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An Inter-Group Conflict and its Relation to Oligopoly Theory

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ABSTRACT

A game theoretical model of inter-group conflicts is revisited. In this model members of each group contribute to secure a public good which becomes then available to all members regardless if they contributed or not, and the groups compete for an exogenous prize simultaneously. We first show that the best response of each group member is mathematically equivalent to that in oligopolies with isoelastic price and linear cost functions. Then a complete equilibrium analysis is given showing that, except in a very special case, there is a unique equilibrium. And finally, a dynamic extension of the game is introduced and analysed, where the players are able to increase their contributions at any time during a given time period.

RESUMEN

Un modelo de juego teórico de conflictos de intergrupos es revisado. En este modelo los miembros de cada grupo contribuyen para asegurar un bien público, el cual queda disponible para todos los miembros sin importar si éstos contribuyen o no, y los grupos compiten por un premio simultaneamente. Mostramos que la mejor respuesta de todo grupo es matemáticamente equivalente a oligopolios con funciones de precio isoelásticas y costo lineal. Un completo análisis de equilibrio es dado, demostrando que, salvo en casos muy especiales, existe un único equilibrio. Finalmente, es presentada y analizada una extensión dinámica del juego, donde los jugadores son capaces de aumentar sus contribuiciones en cualquer momento durante un determinado período.

Key words and phrases: *Oligopoly, n-person games, intergroup conflict.*

Math. Subj. Class.: *91A07, 91A40.*

1 Introduction

In this paper we will investigate an inter-group conflict: a game involving groups of players in which conflict arises in both the group and the individual levels simultaneously. In the classical approach the groups were considered as the players, the payoff of each group was computed as the sum of the payoffs of its members. However, if group members have selfish interest that does not always coincide with group interest, then this "group equilibrium" approach is irrealistic. Therefore it is more appropriate to model how the satisfaction of group objectives affects individual members and to include these consequences into the payoffs of the members. In this case a multiplayer game can be defined in which the players are the members of all groups and therefore we have to consider conflict only among the players. In this study we will follow this approach. The game we will examine in this paper has been introduced and studied by [4] and further analysed in [1].

Assume that the members of n groups ($n \geqq 2$) contribute to a public good and simultaneously the groups compete to win an exogenous prize S. This external prize can be thought of as a mechanism to increase contribution by creating a between group competition. Let $n(i)$ denote the number of members of group i. Each member k of each group i receives an initial endowment of y_{ki}. The decision of each member is to decide on the contributed amount X_{ki}. Then this member will keep $y_{ki} - X_{ki}$ for herself.

The overall contribution of group i is $X_i = \sum_{k=1}^{n(i)} X_{ki}$, which serves two purposes. First, it generates a public good for the group. Let g_i denote the maximal public good that can be generated if all members contribute their entire endowment. Otherwise the generated public good is proportional to the contributed amount: $g_i X_i / Y_i$, where $Y_i = \sum_{k=1}^{n(i)} y_{ki}$ is the total endowment of group i. Second, the group contribution probabilistically determines the group's success in winning the exogenously determined prize. It is assumed that only one group can win the prize and higher group contribution implies higher winning probability. Therefore the payoff of member

k of group i has the following form:

$$\varphi_{ki} = S \cdot \frac{X_i}{X} \cdot \frac{X_{ki}}{X_i} + g_i \frac{X_i}{Y_i} + (y_{ki} - X_{ki}), \tag{1}$$

where $X = \sum_{i=1}^{n} X_i$ is the total contribution of all members of all groups. The first term gives the expected share of the external prize assuming that in the case of winning it, the members' shares are proportional to their contributions. The second term is the public good generated by the group which is assumed to become available equally to all group members, and the third term is that part of the initial endowment which is not contributed. In the first term X_i cancels, and it is not defined for $X = 0$. In this case when no contributions are made, no prize is awarded and no public good is generated. Since no game is played, the payoffs of all members of all groups are zero, or alternatively, we can assume that they can keep their endowments.

In this way a $\sum_{i=1}^{n} n(i)$ – player game is defined, in which the members are the players, the strategy set and payoff function of member k of group i is $[0, y_{ki}]$ and φ_{ki}, respectively.

In this paper we will examine the existence and uniqueness of the Nash equilibrium of this game. The paper is developed as follows. First we will demonstrate a strong analogy between this game and oligopolies with isoelastic price functions. They have mathematically identical best response functions with the only difference that the parameters corresponding to marginal costs are not restricted to positive values. Then the existence and the uniqueness of the equilibrium will be proved. A simple dynamic extension of the game will be introduced next. The last section will conclude the paper.

2 Relation to Oligopoly Games

The game presented above has been introduced to model conflicts between noncooperative groups. Despite the different interpretation, its structure has a close resemblence to a special oligopoly game. In this section we will demonstrate the relationship between these seemingly different lines of research.

Consider a market of N firms producing identical product. Let x_k denote the output of firm k with capacity limit L_k. Assume that the cost of firm k depends on its own output level, $C_k(x_k)$, but the unit price depends on the total production of all firms. Assuming hyperbolic price, $\frac{A}{\sum_{l=1}^{N} x_l}$, and linear cost functions, $\alpha_k + \beta_k x_k$, the profit of firm k can be given as

$$\varphi_k = \frac{A x_k}{\sum_{l=1}^{N} x_l} - (\alpha_k + \beta_k x_k). \tag{2}$$

An N-person noncooperative game is defined above, where the firms are the players, the strategy set of firm k is the closed interval $[0, L_k]$, and its payoff function is φ_k. This family of games, known as oligopoly games, is one of the most frequently discussed topics in mathematical economics [2], [3].

We will next show that games (1) and (2) are very similar, their equilibrium problems are equivalent.

Consider first game (1) and member k of group i, and assume that the contribution of the other participants are known to her. If we introduce notation $Q_{ki} = X - X_{ki}$, which is the total contributions of all others, then clearly,

$$\varphi_{ki} = \frac{SX_{ki}}{X_{ki} + Q_{ki}} + (y_{ki} - X_{ki}) + \frac{g_i X_{ki}}{Y_i} + g_i \frac{\sum_{l \neq k} X_{li}}{Y_i}. \tag{3}$$

Notice that the last term and y_{ki} do not depend on X_{ki}, so the best response of this player depends on only Q_{ki}, and it is the maximizer of function

$$\frac{SX_{ki}}{X_{ki} + Q_{ki}} - \left(1 - \frac{g_i}{y_i}\right) X_{ki} \tag{4}$$

on interval $[0, y_{ki}]$.

Consider next game (2), and for firm k introduce the notation $Q_k = \sum_{l \neq k} x_l$. Then maximizing φ_k is equivalent to the maximization of function

$$\frac{Ax_k}{x_k + Q_k} - \beta_k x_k. \tag{5}$$

Obviously functions (4) and (5) are equivalent with A and β_k being replaced by S and $1 - \frac{g_i}{Y_i}$, respectively. Consequently the best responses of the players are the same and therefore the equilibria of the two games are also equivalent to each other. Notice that in oligopoly theory the marginal cost β_k has to be always positive, while $1 - \frac{g_i}{Y_i}$ can be also zero or negative. Therefore the existence results known from oligopoly theory cannot be directly applied.

3 Best Responses and Equilibria

Our public good contribution game is based on relations (1) and (4) which is mathematically equivalent to a generalized oligopoly game, where marginal costs are not restricted to negative values. As we will see later, the dynamic extension of the public good contribution game is fundamentally different than that of oligopolies.

For the sake of simple notation we will use function (5), which will be denoted by f_k. By simple differentiation,

$$\frac{\partial f_k}{\partial x_k} = \frac{AQ_k}{(x_k + Q_k)^2} - \beta_k \tag{6}$$

and

$$\frac{\partial^2 f_k}{\partial x_k^2} = -\frac{2AQ_k}{(x_k + Q_k)^3} < 0, \tag{7}$$

so f_k is strictly concave in x_k, so the best response of firm k is unique.

Assume first that $Q_k = 0$. Then

$$f_k = \begin{cases} 0 & \text{if} \quad x_k = 0 \\ A - \beta_k x_k & \text{if} \quad x_k > 0, \end{cases}$$

so we have three cases. If $\beta_k > 0$, then firm k's interest is to produce as small as possible positive amount, so no best response exists. This is also the case in game (1), when the players can keep their endowments if no contributions are made by i. If $\beta_k = 0$, then the best response is the entire interval $(0, L_k]$, and if $\beta_k < 0$, then the best response is the maximum feasible amount L_k.

Assume next that $Q_k > 0$. The concavity of f_k implies that the best response of player k is given as

$$R_k(Q_k) = \begin{cases} 0 & \text{if} \quad \frac{\partial f_k}{\partial x_k}\,|_{x_k=0} \leqq 0 \\ L_k & \text{if} \quad \frac{\partial f_k}{\partial x_k}\,|_{x_k=L_k} \geqq 0 \\ z_k^* & \text{otherwise} \end{cases} \tag{8}$$

where z_k^* is the unique solution of equation

$$\frac{\partial f_k}{\partial z_k} = \frac{AQ_k}{(z_k + Q_k)^2} - \beta_k = 0 \tag{9}$$

in interval $(0, L_k)$. Notice that in the case when $\beta_k \leqq 0$, the second case of (8) occurs, so $R_k(Q_k) = L_k$.

Notice that in the case of $Q_k = 0$ we had a similar case, L_k was always a best response but in the case of $\beta_k = 0$ there were infinitely may other best responses in interval $(0, L_k)$. Otherwise with notation

$$z_k^* = \sqrt{\frac{AQ_k}{\beta_k}} - Q_k \tag{10}$$

we have

$$R_k(Q_k) = \begin{cases} 0 & \text{if} \quad z_k^* \leqq 0 \\ L_k & \text{if} \quad z_k^* \geqq L_k \\ z_k^* & \text{otherwise.} \end{cases} \tag{11}$$

This function is illustrated in Figure 1. In order to reduce the equilibrium problem to a single-variable equation we have to rewrite the best response definitions in terms of the total production level $Q = \sum_{k=1}^{N} x_k$ of all players. We assume next again that $Q_k > 0$.

The first case of (11) occurs when $Q_k = Q$, that is, if $\sqrt{\frac{AQ}{\beta_k}} - Q \leqq 0$. This is the case as $Q = 0$ or $Q \geqq \frac{A}{\beta_k}$.

The second case of (11) occurs when $Q_k = Q - L_k$, or

$$\sqrt{\frac{A(Q - L_k)}{\beta_k}} - (Q - L_k) \geqq L_k,$$

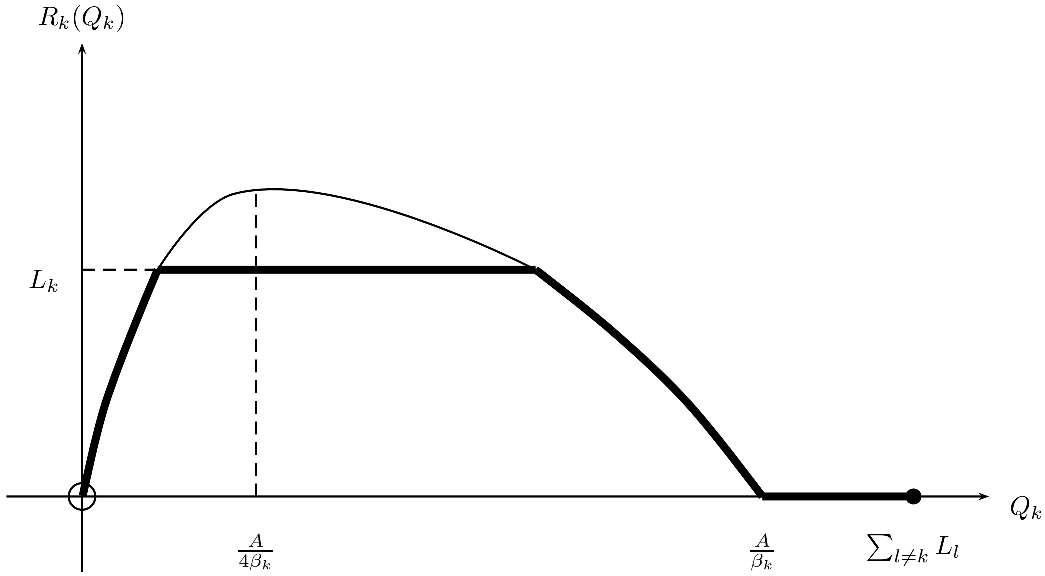

Figure 1: Graph of best response $R_k(Q_k)$

which can be rewritten as

$$Q(1 - Q\frac{\beta_k}{A}) \geqq L_k. \tag{12}$$

The third case occurs when

$$x_k = \sqrt{\frac{A(Q - x_k)}{\beta_k}} - (Q - x_k)$$

or

$$x_k = Q\left(1 - Q\frac{\beta_k}{A}\right). \tag{13}$$

In summary, the best response function of player k is equivalent with the following:

$$\bar{R}_k(Q) = \begin{cases} 0 & \text{if} \quad Q = 0 \quad \text{or} \quad Q \geqq \frac{A}{\beta_k} \\ L_k & \text{if} \quad g_k(Q) \geqq L_k \\ g_k(Q) & \text{otherwise} \end{cases} \tag{14}$$

where

$$g_k(Q) = Q\left(1 - Q\frac{\beta_k}{A}\right) \tag{15}$$

for all k. In analysing the case of $Q_k = 0$ we have seen that in that case zero cannot occur as the best response, so at any equilibrium $Q \neq 0$. Function (14) is illustrated in Figure 2.

In examining the existence and uniqueness of the equilibrium we have to consider several possibilities.

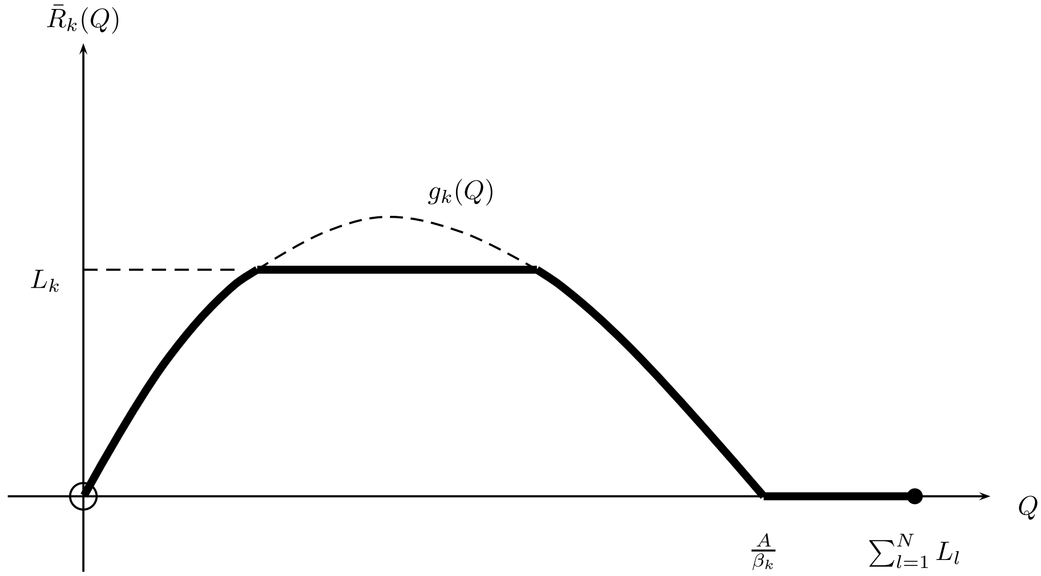

Figure 2: Graph of function $\bar{R}_k(Q)$

Notice first that $Q = 0$, when all $x_k = 0$, cannot be equilibrium. Consider next the case when only one $x_k > 0$ and all other $x_l = 0 \quad (l \neq k)$. Then $Q_k = 0$, and $Q_l > 0$ for $l \neq k$, so $\beta_k \leqq 0$ necessarily. If $\beta_k = 0$, then any $x_k \in (0, L_k]$ is possible and if $\beta_k < 0$, then $x_k = L_k$. For all other players $\beta_l > 0$ and $x_k = Q \geqq \frac{A}{\beta_l}$. Assume next that $x_k > 0$ for at least two players. Then $Q_k > 0$ for all players. Consider next equation

$$H(Q) = \sum_{k=1}^{N} \bar{R}_k(Q) - Q = 0. \tag{16}$$

In the left hand side $\bar{R}_k(Q)$ is either the truncated parabola (14) for $\beta_k > 0$ or the horizontal line L_k for $\beta_k \leqq 0$ (it is also the largest best response if $Q_k = 0$).

We will now prove that equation (16) has a unique solution. Observe first that for a truncated parabola $\bar{R}'_k(0) = 1$. Since $N \geqq 2$, there are either at least two $\bar{R}_k$ functions being truncated parabolas, or for at least one k, $\bar{R}_k(Q) \equiv L_k$. In all cases the right hand side limit of $H(Q)$ at zero is always positive. At $Q = \sum_{l=1}^N L_l$, the value of $H(Q)$ is nonpositive, since $\bar{R}_k(Q)$ cannot be greater than L_k. Since $H(Q)$ is continuous, there is at least one solution $Q > 0$. Assume next that there are two solutions $Q^{(1)}$ and $Q^{(2)}$ $(Q^{(1)} < Q^{(2)})$. We will first prove that there is a $Q^* \in [Q^{(1)}, Q^{(2)}]$ such that $H(Q^*) \leqq 0$ and $H'(Q^*+) \geqq 0$ where $H'(Q^*+)$ denotes the right hand side derivative. Assume that $Q^{(1)}$ does not satisfy these conditions. Then $H'(Q^{(1)}+) < 0$ and there is a small $\varepsilon > 0$ such that $H'(Q^{(1)} + \varepsilon) < 0$ implying that point $(Q^{(1)} + \varepsilon, H(Q^{(1)} + \varepsilon))$ is under the horizontal axis. However $H(Q^{(2)}) = 0$, therefore the graph of $H(Q)$ between $Q^{(1)} + \varepsilon$ and $Q^{(2)}$ cannot be always nonincreasing. Since function H can have only finitely many breakpoints, there has to be a Q^* such that $H(Q^*) \leqq 0$ and $H'(Q^*+) \geqq 0$. In a small neighborhood on the right hand side of Q^*,

$$H(Q) = (C + lQ - AQ^2) - Q \tag{17}$$

where the first term is the sum of l parabolic and some constant segments. If $H(Q^*) \leqq 0$, then

$$C + (l-1)Q^* - AQ^{*2} \leqq 0$$

or

$$A \geqq \frac{C}{Q^{*2}} + \frac{l-1}{Q^*}. \tag{18}$$

Similarly, $H'(Q^*+) \geqq 0$ implies that

$$l - 2AQ^* - 1 \geqq 0,$$

so

$$A \leqq \frac{l-1}{2Q^*}. \tag{19}$$

Relations (18) and (19) are contradictory except in the following special cases. If $l = 0$, then there is no player with truncated parabola as her best response, all have L_k as their best responses with zero derivatives, so $H'(Q^*) = -1$ and hence $H'(Q^*) \geqq 0$ is impossible. If $l = 1$ and $C = 0$, then for one player, $\bar{R}_k(Q^*)$ is truncated parabola and for all other players $\bar{R}_l(Q^*) = 0$. Since all truncated parabolas are under the 45 degree line, there is no larger solution of equation (16) then Q^*. We have therefore contradiction in all cases.

The unique solution of equation (16) gives the unique equilibrium if for at least two players, $x_k > 0$. If at the solution only one x_k is positive, then there is the possibility of infinitely many equilibria. This is the case, when for one player $\beta_k = 0$, all other $\beta_l > 0$, and

$$\max_{l \neq k} \frac{A}{\beta_l} < L_k. \tag{20}$$

Then $\bar{x}_l = 0$ for $l \neq k$ and $\bar{x}_k \in \left[\max_{l \neq k} \frac{A}{\beta_l}; L_k\right]$ are all equilibria. Otherwise for all k, $\bar{x}_k = \bar{R}_k(\bar{Q})$, where $\bar{Q}$ is the solution of equation (16). This is the case in game (1) if for a group $g_i = Y_i$ and this group has only one member.

4 Dynamic Extensions

Consider discrete time scales, $t = 0, 1, 2, \ldots$, and assume that at each time period the players are able to increase their contribution levels, but they cannot decrease the already pledged amounts. At $t = 0$ each player's initial contribution is zero. At each time period each player checks if

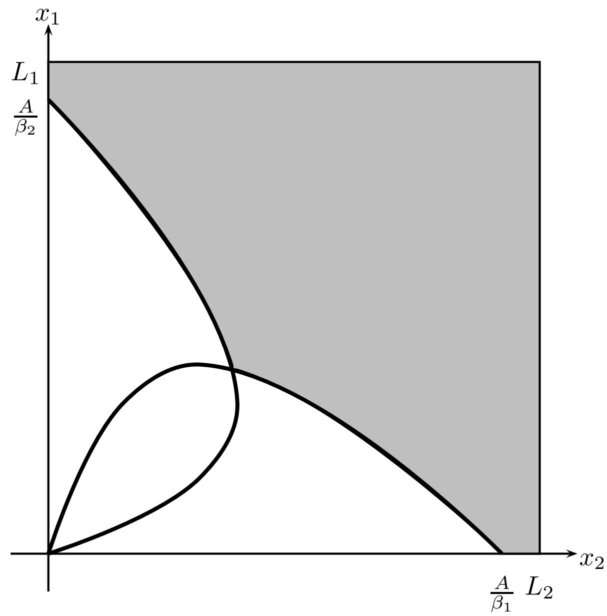

Figure 3: The set of steady states of the dynamical process

additional contribution increases her payoff or not by computing her marginal profit:

$$\frac{\partial \varphi_k}{\partial x_k} = \frac{AQ_k}{(x_k + Q_k)^2} - \beta_k. \tag{21}$$

If $\beta_k < 0$, then this derivative is always positive, so player k will always increase her contribution until it reaches the maximum L_k level. If $\beta_k = 0$, then we have a simple situation, since in the case of $Q_k > 0$ the player's interest is to increase contribution and if $Q_k = 0$, then firm k's interest is to keep the current level, so until Q_k remains zero, firm k will keep her zero initial contribution. If $\beta_k > 0$, then the player stops contributing if

$$\frac{AQ_k}{(x_k + Q_k)^2} - \beta_k \leqq 0 \quad \text{or} \quad x_k = L_k. \tag{22}$$

Since the contributions of each player form a monotonic and bounded sequence, the process always converges regardless of the amounts of the contribution increases. Hence the steady states of the dynamical process can be characterized as follows:

If $\beta_k < 0$, then $x_k = L_k$; if $\beta_k = 0$, then

$$x_k \begin{cases} = 0 & \text{if} \quad Q_k = 0 \\ = L_k & \text{if} \quad Q_k \neq 0; \end{cases}$$

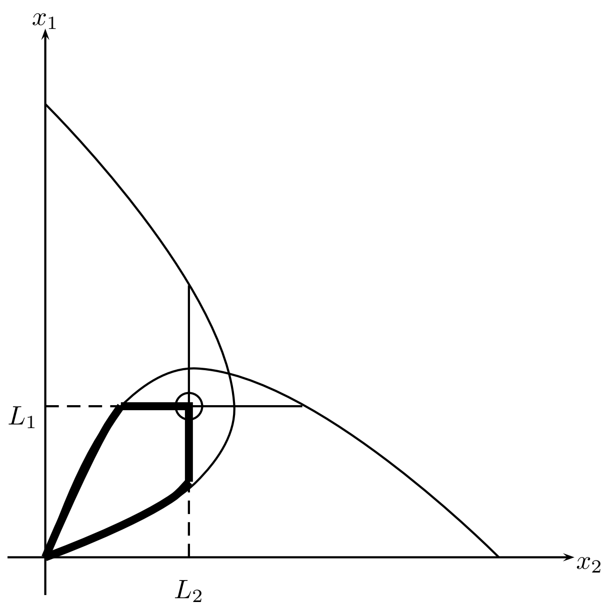

Figure 4: The case of a single steady state

and if $\beta_k > 0$, then

$$x_k \begin{cases} \geqq \sqrt{\frac{AQ_k}{\beta_k}} - Q_k & \text{if} \quad \sqrt{\frac{AQ_k}{\beta_k}} - Q_k < L_k \\ = L_k & \text{otherwise.} \end{cases} \tag{23}$$

Consider the two-person case, when $Q_1 = x_2$ and $Q_2 = x_1$ and assume the general case shown in Figure 1 with both β_1 and β_2 being positive. Figure 3 shows the set of all steady states. We also assume sufficiently large L_1 and L_2 values.

If the L_1 and L_2 values are relatively small, we might have the point (L_1, L_2) as the only steady state, as it is shown in Figure 4.

Any dynamic process starts at the origin and proceeds along a sequence of horizontal and vertical segments. The reached steady state depends on the amounts of the increments.

5 An Example

By assuming that $\beta_k > 0$ for all k, we will first determine the interior equilibrium (when $0 < x_k < L_k$ for all players). Then for all k, $\bar{R}_k(Q) = Q(1 - Q\frac{\beta_k}{A})$.

By adding this relation for all k, a single-variable equation can be obtained for Q:

$$Q = Q \sum_{k=1}^{N} \left(1 - Q\frac{\beta_k}{A}\right). \tag{24}$$

Since $Q = 0$ cannot be an equilibrium, we have

$$N - \frac{Q}{A}\sum_{k=1}^{N}\beta_k = 1,$$

so

$$\bar{Q} = \frac{(N-1)A}{\sum_{k=1}^{N}\beta_k} \tag{25}$$

and therefore

$$\bar{x}_k = \frac{(N-1)A}{\sum_{k=1}^{N}\beta_k}\left(1 - \frac{\beta_k(N-1)}{\sum_{l=1}^{N}\beta_l}\right). \tag{26}$$

In the special case of $N = 2$,

$$\bar{Q} = \frac{A}{\beta_1 + \beta_2} \tag{27}$$

and

$$\begin{aligned}\bar{x}_k &= \frac{A}{\beta_1+\beta_2}\left(1 - \frac{\beta_k}{\beta_1+\beta_2}\right)\\ &= \frac{A\beta_l}{(\beta_1+\beta_2)^2} \quad \text{with} \quad l \neq k.\end{aligned} \tag{28}$$

This is always positive, and is below L_k if

$$\frac{A}{L_k} < \frac{(\beta_1+\beta_2)^2}{\beta_l}. \tag{29}$$

Notice that Figure 3 shows such a case, when the unique equilibrium is the intercept of the two curves.

6 Conclusions

We could show that the multilevel inter-group conflict where groups can compete for an external price while the members are contributing for a common public good is mathematically equivalent to generalized oligopolies with hyperbolic price and linear cost functions. To the best of our knowledge, this similarity has not been noted before in the literature of mathematical economics.

However, the well-known results of oligopoly theory cannot be applied without additional considerations, since the parameter which replaces marginal costs can have zero and negative values, which have no economic sense in oligopoly models.

Except for a very special case the Nash-equilibrium is always unique, and can be obtained by solving a single-variable algebraic equation.

The dynamic extension of the game might have infinitely many steady states, and since the contribution sequences are monotonic and bounded, they always converge. This dynamic model is fundamentally different than dynamic oligopolies ([3]).

A simple formula has been derived for the interior equilibrium of the game under the additional condition that $\beta_k > 0$ or $g_i < Y_i$.

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References

[1] Kugler, T. and Szidarovszky, F., *Modeling inter-group competitions: A game theoretical analysis of between and within group conflicts*, submitted for publication, 2008.

[2] Okuguchi, K., *Expectations and stability in oligopoly models*, Springer-Verlag, Berlin/Heidelberg/New York, 1976.

[3] Okuguchi, K. and Szidarovszky, F., *The theory of oligopoly with multi-product firms*, 2nd edn, Springer-Verlag, Berlin/Heidelberg/New York, 1999.

[4] Rapoport, A. and Amaldoss, W., *Social dilemmas embedded in between-group competitions: Effects of contest and distribution rules*, in M. Foddy, M. Smithson, S. Schneider and M. Hogg, eds *'Resolving social dilemmas'*, Psychology Press, Philadelphia.