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A Characterization of the Product Hardy Space H^1

Luiz Antonio Pereira Gomes
Departamento de Matemática,
Universidade Estadual de Maringá,
Av. Colombo 5790, 87020-000 Maringa - PR, Brazil
email: `lapgomes@uem.br`
and
Eduardo Brandani da Silva
Departamento de Matemática,
Universidade Estadual de Maringá,
Av. Colombo 5790, 87020-000 Maringa - PR, Brazil
email: `ebsilva@wnet.com.br`

ABSTRACT

A characterization of the product space H^1 such as the two parameters space $H_0^{1,2}$ is obtained, where $H_0^{1,2}$ is a particular case of spaces $H_S^{P,Q}$, which are generalizations of spaces studied by J. Peetre and H. Triebel.

RESUMEN

Se obtiene una caracterización del espacio producto H^1 como el espacio a dos parámetros $H_0^{1,2}$, donde $H_0^{1,2}$ es un caso particular de los espacios $H_S^{P,Q}$, los cuales son generalizaciones de los espacios estudiados por J. Peetre y H. Triebel.

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1 Introduction

Recent advances in the theory of product Hardy and BMO spaces (see [10], [11] and [18] for instance) have called the attention of many authors, which have achieve results about old and new problems of this rich area. One of these problems concern to the characterizations of Hardy spaces.

In a fundamental work within the theory of Hardy spaces H^p over product of semi planes (or with two parameters), R. F. Gundy and E. M. Stein [13] proved that two parameters space H^1 may be characterized by double and partial Hilbert transforms, using area integrals and maximal functions, with equivalent norms. After this initial work, several authors obtained other characterizations of the two parameters space H^1. For more details see, for instance, [3], [4], [6], [7], [14], [19], [20] and [21].

In this work a characterization of the Hardy space H^1 over product of semi-planes such as the two parameters space $H_0^{1,2}$ is obtained. This space is a particular case of the two parameters spaces $H_S^{P,Q}$ (when $S = (0,0)$, $P = (1,1)$ and $Q = (2,2)$). The $H_S^{P,Q}$ spaces are generalizations of the one parameter spaces $H_s^{p,q}$, studied by J. Peetre and H. Triebel.

For the one parameter case, a characterization of space H^1, such as that obtained in this work, was initially obtained by J. Peetre [15,16]. Later, H. Triebel obtained in [24] another proof by different arguments and after him, a new proof was achieved by J. L. Rubio de Francia, F. J. Ruiz and J. L. Torrea [17].

One of ingredients for the proof of the H^1 characterization obtained in this work, consists of the theorems about singular integral vector operators contained in [12].

2 Spaces $H^1(I\!R \times I\!R)$ and $H_S^{P,Q}(I\!R \times I\!R)$ in the Product Case

2.1 Notations. The notations and basic results used through this work are introduced here. The letter C always denotes a constant which may assume different values in a sequence of inequalities. $\mathcal{S}(I\!R^2)$ denotes the class of rapidly decreasing functions (at infinity). Let E be a Banach space. $\mathcal{S}'(I\!R^2, E)$ is the class of all continuous linear maps T defined over $\mathcal{S}(I\!R^2)$ with values in E (that is, if $\phi_j \to \phi$ in $\mathcal{S}(I\!R^2)$ then $T(\phi_j) \to T(\phi)$).

If E is a Banach space in relation to the norm $||.||_E$ and $P = (p_1, p_2)$ with $0 < p_1, p_2 \leq \infty$, $L^P(I\!R^2, E)$ is the space of all functions f defined over $I\!R^2$ with values in E, such that $\|f(x)\|_E$ is Lebesgue measurable, and

$$\|f\|_{L^P(I\!R^2,E)} = \left(\int_{I\!R} \left(\int_{I\!R} \|f(x)\|_E^{p_1} \, dx_1 \right)^{p_2/p_1} dx_2 \right)^{1/p_2} < \infty$$

with usual modifications when some of p_i are equal to ∞. We observe that if $p_i = p$, for $i = 1, 2$, then $L^P(I\!R^2, E) = L^p(I\!R^2, E)$.

To avoid any confusion, we write $L^P(E)$ and $\|.\|_{L^P(E)}$ instead of $L^P(I\!R^2, E)$ and $\|.\|_{L^P(I\!R^2,E)}$, and when $E = \mathbb{C}$, the field of complex numbers, L^P and $\|.\|_{L^P}$ are posited.

Given a Banach space E and $Q = (q_1, q_2)$ with $0 < q_1, q_2 \leq \infty$, the (multi-)sequence spaces $\ell^Q(\mathbb{Z}^2, E)$ ($\ell^Q(E)$ to avoid confusion) are defined in a analogous way.

If E is a Banach space, the *Fourier transform* of a function $f \in L^1(I\!R^2, E)$ is defined by

$$\mathcal{F}f(x) = \hat{f}(x) = \iint_{I\!R^2} e^{-2\pi i x\cdot y}\, f(y)\, dy \;\; ,$$

where $x \cdot y = x_1.y_1 + x_2.y_2$. The following notation is used,

$$\square = \{(0,0),\ (1,0),\ (0,1),\ (1,1)\}.$$

2.2 Definition. Let E be a Hilbert space and $f \in L^1(I\!R^2, E)$. Their Hilbert transforms $H_k f$, $k \in \square$, are the elements of $\mathcal{S}'(I\!R^2, E)$ defined by :

(1) $\mathcal{F}(H_{10}f) = -\mathrm{i}\ sgx\ \mathcal{F}(f)(x,y)$,

(2) $\mathcal{F}(H_{01}f) = -\mathrm{i}\ sgy\ \mathcal{F}(f)(x,y)$,

(3) $\mathcal{F}(H_{11}f) = (-\mathrm{i}\ sgx)(-\mathrm{i}\ sgy)\mathcal{F}(f)(x,y)$,

(4) $(H_{00}f) = f$.

Spaces $H^1(I\!R \times I\!R, E)$ and $BMO(I\!R \times I\!R, E)$ are defined, which generalize the product spaces $H^1(I\!R \times I\!R)$ and $BMO(I\!R \times I\!R)$ for the vectorial case.

2.3 Definition. Let E be a Hilbert space. $H^1(I\!R \times I\!R, E)$ is the vector space of function f in $L^1(I\!R^2, E)$ such that their Hilbert transforms, $H_k f$, $k \in \square \setminus \{(0,0)\}$, belong to $L^1(I\!R^2, E)$.

We equipped space $H^1(I\!R \times I\!R, E)$ with the norm:

$$\|f\|_{H^1(I\!R\times I\!R,E)} = \sum_{k\in\square} \|H_k f\|_{L^1(I\!R^2,E)} \;\; ,$$

where $H_{00}f = f$.

2.4 Definition. Let E be a Hilbert space. A function g from $I\!R^2$ to E belongs to $BMO(I\!R \times I\!R, E)$, if it may be represented as

$$g = \sum_{k\in\square} H_k g_k \, , \tag{1}$$

where $H_{00}g_{00} = g_{00}$ and $\sum_{k\in\Box} ||g_k||_{L^\infty(I\!R^2,E)} < \infty$. We equipped the space $BMO(I\!R\times I\!R, E)$ with the norm:

$$||g||_{BMO(I\!R\times I\!R,E)} = \inf\{\sum_{k\in\Box} ||g_k||_{L^\infty(I\!R^2,E)}\} ,$$

where the infimum takes over all representations of g in the form (1).

Chang-Fefferman proved in [6] that for real value functions, the product space $BMO(I\!R\times I\!R)$ is the dual of the product space $H^1(I\!R\times I\!R)$. This result is valid also for spaces $BMO(I\!R\times I\!R, E)$, where E is a Hilbert space; therefore, the product space $BMO(I\!R\times I\!R, E)$ is the dual of the product space $H^1(I\!R\times I\!R, E)$.

Results on the action of singular vector integral operators with product kernel over the product spaces $H^1(I\!R\times I\!R, E)$ and $BMO(I\!R\times I\!R, E)$ are given by the two following theorems. Proofs are provided in Gomes-Silva [12].

2.5 Theorem. Let E, F and G be Banach spaces and k_1 and k_2 kernels in $L^2_{loc}(I\!R^2, L(E, F))$ and $L^2_{loc}(I\!R^2, L(F, G))$, respectively, satisfying

$$\int_{|x-y'|>\gamma|y-y'|} \|k_j(x, y) - k_j(x, y')\|_{L_j}\, dx \le C\cdot\gamma^{-\delta} ,\ j = 1, 2, \tag{1}$$

for every $\gamma \ge 2$ and some $\delta > 0$, where $L_1 = L(E, F)$ and $L_2 = L(F, G)$. Let T_1 and T_2 be bounded linear operators from $L^2(I\!R, E)$ to $L^2(I\!R, F)$ and from $L^2(I\!R, F)$ to $L^2(I\!R, G)$, respectively, satisfying

$$T_1f(x) = \int_{I\!R} k_1(x, u)\, f(u)\, du \ \ , \tag{2}$$

for every $f \in L^2_c(I\!R, E)$, and

$$T_2f(y) = \int_{I\!R} k_2(y, v)\, f(v)\, dv \ \ , \tag{3}$$

for every $f \in L^2_c(I\!R, F)$. Let T be a linear operator from $L^2_c(I\!R^2, E)$ to $M(I\!R^2, G)$ satisfying

$$Tf(x, y) = \iint_{I\!R^2} k_2(y, v)\, k_1(x, u)\, f(u, v)\, du\, dv \ \ , \tag{4}$$

for every $f \in L^2_c(I\!R^2, E)$ and $(x, y) \notin \sup\ f$. Suppose that T has a bounded extension from $L^2(I\!R^2, E)$ to $L^2(I\!R^2, G)$. Then, T has a bounded extension from $H^1(I\!R\times I\!R, E)$ to $L^1(I\!R^2, G)$; that is, there exists a constant $C > 0$, such that

$$\|Tf\|_{L^1(I\!R^2,G)} \le C\, \|f\|_{H^1(I\!R\times I\!R,E)} ,$$

for all $f \in H^1(\mathbb{R} \times \mathbb{R}, E)$.

2.6 Theorem. Let E be a Banach space, F and G Hilbert spaces and k_1 and k_2 kernels in $L^1_{loc}(\mathbb{R}^2, L(E,F))$ and $L^1_{loc}(\mathbb{R}^2, L(F,G))$, respectively, satisfying

$$\int_{|x'-y|>\gamma|x-x'|} \|k_j(x,y) - k_j(x',y)\|_{L_j}\ dx \leq C \cdot \gamma^{-\delta}\ ,\ j = 1, 2, \qquad (1)$$

for every $\gamma \geq 2$ and some $\delta > 0$, where $L_1 = L(E,F)$ and $L_2 = L(F,G)$. Let T_1 and T_2 be bounded linear operators from $L^2(\mathbb{R}, E)$ to $L^2(\mathbb{R}, F)$ and from $L^2(\mathbb{R}, F)$ to $L^2(\mathbb{R}, G)$, respectively, satisfying

$$T_1 f(x) = \int_{\mathbb{R}} k_1(x,u)\ f(u)\ du\ \ , \qquad (2)$$

for every $f \in L^\infty_c(\mathbb{R}, E)$, and

$$T_2 f(y) = \int_{\mathbb{R}} k_2(y,v)\ f(v)\ dv\ \ , \qquad (3)$$

for every $f \in L^\infty_c(\mathbb{R}, F)$. Let T be a linear operator from $L^\infty_c(\mathbb{R}^2, E)$ to $M(\mathbb{R}^2, G)$ satisfying

$$Tf(x,y) = \iint_{\mathbb{R}^2} k_2(y,v)\ k_1(x,u)\ f(u,v)\ du\ dv\ \ , \qquad (4)$$

for every $f \in L^\infty_c(\mathbb{R}^2, E)$ and $(x,y) \notin \sup\ f$. Suppose that T has a bounded extension from $L^2(\mathbb{R}^2, E)$ to $L^2(\mathbb{R}^2, G)$. Then, T is a bounded linear operator from $L^\infty_c(\mathbb{R}^2, E)$ to $BMO(\mathbb{R} \times \mathbb{R}, G)$; that is, there exists a constant $C > 0$, such that

$$\|Tf\|_{BMO(\mathbb{R}\times\mathbb{R},G)} \leq C\, \|f\|_{L^\infty(\mathbb{R}^2,E)}\ ,$$

for all $f \in L^\infty_c(\mathbb{R}^2, E)$.

2.7 Lemma. There exists $\varphi \in S(\mathbb{R})$, such that

(1) $\sup\ \mathcal{F}\varphi = \{t \in \mathbb{R}\ \ :\ \ 2^{-1} \leq |t| \leq 2\}$;

(2) $|\mathcal{F}\varphi(t)| > 0 \quad se\ 2^{-1} < |t| < 2$;

(3) $\sum_{i=-\infty}^{\infty} \mathcal{F}\varphi(2^{-i}t) = 1 \quad se\ t \neq 0$.

For the proof see Berg-Löfströn [2]

2.8 System of Test Functions. Let φ be given as in the Lemma 2.7 and for each $i \in \mathbb{Z}$ let φ_i be the function given by $\varphi_i(t) = 2^i\varphi(2^i t)$. The family $(\varphi_i)_{i\in\mathbb{Z}}$ is called a system of test functions over $\mathbb{R}$. Since $\mathcal{F}\varphi_i(t) = \mathcal{F}\varphi(2^{-i}t)$ for each $i \in \mathbb{Z}$, and from 2.7(1), 2.7(2) and 2.7(3), it follows that

(1) $\sup \mathcal{F}\varphi_i = \{t \in \mathbb{R} \ : \ 2^{i-1} \le |t| \le 2^{i+1}\} \ ; \ i \in \mathbb{Z} \ ;$

(2) $|\mathcal{F}\varphi_i(t)| > 0 \quad \text{se } 2^{i-1} < |t| < 2^{i+1} \ ;$

(3) $\sum_{i=-\infty}^{\infty} \mathcal{F}\varphi_i(t) = 1 \quad se\ t \neq 0 \ .$

2.9 Definition. Let $S = (s_1, s_2)$, $P = (p_1, p_2)$ and $Q = (q_1, q_2)$, such that $s_n \in \mathbb{R}$, $0 < p_n < \infty$ and $0 < q_n \le \infty$, $n = 1,\ 2$. Let $(\varphi_i)_{i\in\mathbb{Z}}$ and $(\psi_j)_{j\in\mathbb{Z}}$ be systems of test functions as in 2.8. Then, $H_S^{P,Q}(\mathbb{R}\times\mathbb{R}) = H_S^{P,Q}(\mathbb{R}\times\mathbb{R}, \varphi, \psi)$ is the vector space of all functions f in $L^P(\mathbb{R}^2) \cap S'(\mathbb{R}^2)$ with real values, satisfying $(2^{s_1 i + s_2 j}\varphi_i\psi_j * f)_{ij} \in L^P(\ell^Q)$.

Spaces $H_S^{P,Q}(\mathbb{R}\times\mathbb{R})$ are equipped with the following quasi-norm (it is a norm if $min\ (p_1, p_2, q_1, q_2) \ge 1)$:

$$\|f\|^{\varphi,\psi}_{H_S^{P,Q}} = \|(2^{s_1 i + s_2 j}\varphi_i\psi_j * f)_{ij}\|_{L^P(\ell^Q)} \, . \tag{1}$$

To avoid any confusion, we simply denote $\|f\|^{\varphi,\psi}_{H_S^{P,Q}}$ by $\|f\|_{H_S^{P,Q}}$.

When $S = (s, s)$, $P = (p, p)$ and $Q = (q, q)$, then

$$\|f\|^{\varphi,\psi}_{H_S^{P,Q}} = \|(2^{s(i+j)}\varphi_i\psi_j * f)_{ij}\|_{L^p(\ell^q)}$$

and the space $H_S^{P,Q}(\mathbb{R}\times\mathbb{R})$ is simply denoted by $H_s^{p,q}(\mathbb{R}\times\mathbb{R})$.

The next result shows that the quasi-norm 2.9(1) is independent of the systems of test functions $(\varphi_i)_{i\in\mathbb{Z}}$ and $(\psi_j)_{j\in\mathbb{Z}}$.

2.10 Theorem. Let $(\alpha_i)_{i\in\mathbb{Z}}$, $(\beta_j)_{j\in\mathbb{Z}}$, $(\varphi_k)_{k\in\mathbb{Z}}$ and $(\psi_l)_{l\in\mathbb{Z}}$ be systems of test functions as in 2.8. Let S, P and Q, as in Definition 2.9. Then the quasi-norms $\|.\|^{\alpha,\beta}_{H_S^{P,Q}}$ and $\|.\|^{\varphi,\psi}_{H_S^{P,Q}}$ are equivalents, that is, there are positive constants C_1 and C_2, such that

$$C_1 \cdot \|f\|^{\alpha,\beta}_{H_S^{P,Q}} \le \|f\|^{\varphi,\psi}_{H_S^{P,Q}} \le C_2 \cdot \|f\|^{\alpha,\beta}_{H_S^{P,Q}} \, . \tag{1}$$

For the proof see Schmeisser-Triebel [22].

2.11 Remark. From the proof of Theorem 2.10 it follows that condition 2.8(3) of the systems $(\alpha_k)_k$ and $(\beta_l)_l$ is unnecessary, that is,

$$\sum_{k=-\infty}^{\infty} \mathcal{F}\alpha_k(t) = \sum_{l=-\infty}^{\infty} \mathcal{F}\beta_l(t) = 1 \quad , \quad t \neq 0 \quad ,$$

to obtain inequalities of the type

$$\|f\|^{\alpha,\beta}_{H^{P,Q}_S} \leq C \cdot \|f\|^{\varphi,\psi}_{H^{P,Q}_S}.$$

From systems $(\alpha_k)_k$ and $(\beta_l)_l$ another kind of condition may be demanded, such as,

$$\sum_{k=-\infty}^{\infty} [\mathcal{F}\alpha_k(t)]^2 = \sum_{l=-\infty}^{\infty} [\mathcal{F}\beta_l(t)]^2 = 1 \quad , \quad t \neq 0 \quad .$$

This will be considered in the next section.

3 The Characterization $H^1(I\!R \times I\!R) = H_0^{1,2}(I\!R \times I\!R)$

3.1 Lemma. Let $\varphi \in S(I\!R)$ such that $\hat{\varphi}(0) = 0$ and $|\hat{\varphi}(t)| > 0$ if $2^{-1} < |t| < 2$. Defining $\varphi_j(x) = 2^j\varphi(2^j x)$, $j \in Z\!\!Z$, one has

(1) $\sum_{j\in Z\!\!Z} |\hat{\varphi}_j(t)|^2 \leq C$;

(2) $\sum_{j\in Z\!\!Z} |\varphi_j(x)|^2 \leq C \cdot |x|^{-2}$;

(3) $(\sum_{j\in Z\!\!Z} |\varphi_j(x-y) - \varphi_j(x)|^2)^{\frac{1}{2}} \leq C \cdot \frac{|y|}{|x|^2}$, if $|x| > 2|y|$.

For the proof see Torrea [23].

3.2 Theorem. Let φ and ψ be as in the Lemma 3.1. Then

$$\|(\varphi_i\psi_j * f)_{ij}\|_{L^1(I\!R^2,\ell^2)} \leq C \cdot \|f\|_{H^1(I\!R\times I\!R)} \tag{1}$$

for all $f \in H^1(I\!R \times I\!R)$.

Proof. Let us consider the linear operator defined on $L^2_c(I\!R^2)$ by

$$Tf = (\varphi_i\psi_j * f)_{ij} \in M(I\!R^2, \ell^2(Z\!\!Z^2)).$$

The operator T is well defined: if $f \in L^2_c(I\!R^2)$, then, by Plancherel's Theorem and by 3.1(1),

$$\begin{aligned}
&\iint_{I\!R^2} \sum_{j\in Z\!\!Z} \sum_{i\in Z\!\!Z} |\varphi_i\psi_j * f(x,y)|^2 dxdy = \\
&= \sum_{j\in Z\!\!Z} \sum_{i\in Z\!\!Z} \iint_{I\!R^2} |\varphi_i\psi_j * f(x,y)|^2 dxdy \\
&= \sum_{j\in Z\!\!Z} \sum_{i\in Z\!\!Z} \iint_{I\!R^2} |\hat{\varphi}_i(s)\hat{\psi}_j(t)\hat{f}(s,t)|^2 dsdt \\
&= \iint_{I\!R^2} (\sum_{i\in Z\!\!Z} |\hat{\varphi}_i(s)|^2)(\sum_{j\in Z\!\!Z} |\hat{\psi}_j(t)|^2)|\hat{f}(s,t)|^2 dsdt \\
&\leq C \cdot \iint_{I\!R^2} |\hat{f}(s,t)|^2 dsdt = C \cdot \|f\|_{L^2(I\!R^2)} , \qquad (2)
\end{aligned}$$

thus, it follows

$$\sum_{j\in Z\!\!Z} \sum_{i\in Z\!\!Z} |\varphi_i\psi_j * f(x,y)|^2 < \infty$$

for almost all (x,y); that is, $Tf(x,y) \in \ell^2(Z\!\!Z^2)$. To show that Tf is a measurable function, it is enough to verify that the map

$$(x,y) \longrightarrow Tf(x,y).\alpha$$

is measurable for all $\alpha \in \ell^2(Z\!\!Z^2)$, since $\ell^2(Z\!\!Z^2)$ is separable. If $\alpha = (\alpha_{ij})_{ij}$

$$\begin{aligned}
Tf(x,y).\alpha &= \sum_{j\in Z\!\!Z} \sum_{i\in Z\!\!Z} (\varphi_i\psi_j * f(x,y))\alpha_{ij} \\
&= \sum_{j\in Z\!\!Z} \sum_{i\in Z\!\!Z} \alpha_{ij}\varphi_i\psi_j * f(x,y) \ ,
\end{aligned}$$

which is measurable. The inequality 3.2(2) shows that T is a bounded operator from $L^2(I\!R^2)$ to $L^2(I\!R^2, \ell^2(Z\!\!Z^2))$.

For each $n \in I\!N$, let us consider the operators T^n, T^n_1 and T^n_2 defined in the following way:

T^n is defined on $L^2_c(I\!R^2)$ by

$$T^n f = (\varphi_i\psi_j * f; \ -n \leq i,j \leq n) \in M(I\!R^2, \ell^2(Z\!\!Z^2)) \ ;$$

T^n_1 is defined on $L^2(I\!R)$ by

$$T^n_1 f = (\varphi_i * f; \ -n \leq i \leq n) \in M(I\!R, \ell^2(Z\!\!Z)) \ ;$$

T_2^n is defined on $L_c^2(\mathbb{R}, \ell^2(\mathbb{Z}))$ by

$$T_2^n g = (\psi_j * g_i; \;\; -n \leq i, j \leq n) \in M(\mathbb{R}, \ell^2(\mathbb{Z}^2)) .$$

Our next step it will be to show that these operators satisfy the hypothesis of Theorem 2.5. Analogously for operator T, it is easy to verify that for each $n \in \mathbb{N}$, T^n is well defined and $T^n f$ is a measurable function. Moreover, from 3.2(2) it follows that operators T^n are all bounded from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2, \ell^2(\mathbb{Z}^2))$, with $\|T^n\|$ bounded by a constant regardless of n.

The operators T_1^n are bounded from $L^2(\mathbb{R})$ to $L^2(\mathbb{R}, \ell^2(\mathbb{Z}))$ with $\|T_1^n\|$ bounded by a constant regardless n, since by the Plancherel's Theorem and 3.1(1),

$$\begin{aligned} \int_{\mathbb{R}} \sum_{i=-n}^{n} |\varphi_i * f(x)|^2 dx &= \sum_{i=-n}^{n} \int_{\mathbb{R}} |\hat{\varphi}_i(s)|^2 |\hat{f}(s)|^2 ds \\ &\leq C \cdot \int_{\mathbb{R}} |\hat{f}(s)|^2 ds \;=\; C \cdot \|f\|_{L^2(\mathbb{R})}. \end{aligned}$$

Now, for each $n \in \mathbb{N}$, the kernel k_1^n defined by

$$k_1^n(x) : \; \lambda \in \mathbb{C} \;\longrightarrow\; k_1^n(x).\lambda = (\varphi_i(x)\lambda; \;\; -n \leq i \leq n) \in \ell^2(\mathbb{Z})$$

is well defined, belongs to $L^2_{loc}(\mathbb{R}, L(\mathbb{C}, \ell^2(\mathbb{Z})))$, verifies the condition 2.5(1) with $L_1 = L(\mathbb{C}, \ell^2(\mathbb{Z}))$ and for all $f \in L_c^2(\mathbb{R})$,

$$T_1^n f(x) = \int_{\mathbb{R}} k_1^n(x-y) f(y) dy. \tag{3}$$

Indeed, k_1^n is well defined: if $\varphi \in S(\mathbb{R})$, then

$$\|k_1^n(x).\lambda\|_{\ell^2(\mathbb{Z})} = (\sum_{i=-n}^{n} |\varphi_i(x)|^2)^{\frac{1}{2}} |\lambda| \;\; \leq \;\; C(n)|\lambda| \tag{4}$$

for all $\lambda \in \mathbb{C}$ and all $x \in \mathbb{R}$. On the other hand, since $L(\mathbb{C}, \ell^2(\mathbb{Z}))$ is isometric in $\ell^2(\mathbb{Z})$, and the map

$$x \in \mathbb{R} \;\longrightarrow\; \sum_{i=-n}^{n} \alpha_i \varphi_i(x)$$

is measurable for all $\alpha = (\alpha_i)_i \in \ell^2(\mathbb{Z})$, it follows that k_1^n is measurable. Now, if $A \subset \mathbb{R}$ is a compact set, then by 3.2(4)

$$\int_A \|k_1^n(x)\|_{L_1}^2 dx \quad \leq \quad C(n)|A| < \infty \ ,$$

where $L_1 = L(\mathbb{C}, \ell^2(\mathbb{Z}))$. This shows that k_1^n belongs to $L^2_{loc}(\mathbb{R}, L_1)$. To prove 3.2(3), since the map

$$u \in \mathbb{R} \ \longrightarrow \ (\varphi_i(u)f(u); \ \ -n \leq i \leq n) \in \ell^2(\mathbb{Z})$$

is integrable, we have

$$\begin{aligned} T_1^n f(x) &= (\varphi_i * f(x); \ \ -n \leq i \leq n) \\ &= (\int_{\mathbb{R}} \varphi_i(x-u)f(u)du; \ \ -n \leq i \leq n) \\ &= \int_{\mathbb{R}} (\varphi_i(x-u)f(u); \ \ -n \leq i \leq n)du \\ &= \int_{\mathbb{R}} k_1^n(x-u)f(u)du. \end{aligned}$$

Finally, if $|x-u'| > \gamma|y-u'|$, with $\gamma \geq 2$, then by 3.1(3) we obtain

$$\begin{aligned} \|k_1^n(x-u) - k_1^n(x-u')\|_{L_1} &= (\sum_{i=-n}^{n} |\varphi_i(x-u) - \varphi_i(x-u')|^2)^{\frac{1}{2}} \\ &= (\sum_{i=-n}^{n} |\varphi_i(x-u'-(u-u')) - \varphi_i(x-u')|^2)^{\frac{1}{2}} \\ &\leq C \cdot \frac{|u-u'|}{|x-u|^2} \ , \end{aligned}$$

where C is a constant regardless of n. Therefore, the kernel k_1^n verifies the condition 2.5(1) with $L_1 = L(\mathbb{C}, \ell^2(\mathbb{Z}))$ and constant C regardless of n. The boundness of the operators T_2^n from $L^2(\mathbb{R}, \ell^2(\mathbb{Z}))$ to $L^2(\mathbb{R}, \ell^2(\mathbb{Z}^2))$, with $\|T_2^n\|$ bounded by a constant regardless of n, follows from 3.1(1) using an analogous reasoning which was done for T_1^n. Now, let us verify that for each $n \in \mathbb{N}$, there exists a kernel k_2^n in $L^2_{loc}(\mathbb{R}, L(\ell^2(\mathbb{Z}), \ell^2(\mathbb{Z}^2)))$, satisfying 2.5(1) with $L_2 = L(\ell^2(\mathbb{Z}), \ell^2(\mathbb{Z}^2))$, such that

$$T_2^n g(y) = \int_{\mathbb{R}} k_2^n(y-v)g(v)dv, \tag{5}$$

for all $g \in L^2_c(\mathbb{R}, \ell^2(\mathbb{Z}))$. Indeed, let k_2^n be defined by

$$k_2^n(y) : \alpha = (\alpha_i)_i \in \ell^2(\mathbb{Z}) \ \longrightarrow \ k_2^n(y).\alpha = (\psi_j(y).\alpha_i; \ \ -n \leq i,j \leq n) \in \ell^2(\mathbb{Z}^2).$$

This function is well defined: if $\psi \in S(I\!R)$, it follows that

$$\begin{aligned} \|k_2^n(y).\alpha\|_{\ell^2(\mathbb{Z}^2)} &= (\sum_{j=-n}^{n}\sum_{i=-n}^{n}|\psi_j(y)\alpha_i|^2)^{\frac{1}{2}} \\ &\leq (\sum_{j=-n}^{n}|\psi_j(y)|^2)^{\frac{1}{2}}(\sum_{i\in\mathbb{Z}}|\alpha_i|^2)^{\frac{1}{2}} \\ &\leq C(n).\|\alpha\|_{\ell^2(\mathbb{Z})} \end{aligned} \tag{6}$$

for all $\alpha = (\alpha_i)_i \in \ell^2(\mathbb{Z})$ and all $y \in I\!R$. The measurability of k_2^n follows from $k_2^n = \sum_{i,j=-n}^{n} k_{2,ij}^n$, where each $k_{2,ij}^n$ is defined by

$$k_{2,ij}^n(y).\alpha = (.....,0,\psi_j(y)\alpha_i,0,.....)$$

and it is measurable, since each ψ_j is measurable. The fact that $\|k_2^n(y)\|_{L_2}^2$ is locally integrable, where $L_2 = L(\ell^2(\mathbb{Z}),\ell^2(\mathbb{Z}^2))$, follows from 3.2(6). The verification of 3.2(5) is analougous to 3.2(3). As in case of the kernel k_1^n, it follows from 3.1(3) that

$$\|k_2^n(y-v) - k_2^n(y-v')\|_{L_2} \leq C \cdot \frac{|v-v'|}{|y-v'|^2}$$

where C is a constant regardless of n. Then, get k_2^n satisfies 2.5(1) with constant regardless of n. To complete the proof that operators T^n, T_1^n and T_2^n satisfy the hypothesis of Theorem 2.5, we observe that the map

$$(u,v) \in I\!R^2 \longrightarrow (\varphi_i(u)\psi_j(v)f(u,v);\ \ -n \leq i,j \leq n) \in \ell^2(\mathbb{Z}^2)$$

is integrable when $f \in L_c^2(I\!R^2)$; then we have

$$T^n f(x,y) = \iint_{I\!R^2} k_2^n(y-v)k_1^n(x-u)f(u,v)dudv$$

for all $f \in L_c^2(I\!R^2)$.

Therefore, by Theorem 2.5,

$$\|T^n f\|_{L^1(I\!R^2,\ell^2(\mathbb{Z}^2))} \leq C \cdot \|f\|_{H^1(I\!R\times I\!R)} \tag{7}$$

for all n and all $f \in H^1(I\!R \times I\!R)$, where C is a constant regardless of n. Finally, applying the theorem of monotone convergence in 3.2(7), 3.2(1) is obtained as requested.

As a consequence of Theorem 3.2, the following result gives us a part of the characterization $H^1(I\!R \times I\!R) = H_0^{1,2}(I\!R \times I\!R)$.

3.3 Corollary. The space $H^1(I\!R \times I\!R)$ is continuously embedded in $H_0^{1,2}(I\!R \times I\!R)$, that is, there is a positive constant C, such that

$$\|f\|_{H_0^{1,2}(I\!R\times I\!R)} \leq C \cdot \|f\|_{H^1(I\!R\times I\!R)}$$

for all $f \in H^1(I\!R \times I\!R)$.

Proof. It is enough to observe the test functions used to define the space $H_0^{1,2}(I\!R\times I\!R)$ satisfies the hypothesis of the Theorem 3.2.

The next theorem will be fundamental to prove the contrary immersion in the Corollary 3.3; that is, the space $H_0^{1,2}(I\!R \times I\!R)$ is continuously embedded in the space $H^1(I\!R \times I\!R)$.

3.4 Theorem. Let φ and ψ be given as in the Lemma 3.1. Then

$$\|(\varphi_i\psi_j * f)_{ij}\|_{BMO(I\!R\times I\!R,\ell^2)} \leq C \cdot \|f\|_{L^\infty(I\!R^2)}\,,$$

for all $f \in L_c^\infty(I\!R^2)$, where $BMO(I\!R\times I\!R, \ell^2)$ is the topological dual of the space $H^1(I\!R\times I\!R, \ell^2)$.

Proof. It is enough to follow the proof of Theorem 3.2, using in this case Theorem 2.6.

3.5 Theorem. Let $\mathcal{O}$ be the space of the real functions $f \in S(I\!R^2)$ with real values, such that

(1) $\hat{f} \in C_c^\infty(I\!R^2)$,

(2) sup $\hat{f} \cap [(I\!R \times \{0\}) \cup (\{0\} \times I\!R)] = \emptyset$.

Then $\mathcal{O}$ is a dense subspace of $H_0^{1,2}(I\!R \times I\!R)$.

Proof. It is enough to adapt the arguments used by H. Sato in [19] to obtain a dense subspace of $H^1(I\!R \times I\!R)$.

3.6 Theorem. A function f in $L^1(I\!R^2)$ belongs to $H^1(I\!R \times I\!R)$ if, and only if f belongs to $H_0^{1,2}(I\!R \times I\!R)$. Moreover, there is a constant $C > 0$, such that

$$C^{-1}.\|f\|_{H^1(I\!R\times I\!R)} \leq \|f\|_{H_0^{1,2}(I\!R\times I\!R)} \leq C \cdot \|f\|_{H^1(I\!R\times I\!R)}. \tag{1}$$

Proof. It is enough to prove the first inequality in 3.6(1), since the second was proved in Corollary 3.3. Let $f \in \mathcal{O}$ and $g \in L_c^\infty(I\!R^2)$ such that $\|g\|_{L^\infty(I\!R^2)} \leq 1$. Let $\alpha = (\alpha_i)_{i\in \mathbb{Z}}$ e $\beta = (\beta_j)_{j\in\mathbb{Z}}$ systems of test functions as given in 2.8, but with the condition 2.8(3) replaced by $\sum_{i=-\infty}^{\infty}[\hat{\alpha}_i(s)]^2 = 1$, $s \neq 0$ and $\sum_{j=-\infty}^{\infty}[\hat{\beta}_j(t)]^2 = 1$, $t \neq 0$ (see remark 2.11). Thus, using the polarization formula and Plancherel's Theorem,

$$\begin{aligned}
&\iint_{I\!R^2} f(x,y)g(x,y)dxdy = \qquad (2)\\
&= \frac{1}{4}[\iint_{I\!R^2}|f+g|^2dxdy - \iint_{I\!R^2}|f-g|^2dxdy]\\
&= \frac{1}{4}[\iint_{I\!R^2}\sum_{i=-\infty}^{\infty}[\hat{\alpha}_i(s)]^2\sum_{j=-\infty}^{\infty}[\hat{\beta}_j(t)]^2|\mathcal{F}(f+g)|^2dsdt -\\
&\quad - \iint_{I\!R^2}\sum_{i=-\infty}^{\infty}[\hat{\alpha}_i(s)]^2\sum_{j=-\infty}^{\infty}[\hat{\beta}_j(t)]^2|\mathcal{F}(f-g)|^2dsdt]\\
&= \frac{1}{4}[\sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty}\iint_{I\!R^2}|\mathcal{F}(\alpha_i\beta_j*(f+g))|^2dsdt -\\
&\quad - \sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty}\iint_{I\!R^2}|\mathcal{F}(\alpha_i\beta_j*(f-g))|^2dsdt]\\
&= \iint_{I\!R^2}\frac{1}{4}[\sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty}(|\alpha_i\beta_j*(f+g)|^2 - |\alpha_i\beta_j*(f-g)|^2)]dxdy\\
&= \iint_{I\!R^2}\sum_{j=-\infty}^{\infty}\sum_{i=-\infty}^{\infty}(\alpha_i\beta_j*f)(\alpha_i\beta_j*g)dxdy.
\end{aligned}$$

Now, by Theorem 3.4,

$$\|(\alpha_i\beta_j*g)_{ij}\|_{BMO(I\!R\times I\!R,\ell^2)} \leq C\cdot\|g\|_{L^\infty(I\!R^2)}\,, \qquad (3)$$

for all $g \in L_c^\infty(I\!R^2)$. This shows that $(\alpha_i\beta_j*g)_{ij} \in BMO(I\!R\times I\!R, \ell^2(\mathbb{Z}^2))$. On the other hand, if we denote by H the Hilbert transform in one variable and with the convention $H_0\varphi = \varphi$ and $H_1\varphi = H\varphi$, then for each $k = (l,m) \in \Box$, we have

$$H_k(\alpha_i\beta_j*f)(x,y) = (H_l\alpha_i\; H_m\beta_j*f)(x,y)\,. \qquad (4)$$

It is enough to prove to $k = (1,0)$, since the another cases are similar. Indeed, by Definition 2.2,

$$\begin{aligned}
\mathcal{F}[H_{10}(\alpha_i\beta_j*f)](s,t) &= -\mathrm{i}\; sgs\; \mathcal{F}(\alpha_i\beta_j*f)(s,t)\\
&= -\mathrm{i}\; sgs\; \hat{\alpha}_i(s)\hat{\beta}_j(t)\hat{f}(s,t)\\
&= \mathcal{F}(H\alpha_i)(s)\;\hat{\beta}_j(t)\;\hat{f}(s,t)\\
&= \mathcal{F}[H\alpha_i.\beta_j*f](s,t)\,,
\end{aligned}$$

and 3.6(4) is obtained to $k=(1,0)$.
Moreover, the sequences $(H\alpha_i)_{i\in\mathbb{Z}}$ and $(H\beta_j)_{j\in\mathbb{Z}}$ are systems of test functions satisfying 2.8(1) and 2.8(2), since $\mathcal{F}[H\alpha_i](s)=-\mathrm{i}\ sgs\ \hat{\alpha}_i(s)$ and $\mathcal{F}[H\beta_j](t)=-\mathrm{i}\ sgt\ \hat{\beta}_j(t)$. Thus, taking into account the Remark 2.11, Theorem 2.10 is applied to obtain

$$\begin{aligned}\|(\alpha_i\beta_j * f)_{ij}\|_{H^1(\mathbb{R}\times\mathbb{R},\ell^2)} &= \sum_{k\in\square}\|(H_k(\alpha_i\beta_j * f))_{ij}\|_{L^1(\mathbb{R}^2,\ell^2)} \\ &= \sum_{(l,m)\in\square}\|(H_l\alpha_i\ H_m\beta_j * f)_{ij}\|_{L^1(\mathbb{R}^2,\ell^2)} \\ &\le C\cdot\|f\|_{H_0^{1,2}(\mathbb{R}\times\mathbb{R})}\ .\end{aligned} \tag{5}$$

This shows that $(\alpha_i\beta_j * f)_{ij}\in H^1(\mathbb{R}\times\mathbb{R},\ell^2(\mathbb{Z}^2))$. Using 3.6(2), 3.6(3) and 3.6(5) and using the fact that $BMO(\mathbb{R}\times\mathbb{R},\ell^2(\mathbb{Z}^2))$ is the dual of $H^1(\mathbb{R}\times\mathbb{R},\ell^2(\mathbb{Z}^2))$,

$$\begin{aligned}\left|\iint_{\mathbb{R}^2} f.g\,dxdy\right| &\le C\cdot\|(\alpha_i\beta_j * f)_{ij}\|_{H^1(\mathbb{R}\times\mathbb{R},\ell^2)}\ \ \|(\alpha_i\beta_j * g)_{ij}\|_{BMO(\mathbb{R}\times\mathbb{R},\ell^2)} \\ &\le C\cdot\|f\|_{H_0^{1,2}(\mathbb{R}\times\mathbb{R})}.\end{aligned}$$

Taking the supremum over all functions g in $L^\infty(\mathbb{R}^2)$, such that $\|g\|_{L^\infty(\mathbb{R}^2)}\le 1$,

$$\|f\|_{L^1(\mathbb{R}^2)}\le C\cdot\|f\|_{H_0^{1,2}(\mathbb{R}\times\mathbb{R})} \tag{6}$$

for all $f\in\mathcal{O}$.

If f belongs to $\mathcal{O}$, then $H_k f$ belongs too, for each $k=(l,m)\in\square$. Therefore, 3.6(6) and Theorem 2.10 implies

$$\begin{aligned}\|f\|_{H^1(\mathbb{R}\times\mathbb{R})} &= \sum_{k\in\square}\|H_k f\|_{L^1(\mathbb{R}^2)} \\ &\le C\cdot\sum_{k\in\square}\|H_k f\|_{H_0^{1,2}(\mathbb{R}\times\mathbb{R})} \\ &= C\cdot\sum_{k\in\square}\|(\varphi_i\psi_j * H_k f)_{ij}\|_{L^1(\mathbb{R}^2,\ell^2)} \\ &= C\cdot\sum_{(l,m)\in\square}\|(H_l\varphi_i\ H_m\psi_j * f)_{ij}\|_{L^1(\mathbb{R}^2,\ell^2)} \\ &\le C\cdot\|(\varphi_i\psi_j * f)_{ij}\|_{L^1(\mathbb{R}^2,\ell^2)} \\ &= C\cdot\|f\|_{H_0^{1,2}(\mathbb{R}\times\mathbb{R})}\ ,\end{aligned} \tag{7}$$

for all $f\in\mathcal{O}$. Finally, we may prove the inequality 3.6(7) is true for all $f\in H_0^{1,2}(\mathbb{R}\times\mathbb{R})$. Let $f\in H_0^{1,2}(\mathbb{R}\times\mathbb{R})$. By Theorem 3.5, $\mathcal{O}$ is dense in $H_0^{1,2}(\mathbb{R}\times\mathbb{R})$; then there exists a sequence $(f_n)_n$ of elements of $\mathcal{O}$ such that $f_n\to f$ in the norm of $H_0^{1,2}(\mathbb{R}\times\mathbb{R})$, from which it follows $(f_n)_n$ is a

Cauchy sequence in $H_0^{1,2}(I\!R \times I\!R)$. From 3.6(7) $(f_n)_n$ is a Cauchy sequence in $H^1(I\!R \times I\!R)$, from which results there exists an element $g \in H^1(I\!R \times I\!R)$ such that $f_n \to g$ in the norm of $H^1(I\!R \times I\!R)$, since $H^1(I\!R \times I\!R)$ is a complete space. By Corollary 3.3, $H^1(I\!R \times I\!R)$ is continuously embedded in $H_0^{1,2}(I\!R \times I\!R)$; then $f_n \to g$ in the norm of $H_0^{1,2}(I\!R \times I\!R)$ and hence $g = f$. Thus, for all $\varepsilon > 0$, there is $n \in I\!N$, such that $\|f_n - f\|_{H^1(I\!R\times I\!R)} < \varepsilon$ and $\|f_n - f\|_{H_0^{1,2}(I\!R\times I\!R)} < \varepsilon$. Therefore, by 3.6(7),

$$\begin{aligned}\|f\|_{H^1(I\!R\times I\!R)} &\leq \|f - f_n\|_{H^1(I\!R\times I\!R)} + \|f_n\|_{H^1(I\!R\times I\!R)} \\ &< \varepsilon + C \cdot \|f_n - f\|_{H_0^{1,2}(I\!R\times I\!R)} + C \cdot \|f\|_{H_0^{1,2}(I\!R\times I\!R)} \\ &< (C+1)\varepsilon + C \cdot \|f\|_{H_0^{1,2}(I\!R\times I\!R)}\,,\end{aligned}$$

for all $\varepsilon > 0$ and $f \in H_0^{1,2}(I\!R \times I\!R)$. This implies that 3.6(7) is true for all $f \in H_0^{1,2}(I\!R \times I\!R)$. Proof is complete.

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