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On a new notion of holomorphy and its applications

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ABSTRACT

This paper devotes a new general notion of holomorphy which works in the continous and discrete cases. With the help of methods of a general operator theory the so called L-holomorphy is introduced. Realizations of this calculus follow. New versions of Taylor- and Taylor–Gontcharov formulae are deduced. The results are applied for the solution of higher order systems of differential equations.

RESUMEN

Este artículo es dedicado a una nueva noción de holomorfía la cual funciona en los casos continuo y discreto. Con la ayuda de métodos de la teoría general de operadores la llamada L-holomorfia es presentada. Realizaciones de este cálculo siguen. Nuevas versiones de fórmulas de Taylor-y Taylor-Gontcharov son deducidas. Los resultados son aplicados para la solución de sistemas de orden superior de ecuaciones diferenciales.

Key words and phrases: *Generalized holomorphic functions, Taylor-Gontcharov formulae, Plemelj projections, higher order boundary value problems.*

1 Introduction.

The aim of this article is to introduce a very general notion of holomorphy by the help of three general operators in Banach spaces which have to satisfy some conditions. This introduction is oriented at the theory of right invertible operators. We refer to the well-known book of V.S. Ryabenskij [11] (1987), W. Schempp and F.J. Delvos [2] (1990) and the article by M. Tasche [16] (1981). The advantage of our approach is the fact that holomorphy can be considered in the continuous and discrete case within one calculus. We continue the line of action we have followed in books [6],[7],[5]. In the second part we present a large number of realisations. Here we use above all results of the common research with K. Gürlebeck confer again in [6], [7] and [4]. Finally, some classes of boundary value problems of higher order will be considered. In that connection new formulae of Taylor- and Taylor-Gontcharov type are obtained. All our considerations take place in the scale of Sobolev and Besov spaces as well as its discrete analogue.

2 A general holomorphy

Let X, Y, Z be Banach spaces. We introduce the bounded linear operators T, Tr and P with the following properties

(i) $T : X \to \operatorname{im} T \subset Y$ is injective.

(ii) $Tr : Y \to Z$ is a generalized trace operator .

(iii) The operator $P : \operatorname{im} Tr \cap Y \to Y$ satisfies the property $PTrPu = Pu$.

Furthermore, we assume

(i) $\operatorname{im} Tr\, T \subset \ker P$,

(ii) $\operatorname{im} T \cap \ker Tr = \{0\}$.

Remark 1. We also have $\operatorname{im} T \cap \operatorname{im} P = \{0\}$. Indeed, let $u \in \operatorname{im} T \cap \operatorname{im} P = \{0\}$ then $u = Pw = Tv$ and

$$u = Pw = PTrPw = PTrTv = 0.$$

Theorem 1. (Mean value formula) *Set* $\operatorname{im} T \oplus \operatorname{im} P =: Y_1 \subset Y$. *There is a unique linear operator* L *with* $\mathcal{D}(L) = Y_1$ *and* $L : \mathcal{D}(L) \to X$, *such that*

$$u = PTru + TLu.$$

Proof. Let $u \in \mathcal{D}(L)$. Then u permits the representation

$$u = Pv + Tw,$$

with $v \in \mathrm{im}Tr \cap Y$, and $w \in X$. Applying PTr from the left it follows

$$PTr\,u = PTrP\,v + PTrT\,w = P\,v.$$

In this way the first item of the desired formula is obtained. In order also to obtain the second item we have to use the injectivity of the operator T. On the linear set $\mathrm{im}\,T$ there exists a linear operator $\tilde{L}$ with

$$\tilde{L}Tw = w.$$

The operator $\tilde{L}$ can be extended to an linear operator L on Y_1 setting

$$Lz := \tilde{L}z_1,$$

where $z = z_1 + z_2$ with $z_1 \in \mathrm{im}\,T$ and $z_2 \in \mathrm{im}\,P$. The additivity follows from

$$L(z + z') = L(z_1 + z_2 + z_1' + z_2') = \tilde{L}(z_1 + z_1') = \tilde{L}z_1 + \tilde{L}z_1' = Lz + Lz'.$$

The monogeneity with a real constant λ is also fulfilled. Indeed, we have

$$L(\lambda z) = \tilde{L}(\lambda z_1) = \lambda\tilde{L}z_1 = \lambda Lz.$$

Now we obtain easily $Lu = LPTr\,u + LT\,w = w$ and our decomposition formula is completely proved. The uniqueness follows from

$$TLu - TL_1u = 0 \quad \text{leads to} \quad Lu = L_1u,$$

where L_1 is another linear operator which has to fulfil the decomposition formula. #

Corollary 1. *The following relations between the operators L, P and T are valid:*

(i) The operator L is the left-inverse to the operator T, i.e. $LT = I$.

(ii) Set $R := TL$ then R is a projection onto Y_1 with $\mathrm{im}R = \mathrm{im}T$.

(iii) It holds $\ker L = \mathrm{im}PTr$.

Proof. The relation (i) follows by the definition of L. Indeed, let $v \in X$, then

$$LTv = \tilde{L}Tv = v.$$

(ii) Obviously, TL fulfils the idempotential property and so we have $R^2 = R$. It is immediately clear that

$$\mathrm{im}\,R \subset \mathrm{im}\,T.$$

Conversely, let $v \in \mathrm{im}\,T$ then $v = Tw$ and

$$Rv = RTw = TLTw = Tw = v,$$

i.e. $\operatorname{im} T \subset \operatorname{im} R$. To prove the relation (iii) we have to argue as follows: Let $u \in \ker L$, then

$$u = PTr\, u + TLu = PTr\, u \in \operatorname{im} PTr.$$

On the other hand it follows from $u \in \operatorname{im} PTr$ that $u = PTr\, v$ with $v \in Y$ and

$$u = PTr\, v = PTrPTr\, v + TLu = PTr\, v + TLu,$$

which leads to $TLu = 0$ and because of the injectivity of the operator $T : X \to \operatorname{im} T$ we conclude $Lu = 0$, i.e. $u \in \ker L$. #

Definition 1. *Elements $u \in \ker L \cap Y$ are called L-*holomorphic. *The operator L is called* algebraic derivative. *The operator PTr is called the* initial projection *and the operator T is denoted as* general Teodorescu transform. *From the point of view of a general operator theory T is also called* algebraical integral.

Corollary 2. *Set $P_r := TrP : \mathrm{im}Tr \cap Y \to Z$ and $Q_r := I - P_r$. The following properties are valid:*

(i) The operators P_r, Q_r are idempotent, i.e. we have $P_r^2 = P_r$ and $Q_r^2 = Q_r$ and furthermore $Q_r P_r = P_r Q_r = 0$.

(ii) An element $\xi \in Z$ is the generalized trace of an element u from $\ker L$ if and only if $P_r \xi = \xi$.

(iii) We have $Q_r \xi = TrTL\, u$.

Proof. (i). It is sufficient to show

$$P_r^2 \xi = TrPTrP\xi = TrP\xi = P_r \xi,$$

with $\xi \in Z$. In order to prove (ii) let $\xi = Tr\, u \in Z$ and $u \in \ker L$. Then we have

$$u = PTr\, u + TL\, u = PTr\, u = P\xi.$$

It now follows $\xi = Tr\, u = TrP\, \xi = P_r \xi$. Conversely, let us assume $\xi = P_r \xi$, then

$$Tr\, u = \xi = P_r \xi = TrP\xi = TrPTr\, u.$$

On the other hand Theorem 1 yields

$$Tr\, u = TrPTr\, u + TrTLu.$$

Hence $TrTL\, u = 0$. Because of $\mathrm{im}T \cap \ker Tr = \{0\}$ follows $TLu = 0$ and such $Lu = 0$, i.e. $u \in \ker L$. For (iii) we have

$$Tr\, u = TrPTr\, u + TrTL\, u.$$

Therfore, it holds

$$TrTL\, u = Tr\, u - TrPTr\, u = \xi - TrP\, \xi = \xi - P_r \xi = Q_r \xi. \quad \#$$

Denotation The operators P_r, Q_r are called general Plemelj projections.

Remark 2. The condition $\mathrm{im}TL \cap \ker Tr = \{0\}$ can be seen as a very general formulation of a maximum principle.

3 Types of L–holomorphy

3.1 L-holomorphy in $\mathbb{R}^1$

A trivial example is given by consideration of all functions $u \in C^1[0,1]$ with

$$L := \frac{d}{dt}\ ,\ T := \int\limits_0^t \cdot d\tau\ ,$$

$P := I$ and $Tr : C^1[0,1] \to \mathbb{R}^1$ with $Tr\, u = u(0)$. Then we get the well-known mean-value theorem:

$$u(t) = u(0) + \int\limits_0^t \dot{u}(\tau)d\tau = PTr\, u + TLu.$$

This is just the main-theorem of differential-integral calculus. The class of all L-holomorphic functions consist of all real constants.

Also a slightly modification of the trace operator and the generalized Teodorescu transfrom does not change the triviality of the class of L-holomorphic functions. Indeed, let $u \in C^1[0,1]$, take $L := \frac{d}{dt}$, $P := I$ and $Tr\, u := \frac{1}{2}[u(0) + u(1)]$, then

$$(Tu)(t) := \int\limits_0^t u(\tau)d\tau - \frac{1}{2}\int\limits_0^1 u(\tau)d\tau\ .$$

Because of $\mathrm{im}PTr = \ker L$ we have again the space of all constants for the class of L-holomorphic functions.

By using the so-called Riemann-Liouville integral of order α (cf. [14],[9]) we obtain a more interesting example. For this reason let $u \in C[0,1],\ 0 < \alpha < 1$. We consider the absolut continuous function

$$(I^{\alpha}_{a+}u)(t) := \frac{1}{\Gamma(\alpha)}\int\limits_0^t \frac{1}{(t-\tau)^{1-\alpha}}u(\tau)d\tau\ ,$$

which has almost everywhere a derivative in $L_1[0,1]$. Take now

$$(Lu)(t) := \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}(I^{1-\alpha}_{a+}u)(t)\ ,\ (Tu)(t) := (I^{\alpha}_{a+}u)(t)$$

and with $n = [\alpha] + 1$

$$(PTru)(t) := \sum_{k=0}^{n-1} \frac{(t-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} \frac{d^{n-k-1}}{dt^{n-k-1}} I_{a+}^{n-\alpha} u(t).$$

$(I_{a+}^{\alpha}u)(t)$ is called Riemann-Liouville fractional integral and $(D_{a+}^{\alpha}u)(t)$ is denoted by Riemann–Liouville fractional derivative. The main-value theorem holds again.

3.2 Notions of holomorphy in the complex plane

The original notion of the holomorphy forms in natural way a class of L-holomorphic function. We have only to set

$$L := \partial_{\overline{z}}\ .$$

In more detailed we have the following: Let $G \subset \mathbb{C}$ be a bounded domain with sufficient smooth boundary curve then the mean-value formula is written as

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{u(t)}{t-z}\, dt - \frac{1}{2\pi i} \int_{G} \frac{1}{t-z} (\overline{\partial} u)(t) d\xi\, d\eta = \begin{cases} u(z) & , \quad z \in G \\ 0 & , \quad z \in \mathbb{C} \setminus \overline{G} \end{cases} .$$

We have only to identify

$$L := \overline{\partial} = \frac{1}{2} \left(\partial_\xi + i \partial_\eta \right) \qquad (t\xi + i\eta)\ ,$$
$$T := -\frac{1}{2\pi i} \int_{G} \frac{1}{t-z} \cdot d\xi\, d\eta\ , \qquad P := \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{t-z} \cdot d\Gamma_t\ .$$

The trace operator Tr is defined as non-tangential limit from inner points tending to the boundary Γ.

Remark 3. It is quite curious that the initial projection acts on the boundary. It seems that "initial values" are "smudged" over the surface.

Another example in the complex plane can be given by

$$L := \overline{\partial}\ ,\ (T\cdot)(z) = -\frac{1}{2\pi i} \int_{G} \left[\frac{1}{t-z} - \frac{1}{t+z} \right] \cdot d\xi\, d\eta$$

$$\text{and} \qquad (P\cdot)(z) = -\frac{1}{2\pi i} \int_{\Gamma} \left[\frac{1}{t-z} - \frac{1}{t+z} \right] \cdot d\Gamma_t\ .$$

The trace operator is definded as before. This model goes back to J. Ryan (cf. [8]).

3.3 L-holomorphy models generated by matrices

A further model for L-holomorphy is given by: Let $\{E_i\}_{i=1}^n$ be a family of orthogonal matrices of order n with entries $0, 1, -1$ as well as the property

$$E_i^* E_j + E_j^* E_i = 0 \qquad (i \neq j)$$

Furthermore, set $E(a) = \sum\limits_{i=1}^n E_i a_i$, $a = (a_1, ..., a_n)^T$ and $E^*(a) = \sum\limits_{i=1}^n E_i^* a_i$ and $\nabla = (\partial_1, ..., \partial_n)^T$.

Take $L := D(\nabla)$, $T := \frac{1}{\sigma_n} \int\limits_G \frac{D^*(y-x)}{|y-x|^n} \cdot dy$ and $P := \frac{-1}{\sigma_n} \int\limits_\Gamma \frac{D^*(y-x)}{|y-x|^n} \cdot d\Gamma_y$ then it holds

$$(Pu)(x) + TL(\nabla)u(x) = \begin{cases} u(x) & , \quad x \in G \\ 0 & , \quad x \in \mathbb{R}^k \setminus \overline{G} \end{cases} .$$

Here σ_n denotes the area of the n-dimensional unit sphere. (cf. [13],[15]).

3.4 Dzuraev's model

Also Dzuraev's model from 1982 [3] is worthy of being mentioned:

Let $u := (u_1, u_2)$, $z = x_2 + ix_3$ and $\frac{\partial}{\partial \overline{z}} := \frac{1}{2}(\frac{\partial}{\partial x_2} + i\frac{\partial}{\partial x_3})$, $y = y_2 + iy_3$. Further, let

$$\overline{\partial}_x = \begin{pmatrix} \frac{\partial}{\partial x_1} & 2\,\frac{\partial}{\partial z} \\ -2\,\frac{\partial}{\partial \overline{z}} & \frac{\partial}{\partial x_1} \end{pmatrix} , \qquad E(y-x) = \frac{-1}{|y-x|^3} \begin{pmatrix} y_1 - x_1 & -(\overline{y} - \overline{z}) \\ y - z & y_1 - x_1 \end{pmatrix}$$

and $n(y) = \begin{pmatrix} n_1 & n_2 - in_3 \\ -(n_2 + in_3) & n_1 \end{pmatrix}$. Then take

$$L := \overline{\partial}_x \ , \ T := \frac{1}{\sigma_3} \int\limits_G E(y-x) \cdot dy \quad \text{and} \quad P := \frac{1}{\sigma_3} \int\limits_\Gamma E(y-x) n(y) \cdot d\Gamma_y \ .$$

The trace operator Tr means in both cases the non-tangential limit to the boundary Γ from inside of G.

4 Quaternionic holomorphic functions

Real Quaternions: The algebra of real quaternions $\mathbb{H}$ is defined by the basis elements

$e_0 = 1$, e_1, e_2, e_3,

which obey the arithmetic rules:

$$e_0^2 = 1\ ,\ e_1e_2 = -e_2e_1 = e_3\ ,\ e_2e_3 = -e_3e_2 = e_1\ ,\ e_3e_1 = -e_1e_3 = e_2\ .$$

Each quaternion $a \in \mathbb{H}$ permits the representation

$$a = \sum_{k=0}^{3} a_k e_k \qquad (a_k \in \mathbb{R}\ ;\ k = 0,1,2,3)\ .$$

Addition and multiplication in $\mathbb{H}$ turn it into a non-commutative number field. The main-involution in $\mathbb{H}$ is called quaternionic conjugation and defined by

$$\overline{e_0} = e_0\ ,\ \overline{e_k} = -e_k \qquad (k = 1,2,3)\ .$$

which can be extended onto $\mathbb{H}$ by $\mathbb{R}$-linearity. Therefore we have

$$\overline{a} = a_0 - \sum_{k=1}^{3} a_k e_k = a_0 - \mathbf{a}.$$

Note that

$$a\overline{a} = \overline{a}a = \sum_{k=1}^{3} a_k^2 =: |a|_{\mathbb{H}}^2\ .$$

If $a \in \mathbb{H} \setminus \{0\}$ then the quaternion

$$a^{-1} := \frac{\overline{a}}{|a|^2}$$

is the inverse to a. For $a, b \in \mathbb{H}$ we have $\overline{abba}$.

Complex quaternions: The set of complex quaternions, which we also need, is denoted by $\mathbb{H}(\mathbb{C})$ and consist of all elements of the form

$$a = \sum_{k=0}^{3} a_k e_k \qquad (a_k \in \mathbb{C}\ ;\ \ k = 0,1,2,3)\ .$$

By definition we state: $ie_k = e_k i,\ k = 0,1,2,3$. Here i denotes the usual imaginary unit in $\mathbb{C}$. Elements of $\mathbb{H}(\mathbb{C})$ can also be represented in the form

$$a = a^1 +\ ia^2 \qquad (a^k \in \mathbb{H};\ \ k = 1,2).$$

Notice that the quaternionic conjugation acts only on the quaternionic units and not on the pure complex number i.

Let $X = W_p^k(G), Y = W_p^{k+1}(G), Z = W_p^{k-(1/p)+1}(\Gamma); k = 0,1,2,...; 1 < p < \infty$. Further, let

$$\begin{aligned}
L &:= D = \sum_{i=1}^{3} \partial_i e_i \quad \text{(Dirac operator (mass zero))},\\
(Tu)(x) &:= -\frac{1}{\sigma_3}\int_G e(x-y)u(y)dy \quad \text{(Teodorescu transform)},\\
(Pu)(x) &:= (F_\Gamma u)(x) = \frac{1}{\sigma_3}\int_\Gamma e(x-y)n(y)u(y)d\Gamma_y \ \text{(Cauchy} - \text{Fueter operator)},\\
(Tru)(\xi) &:= n.t. - \lim_{\substack{z\to\xi\in\Gamma\\ z\in G}} u(z),
\end{aligned}$$

with $e(x) = D\frac{1}{|x|}$ and $n = \sum_{i=1}^{3} e_i n_i$ the outward pointing unit vector of the normal.

The class of L-holomorphic functions are just the solutions of the Mosil–Teodorescu system.

We now consider so called Dirac operators with mass. We will use the same spaces as above. Then the general operators L, T and P are given by

$$\begin{aligned}
L &:= D + i\alpha \quad \text{(Dirac operator with mass)},\\
(Tu)(x) &:= -\frac{1}{\sigma_3}\int_G e_{i\alpha}(x-y)u(y)dy \quad \text{(Teodorescu type transform)},\\
(Pu)(x) &:= \frac{1}{\sigma_3}\int_\Gamma e_{i\alpha}(x-y)n(y)u(y)d\Gamma_y \ \text{(Cauchy} - \text{Fueter} - \text{typeoperator)},\\
(Tru)(\xi) &:= n.t. - \lim_{\substack{z\to\xi\in\Gamma\\ z\in G}} u(z).
\end{aligned}$$

For the description of the kernel function of this new Teodorescu transform we have to use Bessel-functions of third kind so called MacDonald functions. We have

$$e_{i\alpha(x)} := -\left(\frac{i\alpha}{2\pi}\right)^{(3/2)} \left[|x|^{-1/2}K_{3/2}(i\alpha|x|)\omega - K_{1/2}(i\alpha|x|)\right],$$

where $\omega \in S^2$ and $K_\mu(t)$ denotes.

5 Discrete quaternionic holomorphic functions

One advandage of our notion of L-holomorphy is its applicability also on lattices. We will present a calculus which was obtained by K. Guerlebeck in 1988 [4] (cf. also [6]). For this reason we have to represent the domain on the lattice and to define what are inner and outer points relatively to the "discrete boundary" and to say what the discrete boundary means. This boundary has to approximate the original domain. It is necessary to disdinguish between a right and a left parts of the boundary. The approximating discrete domain is here always an axes–parallel polyeder with side faces, edges and corner points. More exactly holds

$$\mathbb{R}^3_h := \{(ih, jh, kh) : i, j, k \textit{ integer},\ h > 0\}, \quad G_h := G \cap \mathbb{R}^3_h,$$
$$\Gamma_h := \{x \in G_h : \mathrm{dist}(x, \mathrm{co}G_h) \leq \sqrt{3}h\}.$$

Let $V^{\pm}_{i,h}x$ the translation of x by $\pm h$ in x_i-direction, then

$$\Gamma_{h,\ell(r)} := \{x \in \Gamma_h : \exists i : V^{\pm}_{i,h}x \notin G_h\} \quad \text{(left(right) side planes)},$$
$$\Gamma_{h,\ell(r);i} := \{x \in \Gamma_h : V^{\pm}_{i,h}x \notin G_h\},$$
$$\Gamma_{h,\ell(r);i,j} := \Gamma_{h,\ell(r);i} \cap \Gamma_{h,\ell(r);j} \quad \text{(left(right) edges)},$$
$$\Gamma_{h,\ell(r);i,j,k} := \Gamma_{h,\ell(r);i,j} \cap \Gamma_{h,\ell(r);k} \quad \text{(left(right) corners)}.$$

Let be $X = W^1_{2,h}(G_h),\ Y = L_{2,h}(G_h),\ Z = W^{\frac{1}{2}}_{2,h}(G_h)$. Then

$$(Lu)(x) := (D^{\pm}_h u)(x) = \sum_{i=1}^{3} e_i[u(V^{\pm}_{i,h}x) - u(x)]\frac{1}{h} \text{ (discr. Dirac operator)},$$
$$(Tu)(x) := (T^{\pm}_h u)(x) \quad \text{(discrete Teodorescu transform)}$$
$$= \left(\sum_{\mathrm{int}G_h \cup \Gamma_{h,\ell(r)}} + \sum_{left(right)\ corners} - \sum_{left(right)\ edges}\right) e^{\pm}_h(x-y)u(y)h^3,$$

where $e^{\pm}_h$ are the discrete fundamental solutions of $D^{\pm}_h$. The discrete Cauchy–Fueter operator is introduced as follows

$$(Pu)(x) := (F^{\pm}_h u)(x) = \sum_{i=1}^{3}\left(-\sum_{s_i} + \sum_{s_{ij}} - \sum_{s_{ijk}}\right) e^{\pm}_h(x - V^{\mp}_{i,h}y)n(y)u(y)h^2$$
$$+ \sum_{i=1}^{3} \sum_{\substack{y \in \Gamma_{h,\ell(r);m,j,k} \\ m \neq j \neq k}} h^{\pm}(x-y)e_i u(y)h^2,$$

where $s_i = \Gamma_{h,\ell;i} \cup \Gamma_{h,r;i},\ s_{ij} := \Gamma_{h,\ell;j} - V^{+}_{i,h}\Gamma_{h,\ell}, \quad s_{ijk} := \Gamma_{h,\ell;j,k} - V^{+}_{i,h}\Gamma_{h,\ell;i,k}$.

The corresponding mean value formulae are given as follows

$$u(x) = (F^{\pm}_h u)(x) + T^{\pm}_h D^{\pm}_h u(x)$$

Much more complicated is to find a suitable discrete fundamental solution, which is given by $E_h(x)$ as solution of a suitable difference equation

$$-\Delta_h E_h(x) = -\sum_{i=1}^{3} D^{-}_{i,h} D^{+}_{i,h} E_h(x) = \delta_h(x) = \begin{cases} h^{-3}, x = 0 \\ 0, x \in \mathbb{R}^3_h \setminus \{0\} \end{cases}$$

expressed by using the Fourier-Transform we have

$$E_h(x) = \frac{1}{\sqrt{2\pi}^3} R_h F\left(\frac{1}{d^2}\right).$$

The function d is defined as follows

$$d^2 = \frac{4}{h^2}\left(\sin^2\frac{h\xi_1}{2} + \sin^2\frac{h\xi_2}{2} + \sin^2\frac{h\xi_3}{2}\right)$$

and $R_h u$ is the restriction of the continuous function u onto the lattice $\mathbb{R}^3_h$. We have $|E_h| \leq C|x|^m$ with a certain $m > 0$ depending on the properties of the difference operator

$$e_h^{\pm}(x) := D^{\mp}_{j,h} E_h(x).$$

6 L-holomorphy on the sphere

Meanwhile is also existing the notion of holomorphy on the sphere. A good reference is doctoral thesis of P. Van Lancker [17] The following operators has to be used $\Gamma_S + \alpha \quad \alpha \in \mathbb{C} \setminus \mathbb{N} \cup (-\mathbb{N})$.

$$\begin{aligned}
L_\alpha : &= \omega(\Gamma_S + \alpha) && \text{(Günter's gradient)},\\
T_\alpha : &= -\int_\Omega E_\alpha(\omega,\xi)\cdot dS(\omega) && \text{(Teodorescu transform)},\\
P_{C,\alpha} : &= -\int_{-C} E_\alpha(\omega,\xi) n(\omega)\cdot dC(\omega) && \text{(Cauchy-Fueter type operator)}.
\end{aligned}$$

A corresponding Borel-Pompeiu formula is given by

$$P_{C,\alpha}u + T_\alpha D_\alpha u = \begin{cases} u & \text{in} \quad \Omega \\ 0 & \text{in} \quad S\setminus\overline{\Omega} \end{cases}.$$

We will consider the fundamental solution of Günter's gradient. Let $\alpha \in \mathbb{C}\setminus\mathbb{N}\cup\{-2-\mathbb{N}\}$. Then

$$E_\alpha(\omega,\xi) = \frac{\pi}{\sigma_3 \sin\pi\alpha} K_\alpha(-\xi,\omega)\omega,$$

where σ_3 is the surface area of the unit sphere. Further, we define

$$K_\alpha(-\xi,\omega)\omega = C_\alpha^{3/2}(\omega\cdot\xi) + \xi\omega C_{\alpha-1}^{3/2}(\omega\cdot\xi),$$

with the so-called Gegenbauer polynomials $C^\mu_\alpha(t)$.

Using Kummer's function ${}_2F_1(a,b;c;z)$ we get the representation

$$C_\alpha^{3/2}(z) = \frac{\Gamma(\alpha+3)}{\Gamma(\alpha+1)}\,\frac{1}{4}\,{}_2F_1(-\alpha,\alpha+3;2;\,\frac{1-z}{z}) \quad z \in \mathbb{C}\setminus\{-\infty,1\}.$$

Kummer's function is for $|z| < 1$ defined by

$$ {}_2F_1(a,b;c;z) := \sum_{k=0}^{\infty} \frac{(a_k)(a_k)}{(c)_k} \frac{z^k}{k!}, \qquad (a)_k \quad = \quad \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}. $$

Solutions of $D_\alpha u = 0$ in Ω are called inner spherical holomorphic functions of order α in Ω. We have

$$ D_\alpha E_\alpha(\omega,\xi) = \delta(\xi-\omega) \ . $$

A good reference for this topic is [1]. Further we introduce a singular integral operator of Bitzadse's type

$$ \begin{aligned} (S_{C,\alpha}u)(\xi) : \quad &= \quad 2\lim_{\varepsilon\to 0} \int\limits_{C\setminus B_\varepsilon(\xi)} E_\alpha(\omega,\xi)n(\omega)u(\omega)dS(\omega) \\ &= \quad 2v.p. \int\limits_{C} E_\alpha(\omega,\xi)n(\omega)u(\omega)dS(\omega). \end{aligned} $$

One can prove the algebraical identity $S_{C,\alpha}^2 = I$. Let $\Omega^+ := \Omega$, $\Omega^- := \mathrm{co}\Omega$. Applying the general trace operator as non-tangential limit on the sphere towards the boundary C we get Plemelj-Sokkotzkij-type formulae.

$$ n.t. - \lim_{\substack{t\to\xi \\ t\in\Omega^\pm}} (F_{C,\alpha}u)(t)\frac{1}{2}[\pm I + S_{C,\alpha}]u(\xi) =: \begin{cases} P_{C,\alpha}u(\xi), t\in\Omega^+ \\ -Q_{C,\alpha}u(\xi), t\in\Omega^- \end{cases} . $$

The operators

$$ Q_{C,\alpha} := \frac{1}{2}[I - S_{C,\alpha}], \qquad P_{C,\alpha} : \frac{1}{2}[I + S_{C,\alpha}] $$

are called Plemelj projections. The space $L_2(\Gamma)$ is now decomposed into the Hardy spaces

$$ \begin{array}{ccccc} L_2(C) = & HS^\alpha(\Omega^+) & \oplus & HS^\alpha(\Omega^-) \\ & \uparrow & & \uparrow \\ & P_{C,\alpha} & & Q_{C,\alpha} \end{array} $$

(cf. [12]).

7 Taylor type formula

Using ideas of the theory of right invertible operators (cf. D. Przeworska-Rolewicz, [10]) one has with $Y_m = \mathcal{D}(L^m) \subset Y$ (m is a natural number) the operators

$$ \begin{aligned} &L^j : Y_m \to X_{m-j}, \quad P : Z_{m-j} \to Y_{m-j}, \quad PTr : Y_{m-j} \to Y_{m-j}, \\ &T^j : X_{m-j} \to Y_m \quad (0 \le j \le m-1). \end{aligned} $$

Here we have $Y_m \subseteq \ldots \subseteq Y_2 \subseteq Y_1$ and $L^0 = T^0 = I$.

Proposition 1. *The following properties are fulfiled*

(i) The operators $T^jPTrL^j \quad (0 \le j \le m-1)$ *are projections on* Y_m.

(ii) The projections $T^jPTrL^j \quad (0 \le j \le m-1)$ *are complementary on* Y_m, *i.e.* $(T^jPTrL^j)(T^kPTrL^k) = (T^kPTrL^k)(T^jPTrL^j) = 0$ *for all* $0 \le j,k \le m-1$ *and* $k \ne j$.

Proof. (i) Indeed, using the assumption $PTrP = P$ and corollary 1 we obtain

$$(T^jPTrL^j)(T^jPTrL^j) = T^jPTrL^jT^jPTrL^j = T^jPTrPTrL^j = T^jPTrL^j,$$

i.e. T^jPTrL^j are projections on Y_m. To prove property (ii) we also use corollary 1. It is immediately clear that $L^jT^j = I$ from $LT = I$. Because of $PTrT = 0$ and $L^jT^j = I$ follows for $j < k$:

$$(T^jPTrL^j)(T^kPTrL^k) = T^jPTrL^jT^kPTrL^k = T^jPTrT^{k-j}PTrL^k = 0,$$

i.e.

$$(T^jPTrL^j)(T^kPTrL^k) = 0 \quad (0 \le j < k \le m).$$

Taking into account relation in the corollary from above, the commutative property is obtained. Indeed, from property $LPTr = 0$ we have

$$(T^kPTrL^k)(T^jPTrL^j) = T^kPTrL^kT^jPTrL^j = T^kPTrL^{k-j}PTrL^j = 0,$$

i.e.

$$(T^kPTrL^k)(T^jPTrL^j) = 0 \quad (0 \le j < k \le m).$$

Hence all $T^jPTrL^j (0 \le j \le m)$ are complementary on Y_m. #

Then the next corollary is clear.

Corollary 3. *The operator*

$$P_m := \sum_{j=0}^{m-1} T^jPTrL^j = T^0PTrL^0 \oplus T^1PTrL^1 \oplus \ldots \oplus T^{m-1}PTrL^{m-1}$$

is a projection on Y_{m-1}.

Corollary 4. *The operators* P_m, T^m *and* L^m *have the following relations*

(i) The operator T^m *is the right-inverse to the operator* L^m, *i.e.* $L^mT^m = I$.

(ii) The operators L^m, P_m *satisfy the property* $L^mP_m = 0$.

(iii) It holds $P_mT^m = 0$.

Proof. The relation (i) is simple to be obtained from corollary 1. To prove (ii), one use assumption $LPTr = 0$ and $L^jT^j = I$ for $0 \le j \le m-1$ as mentioned above then

$$L^mP_m = \sum_{j=0}^{m-1} L^mT^jPTrL^j = \sum_{j=0}^{m-1} L^{m-j}PTrL^j = 0.$$

The same for relation (iii) with assumption $PTrT = 0$:

$$P_mT^m = P_m := \sum_{j=0}^{m-1} T^jPTrL^jT^m = P_m := \sum_{j=0}^{m-1} T^jPTrT^{m-j} = 0.$$

Theorem 2. (The Taylor type formula) *Let L be a right invertible operator that defined from an injection T and an initial operator P. Then for $m = 1, 2, ...$ the following identity holds on Y_m*

$$u = \sum_{j=0}^{m-1} T^jPTrL^ju + T^mL^mu.$$

Proof. We have ker $T^m = \{0\}$ by assumption T is an injection and im $T^m \subset Y_m = \mathcal{D}(L^m)$. Corollary 3 shows that P_m is a projection and $P_mT^m = 0$. Furthermore, it is simple to show that im $T^m \cap$ im $P_m = \{0\}$. Indeed, let $u \in$ im $T^m \cap$ im P_m then

$$u = P_mv = T^mw, \quad (v \in Y_{m-1}, w \in X).$$

Since $P_mT^m = 0$ we get

$$u = P_mv = P_mP_mv = P_mT^mw = 0.$$

Let B be the (unique) right inverse to T^m then (from the mean value formula)

$$u = P_mu + T^mBu \quad \text{with} \quad \mathcal{D}(B) := \mathrm{im}T^m \oplus \mathrm{im}P_m.$$

Now we will show that L^m also satisfies above formula. By applying the mean value formula for L^ju we get

$$L^ju = PTrL^ju + TL^{j+1}u \quad (0 \le j \le m-1)$$

Rewrite in more detail and acting operators $T^j \quad (0 \le j \le m-1)$ to both sides we have

$$\begin{aligned} T^0L^0u &= T^0PTrL^0u + TLu, \\ TLu &= TPTrLu + T^2L^2u, \\ &\dots \\ T^{m-1}L^{m-1}u &= T^{m-1}PTrL^{m-1}u + T^mL^mu. \end{aligned}$$

Sum up all equabilities we obtain

$$\begin{aligned} u = T^0 L^0 u &= T^0 PTrL^0 u + TPTrLu + \ldots + T^{m-1} PTrL^{m-1} u + T^m L^m u \\ &= P_m u + T^m L^m u. \end{aligned}$$

Then the property of uniqueness of right inverse operator leads to

$$B = L^m.$$

This completes the proof of our theorem.

Example 15. (Realisation in R^1) We continue the first example in section 3.1.For all functions $u \in C^1[0,1]$, recall that

$$L := \frac{d}{dt}\ ,\ T := \int\limits_0^t \cdot d\tau\ ,$$

$P := I$ and $Tr : C^1[0,1] \to R^1$ with $Tr\,u = u(0)$. Then we have

$$T^j PTr(L^j u)(t) = (L^j u)(0)\frac{t^j}{j!}$$

and

$$(T^m u)(t) = \int\limits_0^t \frac{(t-\tau)^{m-1}}{(m-1)!} u(\tau) d\tau\ .$$

Hence the theorem 2 yields the classical Taylor's formula

$$u(t) = \sum_{j=0}^{m-1} (L^j u)(0)\frac{t^j}{j!} + \int\limits_0^t \frac{(t-\tau)^{m-1}}{(m-1)!}(L^m u)(\tau) d\tau\ .$$

Example 16. (Taylor formula for fractional operators) In [9] J.D. Munkhammar gave Taylor's formula based on fractional caculus. Let $u(t) \in C^1([a,b])$ then the Riemann-Liouville fractional integral of order α is

$$(Tu)(t) := I^{\alpha}_{a+} u(t) = \frac{1}{\Gamma(\alpha)} \int\limits_a^t \frac{u(s)}{(t-s)^{1-\alpha}} ds\ ,$$

and the Riemann–Liouville fractional derivative of order α as follow

$$(Lu)(t) := D^{\alpha}_{a+} u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int\limits_a^t \frac{u(s)}{(t-s)^{\alpha}} ds$$

where $\alpha \in]0,1[$ and Γ is a well known Gamma function. Hence

$$D^{\alpha}_{a+} I^{\alpha}_{a+} = I.$$

Let $\alpha > 0$, $m \in Z^+$ and $u(t) \in C^{[\alpha]+m+1}([a,b])$, the Taylor formula is

$$u(t) = \sum_{k=-m}^{m-1} \frac{D^{\alpha+k}_{a+} u(t_0)}{\Gamma(\alpha+k+1)} (t-t_0)^{\alpha+k} + I^{\alpha+m}_{a+} D^{\alpha+m}_{a+} u(t)$$

for all $a \le t_0 < t \le b$.

8 Taylor-Gontcharov's formula for high order genaralized Dirac operators

Corollary 5. (The Taylor-Gontcharov's formula) *A generalization of the Taylor formula leads to*

$$u = \sum_{j=0}^{m-1} T_0 T_1 ... T_j P_j L_j ... L_1 L_0 u + T_1 ... T_m L_m ... L_1 u$$

with $L_0 = T_0 = I$.

Example 17. (Realisation on a lattice) Let G_h be the lattice of the bounded domain G and $\Delta_h = D_h^+ D_h^-$ be the discretized Laplace operator. We consider the following problem

$$\begin{aligned} \Delta_h u &= f \quad \text{on} \quad G_h, \\ tr_\Gamma P_{\Gamma_h} u &= g_0 \quad \text{on} \quad \Gamma_h, \\ tr_{\Gamma_h} D_h^- u &= g_1 \quad \text{on} \quad \Gamma_h. \end{aligned}$$

Γ_h is the *"'numerical"' boundary* of G for a meshwidth h. The unique solution is then given by

$$u = F_h^- g_0 + T_h^- F_h^+ (tr_{\Gamma_h} T_h^- F_h^+)^{-1} T_h^- D_h^- g_1 + T_h^- \mathbf{Q}_h T_h^+ f$$

with Bergman projection

$$\mathbf{P}_h = F_h^+ (tr_{\Gamma_h} T_h^- F_h^+)^{-1} tr_{\Gamma_h} T_h^-$$

The operators in Taylor-Gontcharov' s formula are chosen as follows

$$L_1 := D_h^-, \; L_2 := D_h^+, \; P_1 := F_h^-, \; P_2 := F_h^+, \; T_1 := T_h^-, \; T_2 := T_h^+$$

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