \bigcap_{V\in K}\left\{Z\in\mathbb{H} : \langle Z,V\rangle \in \int_{\Omega_0}\left\langle\begin{pmatrix}0\cdot d_1^{-1}\\ d_2^{-1}m_0(d,x,U(x),\nabla U(x))\end{pmatrix}, V(x)\right\rangle dx + \int_\Gamma\left\langle\begin{pmatrix}0\\ m_1(d,x,U(x))\end{pmatrix}, V(x)\right\rangle dx\right\} :=$$

$$\bigcap_{V=(\tilde v,v)\in K}\left\{Z=\begin{pmatrix}0\\ z\end{pmatrix}\in\mathbb{H} : \int_{\Omega_0} d_2^{-1}\underline{c}_0(d)\underline{m}_0(x,U(x),\nabla U(x))v(x)\,dx + \int_\Gamma \underline{c}_1(d)\underline{m}_1(x,U(x))v(x)\,dx \le \langle Z,V\rangle \le \int_{\Omega_0} d_2^{-1}\overline{c}_0(d)\overline{m}_0(x,U(x),\nabla U(x))v(x)\,dx + \int_\Gamma \overline{c}_1(d)\overline{m}_1(x,U(x))v(x)\,dx\right\},$$

respectively. We define weak solutions of problem (3.1)/(3.2) as solutions of the inclusion

$$U - A(d)U - G(d,U) \in M(d,U).$$

Our hypotheses imply in particular (see e.g. [6]):

Proposition 3.1. *$F(d,U) := A(d)U - G(d,U) - M(d,U)$ is an upper semicontinuous map with nonempty compact values. Moreover, F is compact in the sense that if $D_0 \subseteq D_S \cup \{d^*\}$ is compact and $B \subseteq \mathbb{H}$ is bounded then $F(D_0 \times B)$ is precompact.*

3.3 Local and Global Bifurcation Points

Note that $(d,0) \in D_S \times \mathbb{H}$ is always a solution of (3.1)/(3.2). We call a pair $(d,U) \in D_S \times \mathbb{H}$ a *nontrivial solution* if $U = (u,v) \neq 0$, and if (d,u,v) is a weak solution of (3.1)/(3.2). The *local bifurcation points* (in D_S) are the elements of the set

$$B_{\text{local}} := \{d \in D_S : \text{Each neighborhood of } (d,0) \in D_S \times \mathbb{H} \text{ contains a nontrivial solution}\}.$$

We call a point $d \in D_S$ a *global bifurcation point* (with respect to a point $d^* \in C_n \cap \partial D_S$) if there is a connected set $C \subseteq D_S \times (\mathbb{H} \setminus \{0\})$ consisting only of nontrivial solutions such that $(d,0) \in \overline{C}$ and such that C is a global branch in the sense that at least one of the following holds:

1. C is unbounded.
2. C reaches d^*, i.e. $\overline{C}$ contains some point (d^*,U) which is a weak solution of (3.1)/(3.2).

Note that in the second case, we do not exclude $U = 0$, i.e. C might return to the trivial branch at the hyperbola point $d^* \in C_n$. We denote the set of global bifurcation points (with respect to d^*) by $B_{\text{global}}(d^*)$.

Proposition 3.2. *Each global bifurcation point is a local bifurcation point. Moreover, B_{local} is closed in D_S. In particular, $\overline{B_{global}(d^*)} \cap D_S \subseteq B_{local}$.*

In our considerations an important role will be played by the vertical asymptote of the right-most hyperbola

$$\{(d_1, d_2) \in D_S : d_1 = b_{11}/\kappa_1\} \tag{3.5}$$

and the corresponding part to the right of this asymptote, i.e.

$$H := \{(d_1, d_2) \in D_S : d_1 > b_{11}/\kappa_1\}\,. \tag{3.6}$$

The following has been shown in [6]:

Proposition 3.3. *$H \cap B_{local} = \varnothing$.*

We also need another terminology. We say that a point $d \in \partial D_S$ is *n-interior* if $d \in C_n$ and if there is some eigenfunction e of $-\Delta$ for the eigenvalue κ_n, i.e. $e = u$ is a weak solution of (1.5), such that, for some constant $\varepsilon > 0$,

$$\begin{gathered} e \geq \varepsilon > 0 \text{ almost everywhere on } \Omega_0 \text{ and} \\ e \geq \varepsilon > 0 \text{ almost everywhere on } \Gamma. \end{gathered} \tag{3.7}$$

Recall in this connection that we require (3.3)

For the case that Γ is a smooth manifold with boundary and $\Omega_0 = \varnothing$, we replace (3.7) by the milder requirement

$$e(x) > 0 \text{ for almost all } x \in \Gamma. \tag{3.8}$$

We say that $d \in \partial D_S$ is *(n, m)-interior* if $d \in C_n \cap C_m$ and if there is a function e which is a linear combination of eigenfunctions to the eigenvalues κ_n and κ_m such that (3.7) or (3.8) holds, respectively.

If $d \in C_n \cap C_m \cap \partial D_S$ and d is n-interior or m-interior then d is also (n, m)-interior. However, d might be (n, m)-interior without being n-interior or m-interior.

Using the main results from [6], we will prove now:

Lemma 3.1. *Let $d \in \partial D_S$ be n-interior or (n, m)-interior. Then there is an open neighborhood $U_0 \subseteq \mathbb{R}^2$ of d such that $U_0 \cap D_S \cap B_{local} = \varnothing$. Moreover, if the hypotheses are satisfied with $d^* = d$, then each continuous compact path γ in $D_S \cup \{d^*\}$ connecting $d^* = d$ with some point from (3.6) contains some point from $B_{global}(d^*) \subseteq D_S$.*

Lemma 3.1 would follow rather straightforwardly from the results of [6] if we would allow that the connected set C in the definition of global bifurcation points is contained in $(D_S \cup \{d^*\}) \times (\mathbb{H} \setminus \{0\})$. However, it might happen that $C \setminus (\{d^*\} \times \mathbb{H})$ fails to be connected. Therefore, we need some additional arguments. We use the following result which is actually a consequence of Theorem 2.2 (and can also be proved using only the [countable] axiom of dependent choices, see [12]):

Theorem 3.1. *Let X be a regular space, $A \subseteq X$ compact, and $S \subseteq X$ be closed. Suppose that S is locally compact, metrizable and σ-compact. Then for each open set $U \supseteq A$ the following statements are equivalent:*

1. *There is a connected set $C \subseteq S$ which intersects A and is either noncompact or intersects ∂U.*
2. *There is a connected set $C \subseteq (S \cap U) \setminus A$ such that $\overline{C} \cap S$ intersects A and is either noncompact or intersects ∂U.*

Proof of Lemma 3.1. Only the last claim is not immediately contained in some of the results from [6]. To see this last claim, we apply the main result from [6] first to show that there is a connected set $C_0 \subseteq (D_S \cup \{d^*\}) \times (\mathbb{H} \setminus \{0\})$ such that $\overline{C}_0$ intersects $(\gamma \cap D_S) \times \{0\}$ and such that either C_0 is unbounded or $\overline{C}_0$ intersects also $\{d^*\} \times \mathbb{H}$. Moreover, we will arrange it that, in the space $X := \mathbb{R}^2 \times \mathbb{H}$, C_0 has the additional property that closures of bounded subsets of $S := \overline{C}_0$ are compact and consist only of (weak) solutions and satisfies

$$\overline{C}_0 \cap (\mathbb{R}^2 \times \{0\}) = (\gamma \setminus (U_0 \cap D_S)) \times \{0\}\,. \tag{3.9}$$

Indeed, assume that $\gamma = \sigma([a,b])$ with some continuous $\sigma\colon [a,b] \to D_S \cup \{d^*\}$ satisfying $\sigma(a) = d^*$ and $\sigma(b) \in H$. We extend σ to a continuous $\sigma\colon [a,\infty) \to D_S \cup \{d^*\}$ with $\sigma(s) \in H$ for all $s \geq b$ such that both components of $\sigma(s)$ tend to ∞ as $s \to \infty$. For all sufficiently small $s_0 \in (a,b)$ we have $\sigma([a,s_0]) \subseteq U_0$, and by the main result from [6], we find some connected set $C_1 \subseteq [a,\infty) \times (\mathbb{H} \setminus \{0\})$ such that

$$C_0 := \{(\sigma(s),u) : (s,u) \in C_1\}$$

consists only of (nontrivial) weak solutions of (3.1)/(3.2) and such that $\overline{C}_1$ contains some point from $[s_0,b] \times \{0\}$ and such that either C_1 is unbounded or $\overline{C}_1$ contains some point from $\{a\} \times \mathbb{H}$ or from $([a,s_0) \cup (b,\infty)) \times \{0\}$

The set C_0 has all required properties. Indeed, since C_0 consists only of nontrivial solutions, the closure of $\sigma([a,\infty))$ is contained in $\gamma \cup (U_0 \cap D_S) \cup H$, and no point of $(U_0 \cap D_S) \cup H$ is a local bifurcation point, we obtain (3.9). The set C_0 is connected, because it is the image of the connected set C_1 under the continuous map $T(s,u) := (\sigma(s),u)$. The set $\overline{C}_0$ contains $T(\overline{C}_1)$ and thus intersects $T([s_0,b] \times \{0\}) \subseteq (\gamma \cap D_S) \times \{0\}$ and is either unbounded (by our choice of the extension of σ) or intersects $\{d^*\} \times \mathbb{H}$ or $(U_0 \cup H) \times \{0\}$. In the latter case, $\overline{C}_0$ actually intersects $\{d^*\} \times \{0\}$ by (3.9).

To see these remaining properties, recall that with F from Proposition 3.1 the weak solutions of (3.1)/(3.2) are the elements of

$$\{(d,u) \in (D_S \cup \{d^*\}) \times \mathbb{H} : u \in F(d,u)\}\,.$$

Since C_0 is contained in this set, F is upper semicontinuous, and the closure of $\sigma([a,\infty))$ is contained in $D_S \cup \{d^*\}$, also $S = \overline{C}_0$ is contained in this set. The compactness of F described in Proposition 3.1 and our choice of the extension of σ implies that closed bounded subsets of S are compact.

Hence, $S = \overline{C}_0$ has all required properties. In particular, S is locally compact and σ-compact. We apply Theorem 3.1 with $A := (\gamma \setminus U_0) \times \{0\}$ and $U := X \setminus (\{d^*\} \times \mathbb{H})$. The connected set $\overline{C}_0$ witnesses that the first statement of Theorem 3.1 is satisfied: Note that this set indeed intersects A in view of (3.9), because $\overline{C}_0$ intersects $(\gamma \cap D_S) \times \{0\}$.

Hence, also the second statement of Theorem 3.1 holds which means that there is a connected set $C \subseteq (S \cap U) \setminus A$ such that the set $\overline{C}$ contains some point $(d_0, 0)$ with $d_0 \in \gamma \setminus U_0 \subseteq \gamma \cap D_S$ and such that either $\overline{C}$ is noncompact (and thus unbounded) or intersects $\partial U = \{d^*\} \times \mathbb{H}$. Thus, $d_0 \in B_{\text{global}}(d^*)$. $\square$

4 The main result

Theorem 4.1. *Let $D_0 := D_S \setminus H$ where H is from* (3.6).

Let $d^* \in \partial D_S$ be m-interior or (n,m)-interior ($n \leq m$) and such that the hypotheses described at the beginning of Section 3 are satisfied with this d^*. Then there is a connected set $B \subseteq \overline{B_{global}(d^*)} \cap D_0 \subseteq B_{local}$ such that $\overline{B}$ intersects the d_1-axis or some hyperbola C_k "strictly under" d^*.

More precisely, we have $k \geq n$, and the case $C_k = C_m$ is only possible if d^* is an intersection point of two different hyperbolas. In all cases, the intersection $\overline{B} \cap C_k$ does not contain d^* (i.e. is strictly under *d^*).*

Moreover, this branch B satisfies in addition the following:

1. *If C_n is the right-most hyperbola (i.e. if $C_n = C_1$) then B is unbounded.*
2. *Otherwise (i.e. if $C_n \neq C_1$) the set B is unbounded, or $\overline{B}$ intersects some hyperbola C_k "strictly over" d^* (i.e. $k \leq n$, and the case $C_k = C_n$ is only possible if d^* is an intersection point of two different hyperbolas; $\overline{B} \cap C_k$ does not contain d^*).*

Moreover, for any k for which there is some k-interior point we have $\overline{B} \cap C_k = \varnothing$, and for any pair (k,ℓ) for which the intersection point is (k,ℓ)-interior point this intersection point is not contained in $\overline{B}$.

Figure 2 illustrates qualitatively the four possibilities of branches B of bifurcation points if there is some n-interior point with $C_n \neq C_1$; one of these possibilities must (qualitatively) occur according to Theorem 4.1. Similarly, Figure 3 illustrates the two possibilities of branches B if there is some 1-interior point.

In particular, if there are n-interior points for every n, then the last statement of Theorem 4.1 implies that only one possibility can occur: There must be a branch B which is unbounded and such that $\overline{B}$ intersects the d_1-axis (possibly at $(0,0)$).

We point out that (contrary to what the figures might suggest) the theorem does not state that the branch B is pathwise connected, i.e. it might look "weird" (but it is connected in the topological sense).

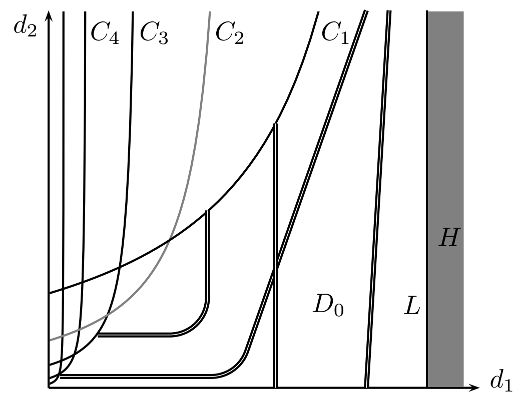

Figure 2: *The four qualitative different possible branches B of bifurcation points if there is some 2-interior point (one of these must occur)*

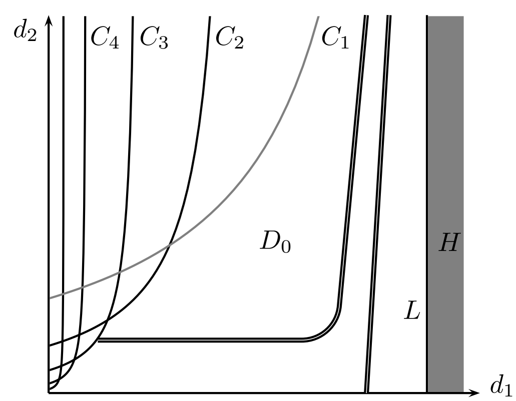

Figure 3: *The two qualitative different possible branches B of bifurcation points if there is some 1-interior point (one of these two must occur)*

Proof. The last statement of Theorem 4.1 is automatically satisfied by the first claim of Lemma 3.1, since $\overline{B_{\text{global}}(d^*)} \subseteq \overline{B_{\text{local}}}$ must be disjoint from any k-interior or (k,ℓ)-interior point.

Using this fact with d^*, we find some open neighborhood $U_0 \subseteq \mathbb{R}^2$ which is disjoint from $\overline{B_{\mathrm{global}}(d^*)} \subseteq \overline{B_{\mathrm{local}}}$.

Let $L_0 \subseteq H$ be some line which is parallel but strictly to the right of the line (3.5). Let $Q, H_0 \subseteq D_S$ be that parts to the left and right of this line L_0, respectively. Lemma 3.1 implies $\overline{B_{\mathrm{local}}} \cap D_S \subseteq Q$.

Using the one-point compactification X of $\overline{Q}$, we consider Q as the disc-interior of some homeomorphically embedded disc, whose boundary corresponds to the union of the d_1-axis, the line L_0, the point ∞, and the "envelope" $E = \mathbb{R}^2_+ \cap \partial D_S$ of all of the hyperbolas C_n. Let A_2 be that part of the boundary which corresponds to L_0 (without the two "boundary points" at ∞ and at the d_1-axis), and let A_4 correspond to $U_0 \cap E$. Let A_1 and A_3 denote the ramining (compact) connected subsets of the boundary of the disc Q.

Now we can apply the disc-cutting theorem with $S = Q \cap \overline{B_{\mathrm{global}}(d^*)}$. In fact, each continuous compact path in Q connecting A_2 with A_4 must intersect S by Lemma 3.1. Hence, the disc-cutting Theorem 2.3 implies the existence of a connected set $B \subseteq S$ with $\overline{B} \cap A_i \neq \varnothing$ for $i = 1, 3$. Since $\overline{B}$ cannot intersect L_0, the property $\overline{B} \cap A_3$ means that either B is unbounded or that $\overline{B}$ intersects some point of some C_k "strictly above" d^*. The property $\overline{B} \cap A_1$ means that $\overline{B}$ intersect some point of some C_k "strictly below" d^* or the d_1-axis. □

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