CUBO A Mathematical Journal
Vol.11, $N^{\underline{o}}$03, (65–77). August 2009

A Localized Heat Source Undergoing Periodic Motion: Analysis of Blow-Up and a Numerical Solution

C.M. Kirk
Department of Mathematics, California Polytechnic State University, San Luis Obispo, Ca 93407
email: `ckirk@calpoly.edu`

ABSTRACT

A localized heat source moves with simple periodic motion along a one-dimensional reactive-diffusive medium. Blow-up will occur regardless of the amplitude or frequency of motion. Numerical results suggest that blow-up is delayed by increasing the amplitude or by increasing the frequency of motion. A brief survey is presented of the literature concerning numerical studies of nonlinear Volterra integral equations with weakly singular kernels that exhibit blow-up solutions.

RESUMEN

Una fuente de calor localizada se mueve con un movimiento periódico simple a lo largo de un medio reactivo-difuso unidimensional. “Blow-up” ocurrirá considerando la amplitud o frecuencia del movimiento. Resultados numéricos sugieren que el “Blow-up” es retardado por aumento de amplitud o frecuencia de movimiento. Un breve informe es presentado de la literatura al respecto de estudios numéricos de ecuaciones integrales de Volterra no lineales con nucleo débilmente singular que exhiben soluciones “Blow-up”.

Key words and phrases: *Moving heat source, blow-up, integral equations, numerical simulation.*

Math. Subj. Class.: *45D05, 45M99, 35K60, 65-04, 65R20.*

Introduction

A number of studies have addressed blow-up in a reactive-diffusive medium due to a localized heat source. Generally this problem is modeled by a parabolic partial differential equation (PDE) with a nonlinear source term. The highly localized nature of the heat source can be represented by a Dirac delta function. See [30] and [31] for surveys of this literature. The general model often takes the form:

$$x^q T_t(x,t) - T_{xx}(x,t) = \delta(x-x_0)F[T(x,t)] \quad \text{for} \quad 0<t \quad \text{and} \quad x \in D \qquad (1)$$
$$T(x,0) = T_0(x) \quad for \quad x \in D$$

with appropriate boundary conditions on D. Usually T_0 is non-negative and continuous. In most studies, $q = 0$. [10] and [12] address degenerate problems with $q \neq 0$. Variations of (1) include systems of equations ([20], [26], [23]), higher dimensions ([11], [19]), and problems that include non-local features ([27], [12]). The model can also include motion of sources ([25], [18], [19], [20]) and sources of varying size and shape [19]. The delta function $\delta(x - x_0)$ reflects the intense localization. Some studies have allowed for less intense, perhaps more realistic, localization [19]. One question of interest is whether or not the solution undergoes blow-up in finite time.

In this article we will examine a localized heat source that moves with simple periodic motion ($x_0(t) = A\cos(\omega t)$) along a one-dimensional reactive-diffusive medium. Blow-up will occur regardless of the amplitude and frequency of motion. This particular motion is suggested by [18] which addresses various types of motion in one-dimension. This problem is initially modeled as a nonlinear parabolic PDE. The analysis is carried out by converting to the corresponding Volterra integral equation (VIE). We develop a numerical method to model the solution. The results of the numerical study suggest that blow-up is delayed either by increasing the amplitude or by increasing the frequency of motion. Intuitively this makes sense since increasing the amplitude allows the heat source to oscillate over a wider spatial domain, allowing the heat more time to dissipate as the source moves into cooler surroundings. Increasing the frequency causes the source to move more quickly, generally allowing the heat less opportunity to accumulate. Furthermore the numerical results agree with the analytical results that can be obtained for this specific kind of motion.

Numerical modeling of nonlinear VIEs with blow-up solutions can be a difficult problem. Little research has been done in this area. Specifically, rigorous numerical analysis of such schemes is essentially non-existent. A brief survey of the literature regarding numerical studies of nonlinear VIEs with weakly singular kernels that exhibit blow-up solutions is presented in the last section of this paper.

Theory

Consider the following nonlinear PDE:

$$T_t(x,t) - T_{xx}(x,t) = \delta(x-x_0)F[T(x,t)] \quad \text{for } 0<t \text{ and } \ x \in (-\infty,\infty)$$
$$T(x,0) = T_0(x) \qquad \text{for} \qquad x \in (-\infty,\infty)$$
$$T(x,t) \to 0 \text{ as } |x| \to \infty$$

This problem models the heating of a reactive-diffusive material by a highly-localized source. The delta function reflects the strong localization of the heat source. T_0 is non-negative, continuous, and approaches 0 as $|x| \to \infty$.

Here we present the conversion from the PDE to the corresponding integral equation. First apply the appropriate one-dimensional free space Green's function.

$$T(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x,t|\xi,s)\delta(\xi - x_0)F[T(x_0,s)]d\xi ds$$
$$+ \int_{-\infty}^{\infty} G(x,t|\xi,0)T_0(\xi)d\xi, \qquad -\infty < x < \infty, t \geq 0$$

Let $x = x_0$ and utilize the delta function property:

$$T(x_0,t) = \int_0^t G(x_0,t|x_0,s)F[T(x_0,s)]ds$$
$$+ \int_{-\infty}^{\infty} G(x_0,t|\xi,0)T_0(\xi)d\xi, \qquad -\infty < x < \infty, t \geq 0$$

Define $u(t) \equiv T(x_0,t)$ and $h(t) \equiv \int_{-\infty}^{\infty} G(x_0,t|\xi,0)T_0(\xi)d\xi$ and

$k(t,s) \equiv G(x_0,t|x_0,s) = \frac{H(t-s)}{2\sqrt{\pi(t-s)}} \exp\left[\frac{-(x_0(t)-x_0(s))^2}{4(t-s)}\right]$ so that:

$$u(t) = h(t) + \int_0^t k(t,s)F(u(s))ds \tag{2}$$

Of interest is whether or not (2) has a blow-up solution. A solution $u(t)$ is a blow-up solution if $u(t) \to \infty$ as $t \to \widehat{t} < \infty$. The following theorem is used to prove the existence of a unique, continuous solution to (2) up to some lower bound time t^*. Various versions of this theorem appear in [32], [24], [18] and [19].

Theorem 1. *(Existence Theorem): Equation (2) has a unique, continuous solution for* $0 \leq t < t^*$, *where* $t^* < \infty$ *can be determined as the smallest root of* $I(t^*) = \Lambda \equiv \max_{0 \leq M < \infty} \left[\frac{M}{F(M+\overline{h})}\right]$ *and* $t^* = \infty$ *if* $I(t) < \Lambda$ *for all* $t \geq 0$.

The proof of this theorem involves the creation of a mapping from the space of continuous functions $u(t)$ that satisfy:

$$0 \leq u(t) \leq M < \infty, 0 \leq t \leq t_1$$

where M is the smallest root of $\frac{M}{F(M+h)} = \frac{1}{F'(M)}$.The mapping is defined as the integral operator K where:

$$K[u(t)] = h(t) + \int_0^t k(t,s)F[u(s)]ds$$

It can be shown that K is a contraction mapping (with the sup norm) from the space into itself if certain conditions are satisfied. Then by the Contraction Mapping Theorem, a unique continuous solution exists. Those required conditions lead to the lower bound on the blow-up time.

The following theorem provides an upper bound on the blow-up time.

Theorem 2. *(Non-existence Theorem): Let $k(t,s)$ be a non-increasing function in t. Whenever there exists a $t^{**} < \infty$ such that $I(t^{**}) = \kappa \equiv \int_h^\infty \frac{dz}{F(z)}$, it follows that (2) can not have a continuous solution for $t \geq t^{**}$.*

A contradiction argument is used to prove non-existence of the solution. This method exploits the non-increasing nature of the typical kernel. The theorem must be modified if the kernel does not have this property. Various versions of this theorem appear in [32], [24], [18] and [19]. The bounds on the blow-up time can then be summarized:

$$\text{If } u(t) \to \infty \text{ as } t \to \widehat{t} < \infty, \text{ then } t^* < \widehat{t} < t^{**}.$$

Now consider a moving source: $x_0 = x_0(t)$ in one-dimension along an infinite rod. Assume $x_0(t)$ is continuously differentiable. This is a reasonable assumption since $x_0(t)$ is a position function. Then:

$$u(t) = h(t) + \int_0^t \frac{1}{2\sqrt{\pi(t-s)}} \exp\left(-\frac{(x_0(t)-x_0(s))^2}{4(t-s)}\right) F[u(s)]ds$$

Specifically we consider the special case of simple periodic motion as suggested by [18]:

$$x_0(t) = A\cos(\omega t), A > 0, \omega > 0.$$

Let $F(z) = e^z$ and $h(t) = 0$ so that

$$u(t) = \int_0^t \frac{1}{2\sqrt{\pi(t-s)}} \exp\left(-\frac{(A\cos(\omega t) - A\cos(\omega s))^2}{4(t-s)}\right) e^{u(s)}ds \tag{3}$$

For this motion, blow-up will always occur. To see this, apply a theorem from [18]:

Theorem 3. *Let $x_0(t)$ be bounded and Lipschitz continuous, with constants $x^{**} > 0$, $v^{**} > 0$ such that $|x_0(t) - x_0(t')| \le v^{**}|t - t'|$, $0 \le t < \infty$, $0 \le t' < \infty$. Then a continuous solution of (3) can not exist for $t \ge t^{**} = \pi\kappa^2 \exp(x^{**}v^{**})$.*

Note, of course, that in our example

$$|x_0(t)| \le A,\ |x_0(t) - x_0\left(t'\right)| \le A\omega\left|t - t'\right|,\ 0 \le t < \infty,\ 0 \le t' < \infty$$

Hence Theorem 3 guarantees that blow-up will necessarily occur. The upper bound is obtained directly from Theorem 3: $t^{**} = \pi\kappa^2 e^{A^2\omega}$. Recall:

$\kappa \equiv \int\limits_{\underline{h}}^{\infty} \frac{dz}{F(z)}$, which for this example becomes: $\kappa = 1$. So then $t^{**} = \pi e^{A^2\omega}$. Now consider the lower bound on the blow-up time. A comparison kernel can be introduced in order to modify Theorem 1 for problems such as this one for which $I(t)$ is difficult to obtain. (Note that a comparison kernel was also needed to prove Theorem 3 since the kernel in this problem is not necessarily non-increasing.) Apply the following theorem from [18]:

Theorem 4. *Let $k(t,s) \le \overline{k}(t,s)$, $0 \le s < t < \infty$, where $\overline{k}(t,s)$ is continuous for $0 \le s < t$ and integrable as $s \to t$. Then Theorem 1 holds with $I(t)$ replaced by $\overline{I}(t)$ where*

$$\overline{I}(t) \equiv \int_0^t \overline{k}(t,s)ds,\ 0 \le t < \infty$$

The appropriate comparison kernel in this case is

$$\overline{k}(t,s) = \frac{1}{2\sqrt{\pi(t-s)}}$$

Since $\overline{h} = 0$ and $F(z) = e^z$, we have $\Lambda = \frac{1}{e}$. The lower bound on the blow-up time is obtained from $\overline{I}(t^*) = \frac{1}{e}$. So $t^* = \frac{\pi}{e^2}$. Together the bounds on the blow-up time are:

$$\frac{\pi}{e^2} < \widehat{t} < \pi e^{A^2\omega}$$

Numerical Method

The numerical computation of the solution to problem (3) is based on a standard product quadrature approach. Discretize the interval of integration $(0,t)$ into subintervals: $(t_i, t_{i+1}), i = 1...n$. $u_i = u(t_i)$ is the approximation of $u(t)$ at $t = t_i$. The nonlinear portion of the integrand is represented by the canonical Lagrange functions. We have $\exp(u(s)) \approx \frac{s-t_{i+1}}{t_i-t_{i+1}}e^{u(t_i)} + \frac{s-t_i}{t_{i+1}-t_i}e^{u(t_{i+1})}$ for $s \in (t_i, t_{i+1})$. The typical integration rule for u_{n+1} can then be written:

$$
\begin{aligned}
u_{n+1} &= \frac{1}{2\sqrt{\pi}}\sum_{i=1}^{n}\int_{t_i}^{t_{i+1}}\frac{1}{\sqrt{t_{n+1}-s}}\exp\left\{\frac{-A^2[\cos(\omega t_{n+1})-\cos(\omega s)]^2}{4(t_{n+1}-s)}\right\}\exp(u(s))ds \\
&= \frac{1}{2\sqrt{\pi}}\sum_{i=1}^{n}\frac{e^{u(t_i)}}{t_i-t_{i+1}}\int_{t_i}^{t_{i+1}}\frac{s-t_{i+1}}{\sqrt{t_{n+1}-s}}\exp\left\{\frac{-A^2[\cos(\omega t_{n+1})-\cos(\omega s)]^2}{4(t_{n+1}-s)}\right\}ds \\
&+\frac{1}{2\sqrt{\pi}}\sum_{i=1}^{n}\frac{e^{u(t_{i+1})}}{t_{i+1}-t_i}\int_{t_i}^{t_{i+1}}\frac{s-t_i}{\sqrt{t_{n+1}-s}}\exp\left\{\frac{-A^2[\cos(\omega t_{n+1})-\cos(\omega s)]^2}{4(t_{n+1}-s)}\right\}ds \qquad (4)
\end{aligned}
$$

for the approximation of the solution at time $t_{n+1.}$ To address the singular behavior when $i = n$, apply the Mean Value Theorem to the term $\cos(\omega t) - \cos(\omega s)$ to obtain:

$$\exp\left\{\frac{-A^2[\cos(\omega t)-\cos(\omega s)]^2}{4(t-s)}\right\} = \exp\left\{\frac{-A^2\omega^2[\sin(\omega\widetilde{t})(t-s)]^2}{4(t-s)}\right\} = \exp\left\{\frac{-A^2\omega^2[\sin(\omega\widetilde{t})]^2}{4}(t-s)\right\}$$

where $s < \widetilde{t} < t$. Then choose a small value $0 < \Delta M_n << (t_n, t_{n+1})$. Consider just the integral from the last term in equation (4) to illustrate this step for $i = n$:

$$
\begin{aligned}
&\int_{t_n}^{t_{n+1}}\frac{s-t_n}{\sqrt{t_{n+1}-s}}\exp\left\{\frac{-A^2[\cos(\omega t_{n+1})-\cos(\omega s)]^2}{4(t_{n+1}-s)}\right\}ds \\
&= \int_{t_n}^{t_{n+1}-\Delta M_n}\frac{s-t_n}{\sqrt{t_{n+1}-s}}\exp\left\{\frac{-A^2[\cos(\omega t_{n+1})-\cos(\omega s)]^2}{4(t_{n+1}-s)}\right\}ds \\
&+ \int_{t_{n+1}-\Delta M_n}^{t_{n+1}}\frac{s-t_n}{\sqrt{t_{n+1}-s}}\exp\left\{\frac{-A^2[\cos(\omega t_{n+1})-\cos(\omega s)]^2}{4(t_{n+1}-s)}\right\}ds
\end{aligned}
$$

Evaluate analytically the integral for $(t_{n+1} - \Delta M_n, t_{n+1})$:

$$\int_{t_{n+1}-\Delta M_n}^{t_{n+1}}\frac{s-t_n}{\sqrt{t_{n+1}-s}}\exp\left\{\frac{-A^2[\cos(\omega t_{n+1})-\cos(\omega s)]^2}{4(t_{n+1}-s)}\right\}ds$$

$$= \int_{t_{n+1}-\Delta M_n}^{t_{n+1}}\frac{s-t_n}{\sqrt{t_{n+1}-s}}\exp\left\{\frac{-A^2\omega^2[\sin(\omega\widetilde{t})]^2}{4}(t_{n+1}-s)\right\}ds$$

$$= \frac{1}{R^{5/2}}[\sqrt{\Delta M_n}R^{3/2}\exp(-R\Delta M_n) + R\sqrt{\pi}\,\mathrm{erf}(\sqrt{R}\sqrt{\Delta M_n})\left\{R(t_{n+1}-t_n)-\tfrac{1}{2}\right\}]$$

where $R \equiv \frac{A^2\omega^2\sin^2\widetilde{t}}{4}$and $t_{n+1} - \Delta M_n < \widetilde{t} < t_{n+1}$. Then the governing equation becomes:

$$u_{n+1} = \frac{1}{2\sqrt{\pi}} \sum_{i=1}^{n} [\frac{e^{u(t_i)}}{t_i - t_{i+1}} \int_{t_i}^{t_{i+1}} \frac{s - t_{i+1}}{\sqrt{t_{n+1} - s}} \exp\left\{ \frac{-A^2[\cos(\omega t_{n+1}) - \cos(\omega s)]^2}{4(t_{n+1} - s)} \right\} ds]$$

$$+ \frac{1}{2\sqrt{\pi}} \sum_{i=1}^{n} [\frac{e^{u(t_{i+1})}}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} \frac{s - t_i}{\sqrt{t_{n+1} - s}} \exp\left\{ \frac{-A^2[\cos(\omega t_{n+1}) - \cos(\omega s)]^2}{4(t_{n+1} - s)} \right\} ds]$$

$$= \frac{1}{2\sqrt{\pi}} \sum_{i=1}^{n-1} [\frac{e^{u(t_i)}}{t_i - t_{i+1}} \int_{t_i}^{t_{i+1}} \frac{s - t_{i+1}}{\sqrt{t_{n+1} - s}} \exp\left\{ \frac{-A^2[\cos(\omega t_{n+1}) - \cos(\omega s)]^2}{4(t_{n+1} - s)} \right\} ds]$$

$$+ \frac{1}{2\sqrt{\pi}} \frac{e^{u(t_n)}}{t_n - t_{n+1}} \int_{t_n}^{t_{n+1}} \frac{s - t_{n+1}}{\sqrt{t_{n+1} - s}} \exp\left\{ \frac{-A^2[\cos(\omega t_{n+1}) - \cos(\omega s)]^2}{4(t_{n+1} - s)} \right\} ds$$

$$+ \frac{1}{2\sqrt{\pi}} \sum_{i=1}^{n-1} [\frac{e^{u(t_{i+1})}}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} \frac{s - t_i}{\sqrt{t_{n+1} - s}} \exp\left\{ \frac{-A^2[\cos(\omega t_{n+1}) - \cos(\omega s)]^2}{4(t_{n+1} - s)} \right\} ds]$$

$$+ \frac{1}{2\sqrt{\pi}} \frac{e^{u(t_{n+1})}}{t_{n+1} - t_i} \int_{t_n}^{t_{n+1}} \frac{s - t_n}{\sqrt{t_{n+1} - s}} \exp\left\{ \frac{-A^2[\cos(\omega t_{n+1}) - \cos(\omega s)]^2}{4(t_{n+1} - s)} \right\} ds$$

$$u_{n+1} = \frac{1}{2\sqrt{\pi}} \sum_{i=1}^{n-1} [\frac{e^{u(t_i)}}{t_i - t_{i+1}} \int_{t_i}^{t_{i+1}} \frac{s - t_{i+1}}{\sqrt{t_{n+1} - s}} \exp\left\{ \frac{-A^2[\cos(\omega t_{n+1}) - \cos(\omega s)]^2}{4(t_{n+1} - s)} \right\} ds]$$

$$+ \frac{1}{2\sqrt{\pi}} \frac{e^{u(t_n)}}{t_n - t_{n+1}} \int_{t_n}^{t_{n+1} - \Delta M_n} \frac{s - t_{n+1}}{\sqrt{t_{n+1} - s}} \exp\left\{ \frac{-A^2[\cos(\omega t_{n+1}) - \cos(\omega s)]^2}{4(t_{n+1} - s)} \right\} ds$$

$$+ \frac{1}{2\sqrt{\pi}} \frac{e^{u(t_n)}}{t_n - t_{n+1}} \frac{1}{R^{5/2}} [\sqrt{\Delta M_n} R^{3/2} \exp(-R\Delta M_n) + -\frac{R}{2}\sqrt{\pi}\,\mathrm{erf}(\sqrt{R}\sqrt{\Delta M_n})]$$

$$+ \frac{1}{2\sqrt{\pi}} \sum_{i=1}^{n-1} [\frac{e^{u(t_{i+1})}}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} \frac{s - t_i}{\sqrt{t_{n+1} - s}} \exp\left\{ \frac{-A^2[\cos(\omega t_{n+1}) - \cos(\omega s)]^2}{4(t_{n+1} - s)} \right\} ds]$$

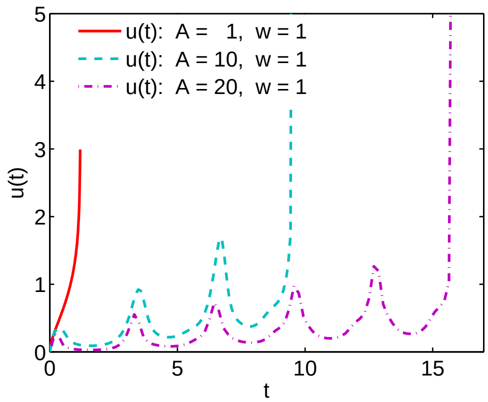
$$+ \frac{1}{2\sqrt{\pi}} \frac{e^{u(t_{n+1})}}{t_{n+1} - t_n} \int_{t_n}^{t_{n+1} - \Delta M_n} \frac{s - t_n}{\sqrt{t_{n+1} - s}} \exp\left\{ \frac{-A^2[\cos(\omega t_{n+1}) - \cos(\omega s)]^2}{4(t_{n+1} - s)} \right\} ds$$

$$+ \frac{1}{2\sqrt{\pi}} \frac{e^{u(t_{n+1})}}{t_{n+1} - t_n} \frac{1}{R^{5/2}} [\sqrt{\Delta M_n} R^{3/2} \exp(-R\Delta M_n) + R\sqrt{\pi}\,\mathrm{erf}(\sqrt{R}\sqrt{\Delta M_n}) \left\{ R(t_{n+1} - t_n) - \tfrac{1}{2} \right\}]$$

where $R \equiv \frac{A^2\omega^2 \sin^2 \widetilde{t}}{4}$ and $t_{n+1} - \Delta M_n < \widetilde{t} < t_{n+1}$. Newton's method is then used to solve the implicit nonlinear equations that arise.

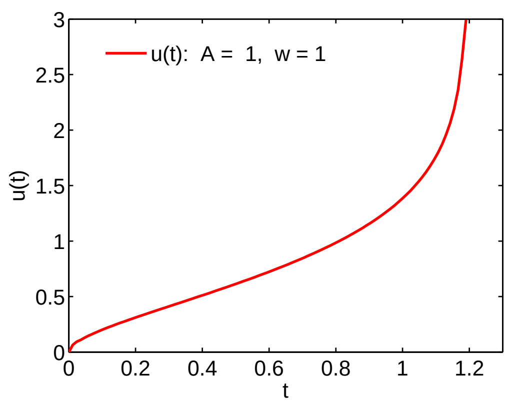
The results of the numerical calculations follow here. For these calculations, $n = 100$ and $\Delta M_n = \frac{1}{100}(\frac{T}{n}) = \frac{T}{10{,}000}$ where T is the stopping time. The numerical code is run in MatLab. The intervals (t_i, t_{i+1}) are chosen to be uniform. First we compare various amplitude values. In Plot 1, we keep frequency fixed at $\omega = 1$ and plot results for $A = 1$, $A = 10$ and $A = 20$. Note that the blow-up time seems to increase from about $\widehat{t} \approx 1.2$ to $\widehat{t} \approx 9.5$ to $\widehat{t} \approx 16$ as the amplitude increases. Intuitively this makes sense since increasing the amplitude allows the heat source to oscillate over a wider spatial domain, allowing the heat more time and space to dissipate as the source moves into cooler surroundings. Note the oscillatory behavior for $A = 10$ and $A = 20$. This behavior is expected since the oscillating source moves more quickly near the center of its path of motion and moves more slowly near the extremes of its path. Hence one would expect the medium (where the source is located) to heat up less readily when the source is near the center of the path and to heat up more readily when the source is near the extremes of the path.

Figure 1:



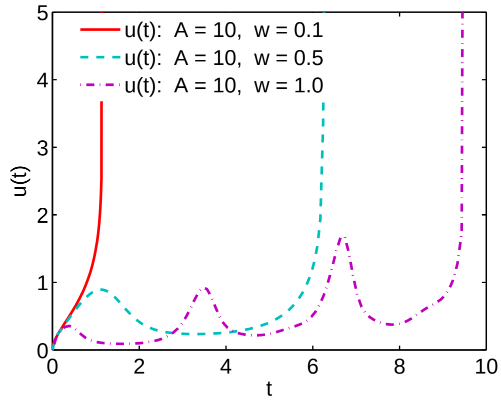
We do not see this oscillatory behavior when $A = 1$. This is because blow-up occurs very quickly, before the source changes direction even once. See Plot 2 for more detailed pre-blow-up behavior for $A = 1$.

Figure 2:



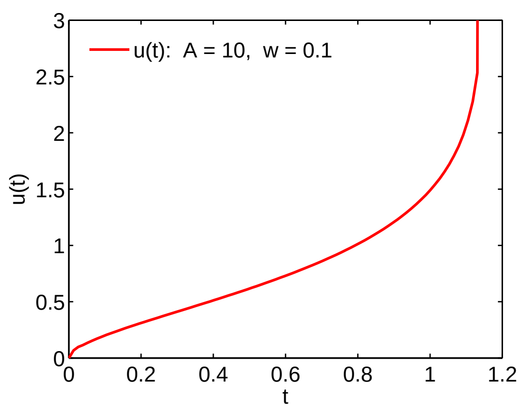
Now vary the frequency ω while keeping the amplitude $A = 10$ fixed. In Plot 3, we show results for $\omega = 0.1$, $\omega = 0.5$ and $\omega = 1$. As frequency increases, the blow-up time seems to increase from about $\widehat{t} \approx 1.1$ to $\widehat{t} \approx 6.2$ to $\widehat{t} \approx 9.5$. This behavior is consistent with intuition. Increasing the frequency causes the source to move more quickly, generally allowing the heat less opportunity to accumulate.

Figure 3:



If the frequency is small enough, the source may move so slowly that the heat accumulates and blow-up occurs before the source changes direction even once. Indeed this seems to be the case with $\omega = 0.1$, where blow-up occurs well before $t = \frac{\pi}{0.1}$ and no oscillatory behavior is seen. See Plot 4. When $\omega = 1$, the source completes an oscillation before blow-up occurs.

Figure 4:



Furthermore these numerical results agree with the analytical results that can be obtained for this specific kind of motion. For example, consider the case $\omega = 1$, $A = 1$, with $\widehat{t} \approx 1.2$ in Plot 2. The numeric blow-up time lies within the analytical bounds: $0.425 < \widehat{t} < 8.539$. These bounds, however, are fairly crude. Also note that the numerical solution remains bounded: $u(t) < M = 1$ for $t < 0.425$, as required by another analytical result we obtained in the Theory section of this paper.

Numerical Approaches to Nonlinear VIEs with Blow-Up Solutions

We have not yet carried out convergence or error analysis of our numerical scheme. In fact, the numerical modeling of nonlinear Volterra integral equations with blow-up solutions can be a difficult problem. Little research has been done in this area. Specifically, rigorous numerical analysis of such schemes is essentially non-existent. Here we present a brief review of the literature regarding numerical studies of nonlinear VIEs of the second kind with weakly singular kernels that exhibit blow-up solutions.

There is a solid body of research on the numerical analysis of nonlinear VIEs of the second kind with weakly singular kernels that *do not* exhibit blow-up solutions. See [4], [5], [6], [7], [1], [16]. These papers provide very good summaries of the existing literature. They also describe in detail some of the important techniques used to address these problems. One successful approach involves collocation solutions, with approximations in the space of piecewise polynomials. Appropriate quadrature formulas are usually used to approximate the integrals involved. Generally large nonlinear systems must then be solved. Special issues arise with the introduction of weakly singular kernels. For example, the order of convergence is reduced near the singularity when polynomial splines and uniform meshes are used. These issues can often be successfully addressed using graded meshes which are finer near the region of the singularity; by carrying out a variable transformation; or by using other basis functions which more closely reflect the nature of the solution near the singularity. See [5], [9], [8], [28], [17]. The convergence, superconvergence, and stability properties of these collocation solutions are then studied. In [5], H. Brunner includes extensive details of these methods. He also lists many references for these methods, as well as for other methods used to address these problems.

Research on numerical methods for nonlinear VIEs of the second kind with weakly singular kernels that *do* exhibit blow-up in finite time is still at the early stages. The development of computational approaches and rigorous analysis of these approaches is limited. See [2] and [3] and [5]. Despite this, researchers have attempted to compute solutions numerically to various interesting applied problems. Without rigorous analysis of errors and convergence, however, these authors often rely upon available analytical information to help verify their numerical results.

It can be useful to employ knowledge of the asymptotic behavior of the solution near the time of blow-up. If the solution is known to exhibit certain asymptotic behavior near blow-up, then the numerical solution can be tracked until it reflects this behavior, suggesting the onset of blow-up. For example, in [21], D. G. Lasseigne and W. E. Olmstead analyze the ignition of a combustible solid due to excessive reactant consumption. An integral equation is obtained that models the perturbation of temperature in the reaction zone. Before carrying out the numerical analysis, an asymptotic analysis is performed on the governing integral equation. Then the behavior of the evolving numerical solution is compared to the expected asymptotic solution form. Eventually, the numerical solution matches the known behavior pattern. This provides extra assurance that thermal runaway is indeed occurring. L. R. Ritter, W. E. Olmstead and V. A. Volpert follow a similar approach in [29] to model numerically the initiation of a polymerization wave. Their goal is to understand how various parameters in a frontal polymerization process determine whether the onset of a wave front will be initiated or inhibited.

In [10] and [12], C. Y. Chan and H. Y. Tian employ computational methods to estimate blow-up time. In [10] they examine a degenerate parabolic problem with a nonlinear source in a one-dimensional material of finite length. They obtain analytical expressions for the bounds on the blow-up time as functions of the length of the domain. Then they use an iterative approach to approximate the solution over the discretized spatial domain at the estimated blow-up time. This estimated blow-up time is then refined, shifted upward or downward, depending on whether the growth of the iterated solution lies within or exceeds certain tolerances. They provide numerical results indicating the blow-up time is a decreasing function of the length of the domain, as expected. These same authors apply a similar technique in [12] to a problem with a nonlinear source of local and nonlocal features. Again they use an iterative process to refine the analytical bounds.

Note that in none of these studies is a rigorous numerical analysis carried out. The authors instead acquire confidence in their numerical solutions partly by making judicious use of known analytical results. Clearly some understanding of the analytical solution is important in choosing the best numerical approach and in acquiring confidence in the numerical results.

H. Brunner proposes in [5] another possible approach to these problems using collocation methods. He suggests the possibility of obtaining two different collocation solutions v_h and u_h generated from different sets of collocation points such that the corresponding iterated numerical solutions satisfy: $v_h^{it} \leq u(t) \leq u_h^{it}$ for a chosen mesh I_h.

Furthermore, if the VIE of interest stems from an originating PDE, one might instead examine the original PDE for blow-up behavior. The research into these corresponding PDEs is more advanced than that of analogous VIEs. See [2] and [3] and [5] for surveys of these studies.

Received: July 28, 2008. Revised: September 5, 2008.

References

[1] Baker C., *A perspective on the numerical treatment of Volterra equations*, Numerical analysis 2000, Vol. VI, Ordinary differential equations and integral equations. J. Comput. Appl. Math., **125**(2000), 217–249.

[2] Bandle, C. and Brunner, H., *Numerical analysis of semilinear parabolic problems with blow-up solutions*, Rev. Real Acad. Cienc. Exact. Fís. Natur. Madrid, **88**(1994), 203–222.

[3] Bandle, C. and Brunner, H., *Blow-up in diffusion equations: A survey*, J. Comp. Appl. Math., **97**(1998), 3–22.

[4] Blom, J.G. and Brunner, H., *The numerical solution of nonlinear Volterra integral equations of the second kind by collocation and iterated collocation methods*, SIAM J. Sci. Statist. Comput., **8**(1987), 806–830.

[5] Brunner, H., *Collocation methods for Volterra integral and related functional differential equations*, Cambridge University Press, 2004.

[6] Brunner, H., *The numerical analysis of functional integral and integro-differential equations of Volterra type*, Acta Numer., **13**(2004), 55–145.

[7] Brunner, H., *The discretization of Volterra functional integral equations with proportional delays*, Differences and differential equations, 3–27, Fields Inst.Commun., **42**, Amer. Math. Soc., Providence, RI, 2004.

[8] Brunner, H., Pedas, A. and Vainikko, G., *The piecewise polynomial collocation method for nonlinear weakly singular Volterra equations*, Math. Comp. **68**(1999), 1079–1095.

[9] Brunner, H. and van der Houwen, P.J., *The numerical solution of Volterra equations*, CWI Monographs **3**, North Holland, 1986.

[10] Chan, C.Y. and Tian, H.Y., *Single-point blow-up for a degenerate parabolic problem due to a concentrated nonlinear source*, Quart. Appl. Math. **61**(2003), 363–385.

[11] Chan, C.Y. and Tian, H.Y., *Multi-dimensional explosion due to a concentrated nonlinear source*, J. Math. Anal. Appl., **295**(2004), 174–190.

[12] Chan, C.Y. and Tian, H.Y., *Single-point blow-up for a degenerate parabolic problem with a nonlinear source of local and nonlocal features*, Appl. Math. Comput., **145**(2003), 371–390.

[13] Chan, C.Y. and Carrillo Escobar, J.C., *Blow-up of the solution for a singular semilinear parabolic problem due to a concentrated nonlinear source*, Proceedings of Dynamic Systems and Applications, **5**, 2008.

[14] Chan, C.Y. and Boonklurb, R., *A blow-up criterion for a degenerate parabolic problem due to a concentrated nonlinear source*, Quart. Appl. Math., **65**(2007), 781–787.

[15] Diogo, T., Franco, N.B. and Lima, P., *High order product integration methods for a Volterra integral equation with logarithmic singular kernel*, Commun. Pure Appl. Anal., **3**(2004), 217–235.

[16] Diogo, T. and Lima, P., *Collocation Solutions of a Weakly Singular Volterra Integral Equation*, TEMA Tend. Mat. Apl. Comput., **8**(2007), 229–238.

[17] Diogo, T., Mckee, S., and Tang, T., *Collocation methods for second-kind Volterra integral equations with weakly singular kernels*, Proc. Roy. soc. Edinburgh, **124A**(1994), 199–201.

[18] Kirk, C.M. and Olmstead, W.E., *Blow-up in a reactive-diffusive medium with a moving heat source*, Z. Angew. Math. Phys., **53**(2002), 147–159.

[19] Kirk, C.M. and Olmstead, W.E., *Blow-up solutions of the two-dimensional heat equation due to a localized moving source*, Anal. Appl., **3**(2005), 1–16.

[20] Kirk, C.M. and Olmstead, W.E., *The influence of two moving heat sources on blow-up in a reactive-diffusive medium*, Z. Angew. Math. Phys., **51**(2000), 1–16.

[21] Glenn Lasseigne, D. and Olmstead, W.E, *Ignition of a Combustible Solid with Reactant Consumption*, SIAM J. Appl. Math., **47**(1987), 332–342.

[22] Levine, H.A., *The role of critical exponents in the blow-up theorems*, SIAM Rev., **32**(1990), 262–288.

[23] Mydlarczyk, W., Okrasi'nski, W. and Roberts, C.A., *Blow-up solutions to a system of nonlinear Volterra equations*, J. Math. Anal. Appl., **301**(2005), 208–218.

[24] Olmstead, W.E. and Roberts, C.A., *Explosion in a diffusive strip due to a concentrated nonlinear source*, Methods Appl. Anal., **1**(1994), 435–445.

[25] Olmstead, W.E., *Critical speed for the avoidance of blow-up in a reactive-diffusive medium*, Z. angew. Math. Phys., **48**(1997), 701–710.

[26] Olmstead, W.E., Roberts, C.A. and Deng, K., *Coupled Volterra equations with blow-up solutions*, J. Integral Equations Appl., **7**(1995), 499–516.

[27] Olmstead, W.E. and Roberts, C.A., *Explosion in a diffusive strip due to a source with local and nonlocal features*, Methods Appl. Analysis, **3**(1996), 345–357.

[28] Pedas, A. and Vainikko, G., *Smoothing transformation and piecewise polynomial collocation for weakly singlular Volterra integral equations*, Computing, **73**(2004), 271–293.

[29] Ritter, L.R., Olmstead, W.E. and Volpert, V.A., *Initiation of free radical polymerization waves*, SIAM J. Appl. Math., **63**(2003), 1831–1848.

[30] Roberts, C.A., *Recent results on blow-up and quenching for nonlinear Volterra equations*, J. Comput. Appl. Math., **205**(2007), 736–743.

[31] Roberts, C.A., *Analysis of explosion for nonlinear Volterra integral equations*, J. Comp. Appl. Math., **97**(1998), 153–166.

[32] Roberts, C.A., Lasseigne, D.G. and Olmstead, W.E, *Volterra equations which model explosion in a diffusive medium*, J. Integral. Eqn. Appl., **5**(1993), 531–546.

[33] Souplet, P., *Blow-up in nonlocal reaction-diffusion equations*, SIAM J. Math. Anal., **29**(1998), 1301–1334.

[34] Stuart, A.M. and Floater, M,S., *On the computation of blow-up*, Europ. J. Appl. Math., **1**(1990), 47–71.