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Uniformly Continuous L^1 Solutions of Volterra Equations and Global Asymptotic Stability

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ABSTRACT

The scalar linear Volterra integro-differential equation

$$x'(t) = -a(t)x(t) + \int_0^t b(t,s)x(s)\,ds \tag{E}$$

is investigated, where a and b are continuous functions. Liapunov functionals are constructed in order to obtain sufficient conditions so that solutions of (E) are absolutely Riemann integrable on $[0, \infty)$ and have bounded derivatives. Then some of these conditions are replaced with less stringent ones while others are eliminated altogether. Under the new conditions, it is shown that one of the Liapunov functionals is uniformly continuous which in turn implies that solutions of (E) are uniformly continuous. We then employ Barbălat's lemma to prove the zero solution of (E) is stable and that all solutions of (E) approach zero as $t \to \infty$. Examples illustrated with numerical solutions are provided.

RESUMEN

Investigamos la siguiente ecuación linear integro-diferencial de Volterra

$$x'(t) = -a(t)x(t) + \int_0^t b(t,s)x(s)\,ds \tag{E}$$

donde a y b son funciones continuas. Funcionales de Liapunov son construidos para obtener condiciones suficientes tal que las soluciones de (E) son absolutamente Riemann integrables sobre el intervalo $[0,\infty)$ y tienen derivadas acotadas. Algumas de estas condiciones son reemplazadas por otras menos rigurosas mientras que otras son eliminadas por completo. Bajo las nuevas condiciones, se demuestra que uno de los funcionales de Liapunov es uniformemente continuo, a su vez, esto implica que las soluciones de (E) son uniformemente continuas. Usamos el lema de Barbălat para provar que la solución nula de (E) es estable y que todas las soluciones de (E) se aproximam a cero cuando $t \to \infty$. Son presentados ejemplos con soluciones numéricas.

Key words and phrases: *Asymptotic stability, Barbălat's lemma, Liapunov functionals, strongly positive definite functionals, uniformly continuous solutions, variation of parameters, Volterra equations.*

Math. Subj. Class.: *45J05, 45M10, 34K20, 45D05.*

1 Introduction

We investigate the asymptotic behavior of solutions of the scalar linear homogeneous Volterra integro-differential equation

$$x'(t) = -a(t)x(t) + \int_0^t b(t,s)x(s)\,ds \tag{1.1}$$

for $t \geq 0$, where a and b are real-valued functions that are continuous on the respective domains $[0,\infty)$ and $\Omega := \{(t,s)\colon 0 \leq s \leq t < \infty\}$. In this setting, a solution of (1.1) satisfying a given initial condition exists on the entire interval $[0,\infty)$ and is unique (cf. Section 2 for more details). Employing Liapunov functionals that were constructed for (1.1) by Burton in [8, p. 122] and [9] and by Becker in [5, p. 34], with some modifications, we obtain a number of conditions involving a and b so that the zero solution of (1.1) is stable and its other solutions approach zero as $t \to \infty$. Typically, one looks for conditions so that the derivative $x'(t)$ is bounded on $[0,\infty)$. However, in this paper we suggest that in investigations of stability more emphasis ought to be placed on the uniform continuity of the solutions. The reason for this derives from the observation: every differentiable function with a bounded derivative is uniformly continuous—but not conversely as attested by the function

$$f(t) = \frac{\sqrt{t}}{1+t}. \tag{1.2}$$

Even though its derivative f' is unbounded on $[0,\infty)$, f is uniformly continuous on $[0,\infty)$. Moreover, $f(t)$ tends to zero as $t \to \infty$. The thesis then in this paper is that conditions yielding uniformly continuous solutions will be less stringent than those yielding solutions with bounded derivatives.

We use the following notation throughout this paper:

- $C[t_0,t_1]$ (resp. $C[t_0,\infty)$) will denote the set of all continuous real-valued functions on $[t_0,t_1]$ (resp. $[t_0,\infty)$).
- For $\varphi \in C[0,t_0]$, $|\varphi|_{t_0} := \sup\{|\varphi(t)|\colon 0 \le t \le t_0\}$.
- $L^1[0,\infty)$ typically denotes the set of all real-valued functions f that are Lebesgue measurable on $[0,\infty)$ and for which the Lebesgue integral $\int_0^\infty |f|$ is finite. However, we use it to denote those functions in $L^1[0,\infty)$ that are also continuous on $[0,\infty)$. For such a function, say g, the improper Riemann integral $\int_0^\infty |g(t)|\,dt$ converges, i.e., $\lim_{t\to\infty}\int_0^t |g(s)|\,ds$ exists and is finite. In short, by $g \in L^1[0,\infty)$ we mean that g is continuous and absolutely Riemann integrable on $[0,\infty)$.
- $L^2[0,\infty)$ will denote the set of all continuous real-valued functions that are square integrable on $[0,\infty)$. That is, $h \in L^2[0,\infty)$ will mean that h is continuous on $[0,\infty)$ and $h^2 \in L^1[0,\infty)$.

In Section 2, we construct a Liapunov functional and use it to find conditions that yield L^2 solutions of (1.1) with bounded derivatives. The idea of replacing those conditions with ones that yield uniformly continuous L^p solutions is broached.

In Section 3, we modify the Liapunov functional so as to lessen the stringency of the conditions in Section 2 and to obtain L^1 solutions of (1.1). Furthermore, a couple of conditions are added so that the derivatives of solutions are bounded on $[0,\infty)$, which enables us to obtain a global asymptotic stability result. A well-known classical result of uniform asymptotic stability is obtained when $a(t)$ is constant and positive and the kernel b is of convolution type, viz., that b depends only on the difference $t-s$.

In Section 4, we show that the conditions that were added to obtain the global asymptotic stability result in Section 3 are actually unnecessary by arguing that the Liapunov functional is uniformly continuous which in turn implies that solutions of (1.1) are uniformly continuous. The upshot is that "bounded derivative" conditions can be replaced with less restrictive "uniform continuity" conditions. One consequence of this is that the usual condition that $a(t)$ be bounded below by a positive constant can be replaced with $a(t)$ being merely nonnegative.

2 Uniformly Continuous L^2 Solutions

In this section, we obtain conditions involving the continuous functions a and b so that all of the solutions of

$$x'(t) = -a(t)x(t) + \int_0^t b(t,s)x(s)\,ds \tag{2.1}$$

belong to $L^2[0,\infty)$. Then we will add more conditions that will drive these L^2 solutions to zero as $t \to \infty$. But first let us define precisely what we mean by solutions of (2.1) and of the related nonhomogeneous equation

$$x'(t) = -a(t)x(t) + \int_0^t b(t,s)x(s)\,ds + f(t), \tag{2.2}$$

where $f\colon [0,\infty) \to \mathbb{R}$ is continuous.

Definition 2.1. A *solution* of (2.1) (resp. (2.2)) on $[0,T)$, where $0 < T \le \infty$, with an initial value $x_0 \in \mathbb{R}$ is a continuous function $x\colon [0,T) \to \mathbb{R}$ that satisfies (2.1) (resp. (2.2)) on $(0,T)$ such that $x(0) = x_0$.

It can be shown that a solution $x(t)$ satisfying an initial condition $x(0) = x_0$ exists on the entire interval $[0,\infty)$ and is unique (cf. [4, p. 5] or [7, pp. 23–27, p. 221]). For the sake of clarity, we will sometimes use the more explicit notation $x(t,0,x_0)$ to denote such a solution. Furthermore, for each $t_0 > 0$ and each continuous initial function $\varphi\colon [0,t_0] \to \mathbb{R}$, there is a unique continuous function $x\colon [0,\infty) \to \mathbb{R}$ that satisfies (2.2) on (t_0,∞) such that $x(t) \equiv \varphi(t)$ on $[0,t_0]$ (cf. [8, p. 179]). When the need for more clarity is warranted, we denote this solution by $x(t,t_0,\varphi)$. Finally, we note that $x(t) = 0$ is a solution of (2.1) for $0 \le t < \infty$, which is called its *zero solution*.

The precise terminology that we use to describe the asymptotic behavior of solutions is given in Definition 2.2 below and in Definition 3.1 in Section 3.

Definition 2.2. The zero solution of (2.1) is

1. *stable* if for every $\epsilon > 0$ and every $t_0 \ge 0$, there exists a $\delta = \delta(\epsilon,t_0) > 0$ such that $\varphi \in C[0,t_0]$ with $|\varphi|_{t_0} < \delta$ implies that $|x(t,t_0,\varphi)| < \epsilon$ for all $t \ge t_0$.

2. *globally asymptotically stable* (*asymptotically stable in the large*) if it is stable and if every solution of (2.1) approaches zero as $t \to \infty$.

Lemma 2.3. *Let $a\colon [0,\infty) \to [0,\infty)$ and $b\colon \Omega \to \mathbb{R}$ be continuous functions. If*

$$a(t) - \int_0^t |b(t,s)|\,ds \ge 0 \tag{2.3}$$

for all $t \ge 0$ and if

$$a(s) - \int_s^t |b(u,s)|\,du \ge 0 \tag{2.4}$$

for all $t \ge s \ge 0$, then the zero solution of (2.1) is stable. Suppose, in addition, that for some $t_1 \ge 0$ there is a constant $k > 0$ such that either

$$a(t) - \int_0^t |b(t,s)|\,ds \ge k \tag{2.5}$$

for all $t \geq t_1$ *or*

$$a(s) - \int_s^t |b(u,s)|\, du \geq k \tag{2.6}$$

for all $t \geq s \geq t_1$. *Then every solution* $x(t)$ *of* (2.1) *belongs to* $L^2[0,\infty)$.

Proof. First we define the Liapunov functional $V\colon [0,\infty) \times C[0,\infty) \to [0,\infty)$ by

$$V(t,\psi(\cdot)) := \psi^2(t) + \int_0^t \left[a(s) - \int_s^t |b(u,s)|\, du\right] \psi^2(s)\, ds. \tag{2.7}$$

It follows from (2.4) that $V(t,\psi(\cdot)) \geq \psi^2(t)$ for all $t \geq 0$.

For any $t_0 \geq 0$ and initial function $\varphi \in C[0,t_0]$, let $x(t) = x(t,t_0,\varphi)$ denote the unique solution of (2.1) on $[0,\infty)$ such that $x(t) = \varphi(t)$ for $0 \leq t \leq t_0$. For brevity, let $V(t) := V(t,x(\cdot))$, that is, the value of the functional V along the solution $x(t)$ at t. Taking the derivative of V with respect to t, we have

$$\begin{aligned}
V'(t) &= 2x(t)x'(t) + a(t)x^2(t) - \int_0^t |b(t,s)| x^2(s)\, ds \\
&= 2x(t)\left[-a(t)x(t) + \int_0^t b(t,s)x(s)\, ds\right] \\
&\qquad\qquad + a(t)x^2(t) - \int_0^t |b(t,s)| x^2(s)\, ds \\
&\leq -a(t)x^2(t) + \int_0^t |b(t,s)| \cdot 2|x(t)||x(s)|\, ds - \int_0^t |b(t,s)| x^2(s)\, ds \\
&\leq -a(t)x^2(t) + \int_0^t |b(t,s)| \left(x^2(t) + x^2(s)\right) ds - \int_0^t |b(t,s)| x^2(s)\, ds.
\end{aligned}$$

Consequently,

$$V'(t) \leq -\left(a(t) - \int_0^t |b(t,s)|\, ds\right) x^2(t) \tag{2.8}$$

for all $t \geq t_0$. Thus, by (2.3), $V'(t) \leq 0$. This, together with $V(t) \geq x^2(t)$, implies that

$$x^2(t) \leq V(t) \leq V(t_0) \tag{2.9}$$

for all $t \geq t_0$. Moreover, from

$$V(t_0) = \varphi^2(t_0) + \int_0^{t_0} \left[a(s) - \int_s^{t_0} |b(u,s)|\, du\right] \varphi^2(s)\, ds \leq |\varphi|_{t_0}^2 M(t_0),$$

where

$$M(t_0) := 1 + \int_0^{t_0} \left[a(s) - \int_s^{t_0} |b(u,s)|\, du\right] ds,$$

this becomes

$$|x(t)| \leq |\varphi|_{t_0} \sqrt{M(t_0)} \tag{2.10}$$

for all $t \geq t_0$. This implies the zero solution is stable: for $\epsilon > 0$, let $\delta = \epsilon/\sqrt{M(t_0)}$. Then for $\varphi \in C[0, t_0]$ with $|\varphi|_{t_0} < \delta$,

$$|x(t)| < \delta\sqrt{M(t_0)} = \epsilon \tag{2.11}$$

for all $t \geq t_0$.

If (2.5) also holds, then (2.8) implies that

$$V'(t) \leq -kx^2(t)$$

for all $t \geq \tau$, where $\tau := \max\{t_0, t_1\}$. Integrating, we obtain

$$V(t) - V(\tau) \leq -k \int_\tau^t x^2(s)\, ds.$$

Consequently,

$$x^2(t) \leq V(t) \leq V(\tau) - k \int_\tau^t x^2(s)\, ds \tag{2.12}$$

for all $t \geq \tau$. If, on the other hand (2.6) holds, then (2.7) and (2.9) imply

$$x^2(t) + k \int_{t_1}^t x^2(s)\, ds \leq V(t) \leq V(t_0) \tag{2.13}$$

for all $t \geq t_1$. Either one, (2.12) or (2.13), implies that $x^2 \in L^1[0, \infty)$. $\square$

We have just proved that under the conditions of Lemma 2.3, the solution $x(t, t_0, \varphi)$ of (2.1) belongs to $L^2[0, \infty)$. It then seems plausible that $x^2(t) \to 0$ as $t \to \infty$. However, by itself convergence of an improper Riemann integral of a function f on $[0, \infty)$ does not ensure that f approaches 0 as $t \to \infty$ (cf. [13, p. 466]). But if f were also known to be uniformly continuous, then it would according to the next lemma attributed to Barbălat [2].

Barbălat's Lemma. *If $f\colon [0, \infty) \to \mathbb{R}$ is both uniformly continuous and Riemann integrable on $[0, \infty)$, then $f(t) \to 0$ as $t \to \infty$.*

A proof of this is given in [16, p. 866]. A proof for a nonnegative f can also be found in [12, p. 89].

Note however that even if an L^2 solution of (2.1) were also uniformly continuous, we could not use Barbălat's lemma to conclude anything since the uniform continuity of a function f does not imply the uniform continuity of f^2, as exemplified by $f(t) = t$ on $[0, \infty)$. Nor does $f \in L^2[0, \infty)$ imply that $f \in L^1[0, \infty)$, as is the case with $f(t) = (t + 1)^{-1}$ on $[0, \infty)$. Nonetheless, an L^2 function f that is uniformly continuous does approach zero. The following proof of this is adapted from the proof of Barbălat's lemma given in the aforementioned reference [16].

Lemma 2.4. *If $f\colon [0, \infty) \to \mathbb{R}$ is uniformly continuous on $[0, \infty)$ and if f^2 is Riemann integrable on $[0, \infty)$, then $f(t) \to 0$ as $t \to \infty$.*

Proof. Suppose to the contrary that $f(t)$ does not approach 0 as $t \to \infty$. Then an $\epsilon > 0$ and a sequence $\{t_n\}_{n=1}^{\infty}$ exists on $[0,\infty)$ with $t_n \to \infty$ such that $|f(t_n)| \geq \epsilon$ for all $n \geq 1$. Since f is uniformly continuous, a $\delta > 0$ exists for this ϵ with the property: for $t \in [0,\infty)$, $|t - t_n| \leq \delta$ implies that

$$|f(t) - f(t_n)| < \frac{\epsilon}{2}$$

for all $n \geq 1$. Hence,

$$|f(t)| \geq |f(t_n)| - |f(t_n) - f(t)| > \frac{\epsilon}{2}$$

for all $t \in [t_n, t_n + \delta]$ and $n \geq 1$. This implies that

$$\int_0^{t_n+\delta} f^2(t)\,dt - \int_0^{t_n} f^2(t)\,dt = \int_{t_n}^{t_n+\delta} f^2(t)\,dt \geq \frac{\delta\epsilon^2}{4} > 0$$

for all $n \geq 1$. However, this is a contradiction because the left-hand side converges to 0 as $n \to \infty$ by the hypothesis that f^2 is integrable on $[0,\infty)$. □

With Lemma 2.4 at our disposal, we can find another condition to add to those of Lemma 2.3 that will ensure that all solutions of (2.1) approach zero.

Theorem 2.5. *Let $a\colon [0,\infty) \to [0,\infty)$ and $b\colon \Omega \to \mathbb{R}$ be continuous functions satisfying conditions* (2.3) *and* (2.4) *of Lemma 2.3. If for some $t_1 \geq 0$ there are positive constants k and K such that either*

$$k + \int_0^t |b(t,s)|\,ds \leq a(t) \leq K \tag{2.14}$$

for all $t \geq t_1$ or

$$k + \int_s^t |b(u,s)|\,du \leq a(s) \leq K \tag{2.15}$$

for all $t \geq s \geq t_1$, then all solutions of (2.1) *are uniformly continuous on $[0,\infty)$ and belong to $L^2[0,\infty)$. Furthermore, the zero solution is globally asymptotically stable.*

Proof. We only need to show that all solutions of (2.1) tend to zero since stability has already been established in Lemma 2.3. To this end, for any $t_0 \geq 0$ and $\varphi \in C[0,t_0]$, consider the corresponding solution $x(t) = x(t,t_0,\varphi)$. By (2.10),

$$|x(t)| \leq |\varphi|_{t_0}\sqrt{M(t_0)}$$

for all $t \geq t_0$. Consequently, as $a(t) \leq K$ for $t \geq t_1$,

$$\begin{aligned}|x'(t)| &\leq a(t)|x(t)| + \int_0^{t_0} |b(t,s)||\varphi(s)|\,ds + \int_{t_0}^t |b(t,s)||x(s)|\,ds \\ &\leq 2K|\varphi|_{t_0}\sqrt{M(t_0)} + K|\varphi|_{t_0}\end{aligned}$$

for $t \geq \tau$, where $\tau = \max\{t_0, t_1\}$. Since $x'(t)$ is bounded on $[\tau,\infty)$, $x(t)$ satisfies a Lipschitz condition on $[\tau,\infty)$. Consequently, it is uniformly continuous on $[\tau,\infty)$. This and the continuity of $x(t)$ on $[0,\infty)$ imply $x(t)$ is uniformly continuous on the entire interval $[0,\infty)$. By Lemma 2.3, $x^2(t) \in L^1[0,\infty)$. Therefore, by Lemma 2.4, $x(t) \to 0$ as $t \to \infty$. □

Example 2.6. Let k be a positive real number. All of the solutions of

$$x'(t) = -\left(k + \frac{1}{1+t}\right) x(t) + \int_0^t \frac{\cos s}{(1+t)^3}\, x(s)\, ds \tag{2.16}$$

are uniformly continuous on $[0, \infty)$ and belong to the set $L^2[0, \infty)$. Moreover, the zero solution is globally asymptotically stable.

Proof. Since $a(t) = k + 1/(1 + t)$ and $b(t, s) = (\cos s)/(1 + t)^3$, we have

$$\begin{aligned} k + \int_0^t |b(t,s)|\, ds &= k + \int_0^t \frac{|\cos s|}{(1+t)^3}\, ds \\ &\le k + \frac{t}{(1+t)^3} < k + \frac{1}{1+t} = a(t) \end{aligned} \tag{2.17}$$

for all $t \geq 0$. Thus, (2.14) is satisfied with $K := k + 1$. Also,

$$\int_s^t |b(u,s)|\, du \le \int_s^t \frac{1}{(1+u)^3}\, du < \frac{1}{2}(1+s)^{-2} < \frac{1}{1+s} < k + \frac{1}{1+s} = a(s)$$

for all $(t, s) \in \Omega$. Therefore, all of the conditions of Theorem 2.5 are satisfied. $\square$

Since the zero solution of (2.16) is globally asymptotically stable for each $k > 0$, no matter how small, it is plausible that the zero solution of

$$x'(t) = -\frac{1}{1+t}\, x(t) + \int_0^t \frac{\cos s}{(1+t)^3}\, x(s)\, ds \tag{2.18}$$

is also globally asymptotically stable. But then again it may not be in light of the scalar equation $x' = -kx$, which has a zero solution that is globally asymptotically stable when $k > 0$ but not for $k = 0$. However, the case for (2.18) is bolstered by the fact that the zero solution of

$$x' = -\frac{x}{1+t} \tag{2.19}$$

is globally asymptotically stable. Additionally, the Maple worksheet [3, cf. (6.1)] at the Maple Application Center (www.maplesoft.com) shows the graphs of two numerical solutions of (2.18) approaching zero. In point of fact, the zero solution of (2.18) is globally asymptotically stable. We prove this next with the aid of the functional (2.20) below, which is the result of conflating, so to speak, the Liapunov function $V(t, x) = (1 + t)x^2$ used by Yoshizawa for (2.19) in [18, p. 59] and the Liapunov functional (2.7).

Example 2.7. The zero solution of (2.18) is globally asymptotically stable.

Proof. Define the Liapunov functional

$$V(t, \psi(\cdot)) := (1+t)\psi^2(t) + \int_0^t \left[\frac{1}{1+s} - \int_s^t \frac{|\cos s|}{(1+u)^2}\, du\right] \psi^2(s)\, ds. \tag{2.20}$$

Notice that $V(t,\psi(\cdot)) \geq \psi^2(t)$ for all $t \geq 0$.

For a given $\varphi \in C[0,t_0]$, let $x(t) = x(t,t_0,\varphi)$ denote the solution of (2.18) with $x(t) = \varphi(t)$ on $[0,t_0]$. For this particular solution, let $V(t) := V(t,x(\cdot))$. Taking the derivative of V with respect to t, we find for $t \geq t_0$ that

$$\begin{aligned}
V'(t) &\leq 2(1+t)x(t)\left[-\frac{1}{1+t}\,x(t) + \int_0^t \frac{\cos s}{(1+t)^3}\,x(s)\,ds\right] + x^2(t) \\
&\quad + \frac{1}{1+t}\,x^2(t) - \int_0^t \frac{|\cos s|}{(1+t)^2}\,x^2(s)\,ds \\
&\leq -x^2(t) + \frac{1}{1+t}\,x^2(t) + \int_0^t \frac{|\cos s|}{(1+t)^2}\,2|x(t)||x(s)|\,ds \\
&\quad - \int_0^t \frac{|\cos s|}{(1+t)^2}\,x^2(s)\,ds \\
&\leq -x^2(t) + \frac{1}{1+t}\,x^2(t) + x^2(t)\int_0^t \frac{|\cos s|}{(1+t)^2}\,ds \\
&\leq -\left(1 - \frac{1}{1+t} - \frac{t}{(1+t)^2}\right)x^2(t),
\end{aligned}$$

which simplifies to

$$V'(t) \leq -\left(\frac{t}{1+t}\right)^2 x^2(t). \tag{2.21}$$

Consequently,

$$x^2(t) \leq V(t) \leq V(t_0) \tag{2.22}$$

for all $t \geq t_0$. This implies that the zero solution of (2.18) is stable by an argument much like the one from (2.9) to (2.11).

It follows from (2.21) that

$$V'(t) \leq -\frac{1}{4}x^2(t) \tag{2.23}$$

for all $t \geq \tau$, where $\tau = \max\{1,t_0\}$. And so

$$x^2(t) \leq V(t) \leq V(\tau) - \frac{1}{4}\int_\tau^t x^2(s)\,ds.$$

Therefore, $x^2(t) \in L^1[0,\infty)$. Furthermore, it follows from (2.18) and (2.22) that $x'(t)$ is bounded on $[t_0,\infty)$. Hence, by the uniform continuity argument in the proof of Theorem 2.5, $x(t)$ is uniformly continuous on $[0,\infty)$. Therefore, $x(t) \to 0$ as $t \to \infty$. □

Remark. T. A. Burton, in a private note, points out that the Liapunov functional (2.20) is *strongly positive definite* in the sense defined by Lakshmikantham and Leela in [15, p. 137]. It appears to be one of the few, if any, such nontrivial strongly positive definite functionals that have appeared in the literature to obtain an asymptotic stability result.

3 Uniformly Continuous L^1 Solutions

Fulfillment of the conditions in Lemma 2.3 in the previous section ensures that the solutions of

$$x'(t) = -a(t)x(t) + \int_0^t b(t,s)x(s)\,ds. \tag{3.1}$$

are in $L^2[0,\infty)$. Now we modify the Liapunov functional (2.7) used to prove this lemma in order to obtain conditions that will ensure that all solutions are in $L^1[0,\infty)$ and to replace the lower bounds for $a(t)$ in (2.14) and (2.15) with less stringent ones. At the same time, we will also find conditions that imply either global asymptotic stability or the following types of stability:

Definition 3.1. The zero solution of (3.1) is

1. *uniformly stable* if for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that $\varphi \in C[0,t_0]$ with $|\varphi|_{t_0} < \delta$ (any $t_0 \geq 0$) implies that $|x(t,t_0,\varphi)| < \epsilon$ for all $t \geq t_0$.

2. *uniformly asymptotically stable* if it is uniformly stable and if there exists an $\eta > 0$ with the following property: for every $\epsilon > 0$, there exists a $T = T(\epsilon) > 0$ such that $\varphi \in C[0,t_0]$ with $|\varphi|_{t_0} < \eta$ (any $t_0 \geq 0$) implies that $|x(t,t_0,\varphi)| < \epsilon$ for all $t \geq t_0 + T$.

3. *uniformly asymptotically stable in the large* if it is uniformly stable and if for every $\eta > 0$ and every $\epsilon > 0$, there exists a $T = T(\eta,\epsilon) > 0$ such that $\varphi \in C[0,t_0]$ with $|\varphi|_{t_0} < \eta$ (any $t_0 \geq 0$) implies that $|x(t,t_0,\varphi)| < \epsilon$ for all $t \geq t_0 + T$. (In other words, $x(t) \equiv 0$ is uniformly asymptotically stable in the large if (2) is true for every $\eta > 0$.)

The Liapunov functional $V(t,\psi(\cdot))$ that we employ in the next lemma to find conditions for the stability of the zero solution of (3.1) eliminates the need for condition (2.3). $V(t,\psi(\cdot))$ was first derived by T. A. Burton, with which he obtained asymptotic stability conditions like some of those in Theorem 2.5 (cf. [8, pp. 122–123]). Especially noteworthy in this regard is Burton's "rough algorithm" (to use his words) for deriving Liapunov functionals that he describes on p. 121 in [8]. In (3.11) below, we tweak $V(t,\psi(\cdot))$ a bit so that we can replace (2.6) with the less restrictive condition that $a(t) \geq k$.

Lemma 3.2. *Let $a\colon [0,\infty) \to [0,\infty)$ and $b\colon \Omega \to \mathbb{R}$ be continuous functions. If*

$$\int_s^t |b(u,s)|\,du \leq a(s) \tag{3.2}$$

for all $t \geq s \geq 0$, then the zero solution of (3.1) is stable. Furthermore, if for some $t_1 \geq 0$ there is a constant $k > 0$ such that

$$a(t) \geq k \tag{3.3}$$

for all $t \geq t_1$ and a constant $\lambda \in (0,1)$ such that

$$\int_s^t |b(u,s)|\,du \leq \lambda a(s) \tag{3.4}$$

for all $t \geq s \geq t_1$, then every solution $x(t)$ of (3.1) belongs to $L^1[0,\infty)$.

Proof. First define the Liapunov functional $V\colon [0,\infty) \times C[0,\infty) \to [0,\infty)$ by

$$V(t,\psi(\cdot)) := |\psi(t)| + \int_0^t \left[a(s) - \int_s^t |b(u,s)|\,du\right] |\psi(s)|\,ds. \tag{3.5}$$

By (3.2), $V(t,\psi(\cdot)) \geq |\psi(t)|$ for all $t \geq 0$.

For any $t_0 \geq 0$ and $\varphi \in C[0,t_0]$, let $x(t) = x(t,t_0,\varphi)$ denote the solution of (3.1) on $[0,\infty)$ with $x(t) = \varphi(t)$ for $0 \leq t \leq t_0$. Then consider $V(t) := V(t,x(\cdot))$ and its derivative. Since $x(t)$ is continuously differentiable on $[t_0,\infty)$, $|x(t)|$ has a right derivative $D_r|x(t)|$ given by

$$D_r|x(t)| = \begin{cases} x'(t)\operatorname{sgn} x(t), & \text{if } x(t) \neq 0 \\ |x'(t)|, & \text{if } x(t) = 0 \end{cases} \tag{3.6}$$

for all $t \geq t_0$ (cf. [14, p. 26]). Thus, the right derivative of V for $t \geq t_0$ is

$$\begin{aligned} D_rV(t) &= D_r|x(t)| + \frac{d}{dt}\int_0^t \left[a(s) - \int_s^t |b(u,s)|\,du\right] |x(s)|\,ds \\ &\leq -a(t)|x(t)| + \int_0^t |b(t,s)||x(s)|\,ds + a(t)|x(t)| - \int_0^t |b(t,s)||x(s)|\,ds \end{aligned}$$

and so

$$D_rV(t) \leq 0. \tag{3.7}$$

Thus,

$$|x(t)| \leq V(t) \leq V(t_0) \tag{3.8}$$

for all $t \geq t_0$, where

$$V(t_0) = |\varphi(t_0)| + \int_0^{t_0} \left[a(s) - \int_s^{t_0} |b(u,s)|\,du\right] |\varphi(s)|\,ds \leq M(t_0)|\varphi|_{t_0}$$

and

$$M(t_0) := 1 + \int_0^{t_0} \left[a(s) - \int_s^{t_0} |b(u,s)|\,du\right] ds. \tag{3.9}$$

For a given $\epsilon > 0$, let $\delta = \epsilon/M(t_0)$. Then for $\varphi \in C[0,t_0]$ with $|\varphi|_{t_0} < \delta$, we have

$$|x(t)| \leq V(t_0) \leq M(t_0)|\varphi|_{t_0} < \delta M(t_0) = \epsilon \tag{3.10}$$

for all $t \geq t_0$, which proves stability.

Now suppose (3.3) and (3.4) also hold. In that case, let $\gamma := \sqrt{\lambda}$ and

$$V_\gamma(t) := |x(t)| + \int_0^t \left[\gamma a(s) - \frac{1}{\gamma}\int_s^t |b(u,s)|\,du\right] |x(s)|\,ds. \tag{3.11}$$

By (3.4),

$$V_\gamma(t) \geq |x(t)| \tag{3.12}$$

for all $t \geq t_1$. And

$$\begin{aligned} D_r V_\gamma(t) &\leq -a(t)|x(t)| + \int_0^t |b(t,s)||x(s)|\,ds \\ &\qquad + \gamma a(t)|x(t)| - \frac{1}{\gamma}\int_0^t |b(t,s)||x(s)|\,ds \\ &\leq -(1-\gamma)a(t)|x(t)| \end{aligned} \tag{3.13}$$

for all $t \geq \tau$, where $\tau := \max\{t_0, t_1\}$. Then, because of (3.3),

$$D_r V(t) \leq -k(1-\gamma)|x(t)|. \tag{3.14}$$

An integration (cf. [14, Cor. 4.1, p. 27]) along with (3.12) yields

$$|x(t)| \leq V_\gamma(t) \leq V_\gamma(\tau) - k(1-\gamma)\int_\tau^t |x(s)|\,ds, \tag{3.15}$$

for all $t \geq \tau$. Therefore, $\int_0^\infty |x(t)|\,dt$ converges. □

If in addition to the conditions of Lemma 3.2, $a(t)$ is bounded from above and condition (3.16) below is met, then all of the solutions of (3.1) tend to zero as $t \to \infty$, as we will now prove.

Theorem 3.3. *Let $a\colon [0,\infty) \to [0,\infty)$ and $b\colon \Omega \to \mathbb{R}$ be continuous functions such that*

$$\int_0^t |b(t,s)|\,ds \leq a(t) \tag{3.16}$$

for all $t \geq 0$ and

$$\int_s^t |b(u,s)|\,du \leq a(s), \tag{3.17}$$

for all $t \geq s \geq 0$. If for some $t_1 \geq 0$ there are positive constants k and K such that

$$k \leq a(t) \leq K \tag{3.18}$$

for all $t \geq t_1$ and a constant $\lambda \in (0,1)$ such that

$$\int_s^t |b(u,s)|\,du \leq \lambda a(s) \tag{3.19}$$

for all $t \geq s \geq t_1$, then all solutions of (3.1) are uniformly continuous on $[0,\infty)$ and belong to $L^1[0,\infty)$. Moreover, the zero solution is globally asymptotically stable.

Proof. By Lemma 3.2, the zero solution of (3.1) is stable. For any $\varphi \in C[0,t_0]$, consider the corresponding solution $x(t) = x(t,t_0,\varphi)$. By (3.8),

$$|x(t)| \leq V(t_0)$$

for all $t \geq t_0$. This, together with (3.16) and (3.18), applied to (3.1) gives

$$\begin{aligned}|x'(t)| &\leq a(t)|x(t)| + \int_0^{t_0} |b(t,s)||\varphi(s)|\,ds + \int_{t_0}^{t} |b(t,s)||x(s)|\,ds \\ &\leq KV(t_0) + a(t_0)|\varphi|_{t_0} + V(t_0)a(t) \leq 2KV(t_0) + a(t_0)|\varphi|_{t_0}\end{aligned}$$

for all $t \geq \tau$, where as before $\tau = \max\{t_0, t_1\}$. In short, $x'(t)$ is bounded on $[\tau, \infty)$. Consequently, by the uniform continuity argument in the proof of Theorem 2.5, $x(t)$ is uniformly continuous on $[0, \infty)$. Also, by Lemma 3.2, $x \in L^1[0, \infty)$. Therefore, by Barbălat's lemma, $x(t) \to 0$ as $t \to \infty$. □

Example 3.4. For any positive constants k and β, the zero solution of

$$x'(t) = -\left(k + \frac{1+\beta}{1+t}\right) x(t) + \int_0^t \frac{\cos s}{(1+t)^2}\, x(s)\,ds, \tag{3.20}$$

is globally asymptotically stable.

Proof. Condition (3.18) is satisfied since $a(t) = k + \frac{1+\beta}{1+t}$ is bounded by positive constants. Condition (3.16) is also satisfied since

$$\int_0^t |b(t,s)|\,ds = \int_0^t \frac{|\cos s|}{(1+t)^2}\,ds \leq \frac{t}{(1+t)^2} < \frac{1}{1+t} < a(t),$$

for all $t \geq 0$. Furthermore,

$$\begin{aligned}\int_s^t |b(u,s)|\,du \leq \int_s^t \frac{1}{(1+u)^2}\,du &< \frac{1}{1+s} \\ &< \frac{1}{1+\beta}\left(k + \frac{1+\beta}{1+s}\right) = \frac{1}{1+\beta}\,a(s)\end{aligned}$$

for all $t \geq s \geq 0$. Therefore, all the conditions of Theorem 3.3 are satisfied. □

As the next example shows, the conditions of Theorem 3.3 are easily met when $a(t)$ is a positive constant and b is of convolution type (i.e., b depends only on the difference $t-s$). Uniform asymptotic stability of the zero solution for this case was originally established by Burton and Mahfoud [10, p. 146]. A proof for a positive function b can also be found in Burton's monograph [7, pp. 55-57].

Example 3.5. Let a be a positive constant. If $b\colon [0, \infty) \to \mathbb{R}$ is continuous and

$$\int_0^\infty |b(t)|\,dt < a, \tag{3.21}$$

then the zero solution of

$$x'(t) = -ax(t) + \int_0^t b(t-s)x(s)\,ds \tag{3.22}$$

is globally asymptotically stable. In point of fact, it is uniformly asymptotically stable in the large.

Proof. Define λ by

$$\lambda := \frac{1}{a}\int_0^\infty |b(t)|\,dt.$$

It follows from (3.21) that $\lambda < 1$ and

$$\int_s^t |b(u-s)|\,du = \int_0^{t-s} |b(v)|\,dv \le \lambda a$$

for all $t \ge s \ge 0$. Clearly then, conditions (3.16)–(3.19) are satisfied. Therefore, by Theorem 3.3, the zero solution of (3.22) is globally asymptotically stable.

To show that the zero solution is also uniformly asymptotically stable in the large, let $z(t)$ denote the *principal solution*, i.e., the solution of (3.22) with $z(0) = 1$. By Lemma 3.2,

$$z(t) \in L^1[0,\infty). \tag{3.23}$$

Furthermore, (3.5) and (3.8) imply that

$$|z(t)| \le 1 \tag{3.24}$$

for $0 \le t \le \infty$. At this point we could invoke Miller's [17, pp. 493–498] classic result: for $\alpha \in \mathbb{R}$ and $b \in L^1[0,\infty)$, the zero solution of

$$x'(t) = \alpha x(t) + \int_0^t b(t-s)x(s)\,ds$$

is uniformly asymptotically stable if and only if $z(t) \in L^1[0,\infty)$. It would then follow from (3.21) and (3.23) that the zero solution of (3.22) is uniformly asymptotically stable. However, in order to show that it is in fact uniformly asymptotically stable in the large, we present a self-contained proof of that next.

For $t_0 \ge 0$ and $\varphi \in C[0,t_0]$, let $x(t) = x(t,t_0,\varphi)$ denote the unique solution of (3.22) with $x(t) = \varphi(t)$ for $0 \le t \le t_0$. Hence,

$$x'(t) = -ax(t) + \int_0^{t_0} b(t-s)\varphi(s)\,ds + \int_{t_0}^t b(t-s)x(s)\,ds$$

for $t > t_0$. Then, for $x(t+t_0) := x(t+t_0,t_0,\varphi)$, it follows from the chain rule that

$$\frac{d}{dt}x(t+t_0) = -ax(t+t_0) + \int_0^t b(t-s)x(s+t_0)\,ds + f(t) \tag{3.25}$$

for $t > 0$, where

$$f(t) := \int_0^{t_0} b(t+u)\varphi(t_0-u)\,du. \tag{3.26}$$

In other words, $x(t+t_0)$ is the unique solution of

$$y'(t) = -ay(t) + \int_0^t b(t-s)y(s)\,ds + f(t) \tag{3.27}$$

with $y(0) = x(t_0) = \varphi(t_0)$.

By the variation of parameters formula (cf. [4, p. 14] or [7, p. 31, p. 223]), we have

$$y(t) = z(t)\varphi(t_0) + \int_0^t z(t-s)f(s)\,ds,$$

or as $y(t) = x(t+t_0)$,

$$x(t+t_0) = z(t)\varphi(t_0) + \int_0^t z(t-s)\left\{\int_0^{t_0} b(s+u)\varphi(t_0-u)\,du\right\} ds. \tag{3.28}$$

Using (3.28), we now show that the zero solution of (3.22) is uniformly stable. Using (3.21) and (3.24), we obtain

$$|x(t+t_0)| \le |\varphi|_{t_0}\left[|z(t)| + \int_0^t |z(t-s)|\left\{\int_s^{s+t_0} |b(v)|\,dv\right\} ds\right] \le |\varphi|_{t_0}\left[1 + a\int_0^\infty |z(t)|\,dt\right]. \tag{3.29}$$

For any $\epsilon > 0$, take

$$\delta := \epsilon\left[1 + a\int_0^\infty |z(t)|\,dt\right]^{-1}.$$

Then from (3.29) it follows that $|x(t+t_0)| < \epsilon$ for $t \ge 0$, or that $|x(t)| < \epsilon$ for $t \ge t_0$, for any $\varphi \in C[0,t_0]$ with $|\varphi|_{t_0} < \delta$.

Now that uniform stability has been established, we must show that the rest of Definition 3.1 (3) holds. Choose any $\eta > 0$. For any $t_0 \ge 0$, choose any $\varphi \in C[0,t_0]$ with $|\varphi|_{t_0} < \eta$. Then it follows from (3.29) that

$$|x(t+t_0)| \le \eta\left[|z(t)| + \int_0^t |z(t-s)|g(s)\,ds\right] \tag{3.30}$$

where

$$g(s) := \int_s^{s+t_0} |b(v)|\,dv. \tag{3.31}$$

Since $b \in L^1[0,\infty)$, $g(s) \to 0$ as $s \to \infty$. This and (3.23) imply that

$$\int_0^t |z(t-s)|g(s)\,ds \to 0 \quad \text{as} \quad t \to \infty \tag{3.32}$$

since the convolution of two functions approaches 0 as $t \to \infty$ if one of them belongs to $L^1[0,\infty)$ and the other one approaches 0 as $t \to \infty$. By Theorem 3.3,

$$z(t) \to 0 \quad \text{as} \quad t \to \infty. \tag{3.33}$$

This and (3.32) imply that the right-hand side of (3.30) approaches 0 as $t \to \infty$. Consequently, for every $\epsilon > 0$, there is a $T = T(\eta,\epsilon) > 0$ such that $|x(t+t_0)| < \epsilon$ for $t \ge T$. In short, we have shown that for a given $\eta > 0$ and $\epsilon > 0$, and any $t_0 \ge 0$, a $T > 0$ exists (independent of t_0) such that

$$|x(t,t_0,\varphi)| < \epsilon$$

for all $t \ge t_0 + T$ if $\varphi \in C[0,t_0]$ and $|\varphi|_{t_0} < \eta$. □

4 A Uniformly Continuous Liapunov Functional

We will establish in the next theorem that the conclusions reached in Theorem 3.3 are in fact valid under the conditions of Lemma 3.2. In other words, the conditions that were added to Lemma 3.2 to obtain Theorem 3.3 are superfluous. This result depends on the following two lemmas, the first of which is a consequence of the definition of uniform continuity and appears as an exercise in Boas [6, p. 119].

Lemma 4.1. *Let $f \in C[0,\infty)$. If $\lim_{t\to\infty} f(t)$ exists and is finite, then f is uniformly continuous on $[0,\infty)$.*

Consequently, we have:

Lemma 4.2. *Let $f \in C[0,\infty)$. If f is Riemann integrable on $[\tau,\infty)$ for some $\tau \geq 0$, then $\int_0^t f(s)\,ds$ is uniformly continuous on $[0,\infty)$.*

Theorem 4.3. *Let $a\colon [0,\infty) \to [0,\infty)$ and $b\colon \Omega \to \mathbb{R}$ be continuous functions. If*

$$\int_s^t |b(u,s)|\,du \leq a(s) \tag{4.1}$$

for all $t \geq s \geq 0$, then the zero solution of

$$x'(t) = -a(t)x(t) + \int_0^t b(t,s)x(s)\,ds \tag{4.2}$$

is stable. Furthermore, if for some $t_1 \geq 0$ there is a constant $k > 0$ such that

$$a(t) \geq k \tag{4.3}$$

for all $t \geq t_1$ and a constant $\lambda \in (0,1)$ such that

$$\int_s^t |b(u,s)|\,du \leq \lambda a(s) \tag{4.4}$$

for all $t \geq s \geq t_1$, then every solution $x(t)$ of (4.2) belongs to $L^1[0,\infty)$ and is uniformly continuous on $[0,\infty)$. Moreover, the zero solution is globally asymptotically stable.

Proof. We have already established with Lemma 3.2 that (4.1) implies the stability of the zero solution of (4.2). Revisiting the proof of the lemma, let $x(t) = x(t,t_0,\varphi)$ be the solution corresponding to an initial function $\varphi \in C[0,t_0]$. Now consider $V(t,\psi(\cdot))$ defined by (3.5). By (3.7), $V(t) = V(t,x(\cdot))$ is decreasing on $[t_0,\infty)$. Consequently, as $V(t) \geq 0$, $\lim_{t\to\infty} V(t)$ exists and is finite. Therefore, by Lemma 4.1, V is uniformly continuous on $[0,\infty)$.

Now consider $V_\gamma(t)$ defined by (3.11). An integration of (3.13) yields

$$V_\gamma(t) - V_\gamma(\tau) \leq -(1-\gamma)\int_\tau^t a(s)|x(s)|\,ds.$$

Hence,

$$\int_\tau^t a(s)|x(s)|\,ds \le \frac{V_\gamma(\tau)}{1-\gamma}$$

for all $t \ge \tau$. And so $a(t)|x(t)|$ is Riemann integrable on $[\tau, \infty)$. By Lemma 4.2,

$$\int_0^t a(s)|x(s)|\,ds$$

is uniformly continuous on $[0, \infty)$. This suggests that the integral terms in $V(t)$, namely

$$W(t) := \int_0^t \left[a(s) - \int_s^t |b(u,s)|\,du\right] |x(s)|\,ds,$$

may also be uniformly continuous on $[0, \infty)$. We now prove that this is the case. First define $h\colon [0,\infty) \times [0,\infty) \to [0,\infty)$ by

$$h(t,s) := \begin{cases} a(s) - \int_s^t |b(u,s)|\,du & \text{if } t \ge s, \\ a(s) & \text{if } t < s. \end{cases} \tag{4.5}$$

For a fixed $t^* \in [0, \infty)$,

$$0 \le h(t^*, s)|x(s)| \le a(s)|x(s)|.$$

By a comparison test,

$$\int_0^\infty h(t^*, s)|x(s)|\,ds \le \int_0^\infty a(s)|x(s)|\,ds < \infty.$$

Consequently, the improper integral

$$\int_0^\infty h(t,s)|x(s)|\,ds$$

defines a function, call it $w(t)$, on the interval $[0, \infty)$.

For $t_2 \ge t_1 \ge 0$, $h(t_2, s) \le h(t_1, s)$. This implies w is decreasing on $[0, \infty)$. As $w(t) \ge 0$, $w(t)$ approaches a finite limit, say L, as $t \to \infty$. This in turn implies that $W(t) \to L$ as $t \to \infty$ because

$$\begin{aligned}
|W(t) - L| &\le |W(t) - w(t)| + |w(t) - L| \\
&= \left| \int_0^t h(t,s)|x(s)|\,ds - \int_0^\infty h(t,s)|x(s)|\,ds \right| + |w(t) - L| \\
&= \left| - \int_t^\infty h(t,s)|x(s)|\,ds \right| + |w(t) - L| \\
&= \int_t^\infty a(s)|x(s)|\,ds + |w(t) - L|.
\end{aligned}$$

By Lemma 4.1, W is uniformly continuous on $[0, \infty)$.

We have established that V and W are uniformly continuous on $[0, \infty)$. Hence, so is the difference $V(t) - W(t) = |x(t)|$. By Lemma 3.2, $|x| \in L^1[0, \infty)$. By Barbălat's lemma, $|x(t)| \to 0$ as $t \to \infty$. By Lemma 4.1, $x(t)$ is uniformly continuous on $[0, \infty)$. $\square$

Example 4.4. Every solution of

$$x'(t) = -(t+1)x(t) + \int_0^t \frac{2t}{(1+t^2-s^2)^2}\, x(s)\, ds \tag{4.6}$$

belongs to $L^1[0,\infty)$ and is uniformly continuous on $[0,\infty)$ and its zero solution is globally asymptotically stable.

Proof. As $a(t) = t+1$, (4.3) is satisfied with $k = 1$. As $b(t,s) = 2t(1+t^2-s^2)^{-2}$,

$$\int_s^t |b(u,s)|\, du = \int_s^t \frac{2u}{(1+u^2-s^2)^2}\, du = 1 - \frac{1}{1+t^2-s^2}.$$

Clearly then (4.1) is satisfied as

$$\int_s^t |b(u,s)|\, du \le 1 + s = a(s)$$

for all $t \ge s \ge 0$. Also, for $t \ge s \ge 2$,

$$\int_s^t |b(u,s)|\, du \le 1 - \frac{1}{1+t^2-s^2} < 1 + \frac{1}{s} = \frac{1}{s}a(s) \le \frac{1}{2}a(s).$$

In other words, (4.4) is also satisfied with $t_1 = 2$ and $\lambda = 1/2$. $\square$

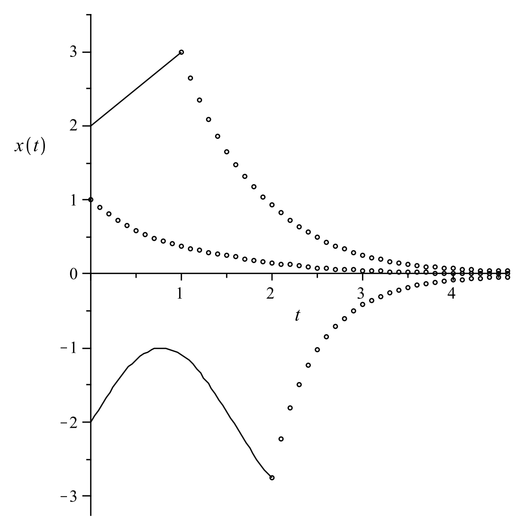

Figure 1: Three numerical solutions of (4.6).

The Maple worksheet [3], which can be found at the Maple Application Center website, uses the implicit trapezoidal rule and Newton's method for nonlinear systems to numerically approximate solutions of scalar Volterra integro-differential equations and draw their respective graphs. It was used to compute the three numerical solutions of (4.6) that are shown in Fig. 1. A step size of $h = 0.1$ was used. One of the solutions satisfies the initial condition $x(0) = 1$. The other two solutions correspond to the initial functions $\varphi(t) = 2 + t$ on the initial interval $[0, 1]$ and $\varphi(t) = -2 + \sin(2t)$ on $[0, 2]$.

Finally, we add two integral conditions to Theorem 4.3 so that we can drop (4.3), namely, the condition that $a(t)$ be eventually bounded below by a positive constant.

Theorem 4.5. *Let $a\colon [0,\infty) \to [0,\infty)$ and $b\colon \Omega \to \mathbb{R}$ be continuous functions satisfying conditions* (4.1) *and* (4.4). *If a constant L and a nonnegative function $p \in L^1[0,\infty)$ exist such that*

$$\int_0^t e^{-\int_\xi^t a(u)\,du}\, d\xi \le L \tag{4.7}$$

for all $t \ge 0$ and

$$\int_s^t e^{-\int_\xi^t a(u)\,du} |b(\xi, s)|\, d\xi \le p(s) \tag{4.8}$$

for all $t \ge s \ge 0$, then the zero solution of (4.2) *is globally asymptotically stable.*

Proof. By Lemma 3.2, the zero solution of (4.2) is stable because of (4.1). We will show that all of its solutions approach zero as $t \to \infty$ by comparing them to the solutions of the equations

$$y'(t) = -\Big(a(t) + \frac{1}{k}\Big) y(t) + \int_0^t b(t,s) y(s)\, ds, \tag{4.9}$$

where $k \in \mathbb{N}$ (the set of natural numbers). Note that for a given $k \in \mathbb{N}$, (4.9) has a globally asymptotically stable zero solution on account of $a(t)+1/k$ and $b(t,s)$ satisfying all of the conditions of Theorem 4.3.

For any $t_0 \ge 0$ and $\varphi \in C[0, t_0]$, let $x(t)$ be the solution of (4.2) with $x(t) = \varphi(t)$ for $0 \le t \le t_0$. For $k \in \mathbb{N}$, let $y_k(t)$ denote the solution of (4.9) with the same initial function—i.e., $y_k(t) = \varphi(t)$ for $0 \le t \le t_0$. Now consider the difference $x(t) - y_k(t)$. For $t \ge t_0$,

$$\frac{d}{dt}[x(t) - y_k(t)] = -a(t)[x(t) - y_k(t)] + \frac{1}{k} y_k(t) + \int_0^t b(t,s)[x(s) - y_k(s)]\, ds.$$

Multiplying this by

$$\mu(t) := \exp\left(\int_0^t a(v)\, dv\right)$$

and replacing $a(t)\mu(t)$ with $\mu'(t)$, we obtain

$$\frac{d}{dt}\left(\mu(t)[x(t) - y_k(t)]\right) = \frac{1}{k}\mu(t) y_k(t) + \mu(t)\int_0^t b(t,s)[x(s) - y_k(s)]\, ds.$$

Then an integration from t_0 to t yields

$$x(t) - y_k(t) = \frac{1}{k}\int_{t_0}^t \frac{\mu(\xi)}{\mu(t)} y_k(\xi)\, d\xi + \int_{t_0}^t \frac{\mu(\xi)}{\mu(t)} \int_0^\xi b(\xi,s)[x(s) - y_k(s)]\, ds\, d\xi$$

for all $t \geq t_0$. As $x(t) \equiv y_k(t)$ on $[0, t_0]$, it follows that

$$|x(t) - y_k(t)| \leq \frac{1}{k} \int_0^t \frac{\mu(\xi)}{\mu(t)} |y_k(\xi)|\, d\xi + \int_0^t \frac{\mu(\xi)}{\mu(t)} \int_0^\xi |b(\xi, s)|\, |x(s) - y_k(s)|\, ds\, d\xi \tag{4.10}$$

for all $t \geq 0$.

By (3.9) and (3.10),

$$|y_k(t)| \leq M_k(t_0)|\varphi|_{t_0} \tag{4.11}$$

for all $t \geq t_0$, where

$$M_k(t_0) := 1 + \int_0^{t_0} \left[a(s) + \frac{1}{k} - \int_s^{t_0} |b(u, s)|\, du\right] ds.$$

This holds in fact for all $t \geq 0$ as $M_k(t_0) \geq 1$. Moreover, the upper bound in (4.11) can be replaced by one that is independent of k; that is,

$$|y_k(t)| \leq M_1(t_0)|\varphi|_{t_0}$$

for all $t \geq 0$ and $k \in \mathbb{N}$. This and (4.7) imply

$$\int_0^t \frac{\mu(\xi)}{\mu(t)} |y_k(\xi)|\, d\xi = \int_0^t e^{-\int_\xi^t a(u)\, du} |y_k(\xi)|\, d\xi \leq LM_1(t_0)|\varphi|_{t_0} \tag{4.12}$$

for all $t \geq 0$. As for the iterated integral in (4.10), by interchanging the order of integration and applying (4.8), we obtain

$$\begin{aligned} \int_0^t \frac{\mu(\xi)}{\mu(t)} \int_0^\xi |b(\xi, s)|\, |x(s) - y_k(s)|\, ds\, d\xi \\ = \int_0^t \left(\int_s^t e^{-\int_\xi^t a(v)\, dv} |b(\xi, s)|\, d\xi\right) |x(s) - y_k(s)|\, ds \\ \leq \int_0^t p(s)|x(s) - y_k(s)|\, ds. \end{aligned} \tag{4.13}$$

It follows then from (4.10), (4.12), and (4.13) that

$$|x(t) - y_k(t)| \leq \frac{1}{k} LM_1(t_0)|\varphi|_{t_0} + \int_0^t p(s)\, |x(s) - y_k(s)|\, ds$$

for $t \geq 0$. By Gronwall's inequality,

$$|x(t) - y_k(t)| \leq \frac{1}{k} LM_1(t_0)|\varphi|_{t_0}\, e^{\int_0^t p(s)\, ds}.$$

Consequently,

$$|x(t)| \leq |y_k(t)| + \frac{1}{k} LM_1(t_0)|\varphi|_{t_0}\, e^{\int_0^\infty p(s)\, ds} < \infty \tag{4.14}$$

for all $t \geq 0$. This with $y_k(t) \to 0$ as $t \to \infty$ for every $k \in \mathbb{N}$ implies that $x(t) \to 0$ as $t \to \infty$. $\square$

Example 4.6. The zero solution of

$$x' = -a(t)x \tag{4.15}$$

is globally asymptotically stable if a constant $\alpha > 0$ exists such that

$$\int_{t_0}^{t} a(u)\,du \geq \alpha(t - t_0) \tag{4.16}$$

for $t \geq t_0 \geq 0$.

Proof. Condition (4.7) is satisfied with $L = 1/\alpha$. The other three conditions in Theorem 4.5 are trivially satisfied as $b(t, s)$ in (4.2) is identically equal to zero. ☐

Remark. In fact under condition (4.16), the zero solution of (4.15) is uniformly asymptotically stable in the large (cf. [11, p. 88]).

Example 4.7. The zero solution of

$$x'(t) = -tx(t) + \int_0^t b(t, s)\,x(s)\,ds, \tag{4.17}$$

where $b\colon \Omega \to \mathbb{R}$ is the function

$$b(t, s) = \begin{cases} 0, & \text{if } 0 \leq s < \frac{1}{3} \\ \dfrac{(3t-1)(3s-1)}{(1+t+s)^5}, & \text{if } s \geq \frac{1}{3}, \end{cases} \tag{4.18}$$

is globally asymptotically stable.

Proof. Since $a(t) = t$,

$$\int_0^t e^{-\int_\xi^t a(u)\,du}\,d\xi = e^{-t^2/2}\int_0^t e^{\xi^2/2}\,d\xi = \sqrt{2}D(t/\sqrt{2}), \tag{4.19}$$

where D is Dawson's integral, namely,

$$D(t) := e^{-t^2}\int_0^t e^{\xi^2}\,d\xi. \tag{4.20}$$

It can be shown using an elementary argument that D is bounded on $[0, \infty)$, but that is a long established fact (cf. [1, p. 298]). From the information in [1] or through the use of a computer algebra system, we find that (4.19) has a absolute maximum value of $0.7651\ldots$ at $t = 1.3069\ldots$. Consequently, (4.7) holds with $L = 0.8$.

For $t \geq s \geq 1/3$,

$$\begin{aligned} \int_s^t |b(u, s)|\,du &= \int_s^t \frac{(3u-1)(3s-1)}{(1+u+s)^5}\,du \\ &\leq (3s-1)\int_s^t \frac{(3u-1)}{(1+u)^5}\,du \leq \left[\frac{3s-1}{(1+s)^4}\right] s \leq 0.2s. \end{aligned} \tag{4.21}$$

Since $b(t,s) = 0$ for $0 \le s < 1/3$, we conclude

$$\int_s^t |b(u,s)|\,du \le \lambda a(s) \tag{4.22}$$

for all $t \ge s \ge 0$, where $\lambda = 0.2$. Thus, (4.1) and (4.4) hold for all $t \ge s \ge 0$.

From (4.21) we see that

$$\int_s^t e^{-\int_\xi^t a(u)\,du} |b(\xi,s)|\,d\xi \le \int_s^t |b(\xi,s)|\,d\xi \le p(s), \tag{4.23}$$

where

$$p(s) := \begin{cases} 0, & \text{if } 0 \le s < \frac{1}{3} \\ \left[\dfrac{3s-1}{(1+s)^4}\right] s, & \text{if } s \ge \frac{1}{3}. \end{cases} \tag{4.24}$$

Since $p \in L^1[0,\infty)$, condition (4.8) holds. □

The graphs of four numerical solutions of (4.17) computed with the Maple worksheet [3] are shown in Fig. 2. One of the solutions satisfies the initial condition $x(0) = 2$. The others issue from the initial functions $\varphi(t) = -5$, $\varphi(t) = -2 + \sin(4t)$, and $\varphi(t) = 3 + 2t$ on the initial interval $[0,1]$.

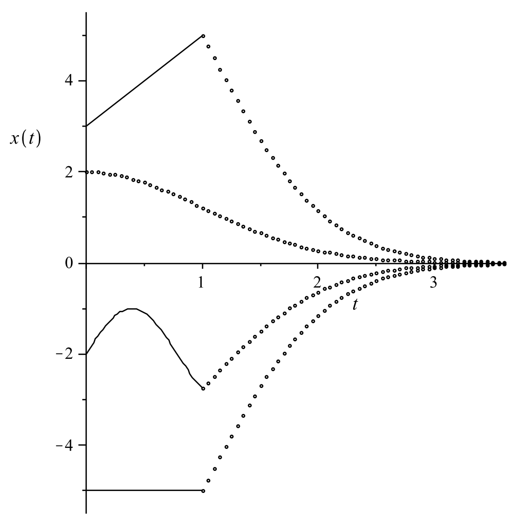

Figure 2: Four numerical solutions of (4.17).

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