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Small Data Global Existence and Scattering for the Mass-Critical Nonlinear Schrödinger Equation with Power Convolution in $\mathbb{R}^3$

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ABSTRACT

The main purpose of the present paper is to consider the well-posedness of the L^2-critical nonlinear Schrödinger equation of a Hartree type

$$i\partial_t\psi + \triangle\psi = (|x|^{-1} * |\psi|^{\frac{8}{3}})\psi, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3.$$

More precisely, we shall establish the local existence of solutions for initial data ψ_0 in $L^2(\mathbb{R}^3)$, as well as the existence of global solutions for small initial data. Moreover, we shall prove the existence of scattering operator.

RESUMEN

El principal objetivo del artículo es considerar si la ecuación de Schrödinger no lineal L^2- crítica del tipo Hartree

$$i\partial_t\psi + \triangle\psi = (|x|^{-1} * |\psi|^{\frac{8}{3}})\psi, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3.$$

está bien puesta o no. En efecto, estableceremos la existencia local de soluciones para datos iniciales ψ_0 en $L^2(\mathbb{R}^3)$, así como la existencia de soluciones globales para datos iniciales pequeños. Más aún, probaremos la existencia del operador de scattering.

Key words and phrases: *Nonlinear Schrödinger equation, power convolution, Hartree equation, local and global existence.*

Math. Subj. Class.: *35A05, 35Q55.*

1 Introduction

In this paper we consider the Cauchy problem for the defocussing mass-critical nonlinear Schrödinger equation of a Hartree type

$$i\partial_t\psi + \triangle\psi = (|x|^{-1} * |\psi|^{\frac{8}{3}})\psi, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \tag{1.1}$$

$$\psi(0,x) = \psi_0(x), \tag{1.2}$$

where $*$ denotes the usual convolution operator in $\mathbb{R}^3$. Here $\psi = \psi(t,x)$ is a complex valued function and the initial value $\psi_0 : \mathbb{R}^3 \mapsto \mathbb{C}$ is given.

Equation (1.1) can be written in terms of the wave function ψ and the potential V as the Schrödinger-Poisson system of the form

$$i\partial_t\psi + \triangle\psi = V\psi, \tag{1.3}$$

$$\triangle V = -4\pi|\psi|^{\frac{8}{3}}, \tag{1.4}$$

where the $(-)$ sign in the Poisson equation (1.4) corresponds to the repulsive type interaction. Equation (1.1) is known as the Schrödinger equation with nonlocal power nonlinearity of a Hartree type.

The main purpose of the present work is to obtain local and global existence, well-posedness and scattering of solutions to (1.1)–(1.2). When studying the problem of local and global existence of solution to the nonlinear Schrödinger equation, one is primarily interested in the scaling symmetry of the equation under the transformation

$$\psi_\lambda(t,x) = \frac{1}{\lambda^a}\psi(\frac{t}{\lambda^2}, \frac{x}{\lambda}), \quad \lambda > 0, \tag{1.5}$$

with some constant a depending on the equation. On the other hand, after the above scaling the L^p-norm has dimension, namely

$$\|\psi_\lambda\|_{L^p(\mathbb{R}^3)} = \frac{1}{\lambda^{a-\frac{3}{p}}}\|\psi\|_{L^p(\mathbb{R}^3)}. \tag{1.6}$$

A simple calculation shows that for $a = \frac{3}{2}$ the scaling transformation (1.5) leaves equation (1.1) unperturbed and, in addition, preserves the L^2-norm.

The above scaling symmetry of (1.1) is closely related to the so-called pseudoconformal symmetry,

$$\mathcal{P}[\psi](\tau,y) = \phi(\tau,y) = \frac{1}{\tau^{\frac{3}{2}}}e^{i\frac{y^2}{4\tau}}\overline{\psi\left(\frac{1}{\tau}, \frac{y}{\tau}\right)}. \tag{1.7}$$

This is a symmetry in the sense that, if $\psi(t,x)$ is a solution to (1.1) on $(t,x) \in [t_1,t_2] \times \mathbb{R}^3$, then $\phi(\tau,y)$ is a solution to the same equation on $(\tau,y) \in [\frac{1}{t_2}, \frac{1}{t_1}] \times \mathbb{R}^3$. In general, the scaling and the pseudoconformal symmetries (1.5) and (1.7) relate (1.1) to a wide class of equations, referred to as the mass-critical (L^2-critical or pseudoconformal) nonlinear Schrödinger equations. The name comes from the fact that the transforms (1.5) and (1.7) leave both the equation and the mass (the L^2-norm) invariant. Mass is one of the basic structures used in physics and is defined by

$$M(\psi(t)) = \int_{\mathbb{R}^3} |\psi(t,x)|^2 dx. \tag{1.8}$$

For (1.1), we shall prove (see Corollary 2.4 bellow) that the mass is a conserved quantity, i.e.

$$M(\psi(t)) = \|\psi(t)\|^2_{L^2(\mathbb{R}^3)} = \|\psi_0\|^2_{L^2(\mathbb{R}^3)} = M(\psi_0). \tag{1.9}$$

As in the papers [1,4,11,13,18–20], our results make use of mixed spaces of the type $L^q([0,T], L^r(\mathbb{R}^3))$ for admissible q and r. Thus, we make the following definition.

Definition 1. *We say that the pair (q,r) of exponents is Schrödinger–admissible if q and r satisfy*

$$\frac{2}{q} = 3\Big(\frac{1}{2} - \frac{1}{r}\Big) \quad , \quad 2 \le q \le \infty. \tag{1.10}$$

In the frame of the mass-critical NLS, equation (1.1) is similar to the Schrödinger equation with local (pure power) nonlinearity, which in an arbitrary spatial dimension $n \ge 1$ has the form

$$i\partial_t\psi + \triangle\psi = |\psi|^{\frac{4}{n}}\psi, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n. \tag{1.11}$$

The problems of global existence and well-posedness for solutions to (1.11) have been intensively studied, see for example [1–4, 18–21]. The local theory for (1.11) is due to Cazenave and Weissler, who in [4] constructed local-in-time solutions for arbitrary initial data in $L^2(\mathbb{R}^n)$ and also constructed global solutions for small initial data. Their results can be summarized as follows: Given $\psi_0 \in L^2(\mathbb{R}^n)$, there exists a unique local solution ψ to (1.11) with $\psi(0,x) = \psi_0(x)$. The solution ψ has a conserved mass $M(\psi(t)) = M(\psi_0)$. Moreover, if $M(\psi_0)$ is sufficiently small depending on n, then ψ is a global solution and

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}^3} |\psi(t,x)|^{\frac{2(n+2)}{n}} dxdt \le M(\psi_0). \tag{1.12}$$

The condition that $\psi \in L^{2(n+2)/n}_{t,x}$ is natural for (1.11). This balanced space appears in the original Strichartz inequality [17] and it is necessary in order to ensure local existence and uniqueness of solutions to (1.11).

Following the strategy developed for (1.11), we aim to establish the local well-posedness theory for (1.1) and to construct global solutions for sufficiently small L^2-initial data. More precisely we shall use the following

Definition 2. *A function $\psi : [0,T^*) \times \mathbb{R}^3 \mapsto \mathbb{C}$, $0 < T^* \le \infty$ is a $L^2(\mathbb{R}^3)$ solution to (1.1) if $\psi \in C^0([0,T], L^2(\mathbb{R}^3)) \cap L^{14/3}([0,T], L^{14/5}(\mathbb{R}^3))$ for $0 < T < T^*$, and we have the Duhamel's integral representation*

$$\psi(t) = U(t)\psi_0 - i\int_0^t U(t-s)(|x|^{-1} * |\psi(s)|^{\frac{8}{3}})\psi(s)ds, \tag{1.13}$$

for any $t \in [0,T]$. *Here* $U(t) = e^{it\Delta}$ *is the free Schrödinger evolution group defined via the Fourier transform*

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x)dx, \tag{1.14}$$

by

$$\widehat{e^{it\Delta} f}(\xi) = e^{-it|\xi|^2} f(\xi). \tag{1.15}$$

We say that ψ *is a global solution to* (1.1) *if* $T^* = \infty$.

The first main result of the present paper is the following

Theorem 1.1. *For every initial data* $\psi_0 \in L^2(\mathbb{R}^3)$ *there exists a unique maximal solution* $\psi \in C^0([0,T^*), L^2(\mathbb{R}^3)) \cap L^{\frac{14}{3}}([0,T^*), L^{\frac{14}{5}}(\mathbb{R}^3))$ *of* (1.1). *Furthermore:*

(i) $\psi \in L^q([0,T], L^r(\mathbb{R}^3))$, *for* $0 < T < T^*$ *and every admissible pair* (q,r);

(ii) *the mass is conserved, i.e.* $M(\psi(t)) = M(\psi_0)$ *for* $t \in [0,T^*)$;

(iii) *there exists a constant* $\varepsilon > 0$ *sufficiently small, such that if* $\|\psi_0\|_{L^2(\mathbb{R}^3)} < \varepsilon$, *then* $T^* = \infty$ *and* $\psi \in L^q(\mathbb{R}_+, L^r(\mathbb{R}^3))$ *for every admissible pair* (q,r);

(iv) *if* $T^* < \infty$, *then* $\|\psi\|_{L^q([0,T^*),L^r(\mathbb{R}^3))} = \infty$ *for every* $r > 14/5$;

(v) ψ *depends continuously on the initial data* $\psi_0 \in L^2(\mathbb{R}^3)$ *in the space* $\psi \in C^0([0,T^*), L^2(\mathbb{R}^3)) \cap L^{\frac{14}{3}}([0,T^*), L^{\frac{14}{5}}(\mathbb{R}^3))$.

There exists an extensive literature on the scattering theory for the Schrödinger equation with convolution nonlinearity [8, 14, 15] and for the Hartree equation [5–7, 9, 10], of which the existence of a wave operator is the question of crucial importance. Let $v(t) = U(t)\psi_+$ be a solution to the free Schrödinger equation

$$i\partial_t v + \triangle v = 0, \tag{1.16}$$

with initial data $\psi_+ \in X$ (called the asymptotic state), where $X = X_{\psi_0}$ is a suitable Banach space, depending on the initial data. That question can be formulated as follows. Does there exist a solution of (1.1)–(1.2), which behaves asymptotically as v when $t \to \infty$ in a suitable sense, depending on the choice of the space X? If that is the case, then the map $\Omega_+ : X \mapsto X$ is called the wave operator for positive times. In other words, a global strong X-solution ψ to the nonlinear equation (1.1) with an initial data ψ_0 scatters in X to a solution $v(t) = U(t)\psi_+$ if we have

$$\lim_{t\to\infty} \|\psi(t) - U(t)\psi_+\|_X = 0, \tag{1.17}$$

or equivalently (by using the unitarity of $U(t)$)

$$\lim_{t\to\infty} \|U(-t)\psi(t) - \psi_+\|_X = 0. \tag{1.18}$$

Suppose that for every asymptotic state $\psi_+ \in X$, there exists a unique initial data $\psi_0 \in X$, whose corresponding X-wellposed solution is global and scatters to $v(t)$ as $t \to \infty$. Then, we can define the wave operator $\Omega_+ : X \mapsto X$ in the sense of the space X by

$$\Omega_+ \psi_+ = \psi_0. \tag{1.19}$$

The problem of the existence of ψ for given ψ_+ is referred to as the problem of existence of the wave operator. When the wave operator Ω_+ is injective, we say that the Cauchy problem (1.1)–(1.2) is asymptotically complete in X. The same problem can be constructed for negative times, but for definiteness, hereinafter, we shall restrict our attention only to positive time.

A standard way to construct the wave operator Ω_+ consists in solving the Cauchy problem for (1.1) with initial data ψ_+ at $t = \infty$ in the form of the integral equation

$$\psi(t) = U(t)\psi_0 + i \int_t^\infty U(t-s)(|x|^{-1} * |\psi(s)|^{\frac{8}{3}})\psi(s)ds. \tag{1.20}$$

One usually solves (1.20) by a contraction method in a neighborhood of infinity in time (or in the time interval $[T, \infty)$ for T sufficiently large) and then continues that solution to all times. Thus, the problem is an immediate consequence of the global well-posedness and uses the results of Theorem 1.1.

With our second main result, we shall construct scattering theory in $L^2(\mathbb{R}^3)$ with small initial data. In fact, we shall prove the following

Theorem 1.2. *Let $\varepsilon > 0$ be sufficiently small and consider the ball $B_\varepsilon = \{\psi \in L^2(\mathbb{R}^3); \|\psi\|_{L^2} < \varepsilon\}$.*

Let $\psi \in C^0([0, T^*), L^2(\mathbb{R}^3)) \cap L^{\frac{14}{3}}([0, T^*), L^{\frac{14}{5}}(\mathbb{R}^3))$ be the unique maximal solution of (1.1)*, given by part(iii) of Theorem 1.1. Then we have:*

(i) *for any $\psi_\pm \in B_\varepsilon$, there exists a unique $\psi_0 \in B_\varepsilon$, such that*

$$\lim_{t \to \pm\infty} \|U(-t)\psi(t) - \psi_\pm\|_{L^2} = 0; \tag{1.21}$$

(ii) *for any $\psi_0 \in B_\varepsilon$, there exists unique $\psi_\pm \in B_\varepsilon$, such that* (1.21) *is satisfied;*

(iii) *the wave operators $\Omega_\pm : \psi_\pm \mapsto \phi_0$ and the scattering operator $S = \Omega_+^{-1} \circ \Omega_-$ are homeomorphisms from B_ε onto itself and isometric in the $L^2(\mathbb{R}^3)$ norm.*

The paper is organized as follows. In Section 2 we state some useful results and prove Theorem 1.1. In Section 3 we prove Theorem 1.2 and give some generalization notes about the scattering problems for (1.1).

We shall conclude this section by giving some of the notations, used in the paper. As usual, $L^r(\mathbb{R}^n) = \{\varphi \in \mathcal{S}'; \|\varphi\|_{L^r} < \infty\}$, where $\|\varphi\|_{L^r} = (\int |\varphi(x)|^r dx)^{\frac{1}{r}}$ if $1 \le r < \infty$ and $\|\varphi\|_{L^\infty} =$ ess.sup$\{|\varphi(x)|; x \in \mathbb{R}^n\}$ if $r = \infty$. We use r' for denoting the exponent dual to r and defined by $1/r + 1/r' = 1$. Given Lebesgue exponents q, r and a function $f(t,x)$ in $L^q(\mathbb{R}, L^r(\mathbb{R}^3))$, we write $\|f\|_{L^q(\mathbb{R}, L^r(\mathbb{R}^3))} = (\int \|f(t)\|^q_{L^r(\mathbb{R}^3)} dt)^{\frac{1}{q}}$.

2 The local existence result

We shall start this section by collecting some preliminaries and useful results.

As we shall see, the $L_t^{\frac{14}{3}} L_x^{\frac{14}{5}}$ norm in space-time plays a fundamental role. This is better understood if we recall some of the estimates available for the corresponding linear problem. We begin by recalling the following properties of the free Schrödinger evolution group $U(t) = e^{it\triangle}$ (see for instance [12, 13, 22]).

Lemma 2.1. *Let (q, r) be an admissible pair. Then, for every $\varphi \in L^2(\mathbb{R}^3)$ the following estimate holds*

$$\|U(t)\varphi\|_{L^q(\mathbb{R},L^r(\mathbb{R}^3))} \leq C_0\|\varphi\|_{L^2(\mathbb{R}^3)}. \tag{2.1}$$

Moreover, for every admissible pair (θ, ρ) and $f \in L^{\theta'}([0, T], L^{\rho'}(\mathbb{R}^3))$ we have

$$\left\| \int_0^{\cdot} U(\cdot - s)f(s)ds \right\|_{L^q([0,T],L^r(\mathbb{R}^3))} \leq C\|f\|_{L^{\theta'}([0,T],L^{\rho'}(\mathbb{R}^3))}, \tag{2.2}$$

for $0 < T \leq \infty$. Here the constants $C_0, C > 0$ and depend only on the spatial exponents r and ρ.

The classical Strichartz estimates for the Schrödinger equation [17] are one of the main tools in the study of local and global existence, time decay and scattering both for the linear and the nonlinear equation, due to the fact that they fit the assumptions required by the contraction argument (see for instance [5, 13, 17, 18, 22]).

The Strichartz type estimates for the inhomogeneous Schrödinger equation

$$i\partial_t v(t, x) + \triangle v(t, x) = F(t, x), \quad v(0, x) = f(x), \tag{2.3}$$

in $\mathbb{R}_+ \times \mathbb{R}^3$ are given, up to the end-point, namely the pair $(q, r) = (2, 6)$ by the following result due to Keel and Tao [13].

Lemma 2.2. *If (q, r) and $(\tilde{q}, \tilde{r})$ satisfy* (1.10)*, then the solution to the Cauchy problem* (2.3) *satisfies the estimate*

$$\begin{aligned} &\|v\|_{L^q([0,T^*),L^r(\mathbb{R}^3))} + \|v\|_{L^\infty([0,T^*),L^2(\mathbb{R}^3))} \\ &\qquad\qquad \leq C(\|F\|_{L^{\tilde{q}'}([0,T^*),L^{\tilde{r}'}(\mathbb{R}^3))} + \|f\|_{L^2(\mathbb{R}^3)}). \end{aligned} \tag{2.4}$$

Very important tool in our functional analysis background is the following lemma.

Lemma 2.3. *(Hardy-Littlewood-Sobolev Inequality) For $0 < \alpha < 3$ consider the Riesz potential*

$$I_\alpha(g)(x) = \int_{\mathbb{R}^3} \frac{g(y)}{|x - y|^{3-\alpha}} dy. \tag{2.5}$$

Then for any $1 < \theta < r < \infty$ and $g \in L^r(\mathbb{R}^3)$, we have

$$\|I_\alpha(g)\|_{L^\theta} \leq C\|g\|_{L^r}, \tag{2.6}$$

where $\frac{1}{\theta} = \frac{1}{r} - \frac{\alpha}{3}$.

For the proof of Lemma 2.3, see equation (31), Chapter VIII.4.2 in Stein [16].

The first important result in the present study of (1.1) is the following conservation law.

Lemma 2.4. *Let* $\psi \in C^0([0,T^*), L^2(\mathbb{R}^3))$ *be a solution to* (1.1) *with initial data* $\psi(0) = \psi_0 \in L^2(\mathbb{R}^3)$. *Then*

$$M(\psi(t)) = M(\psi_0), \tag{2.7}$$

for any $0 \le t < T^*$.

Proof. We multiply (1.1) by $\overline{\psi}$ to get

$$i\overline{\psi}\partial_t\psi + \overline{\psi}\triangle\psi = (|x|^{-1} * |\psi|^{\frac{8}{3}})|\psi|^2. \tag{2.8}$$

We conjugate (2.8) and subtract the result from the above expression to obtain

$$i\partial_t|\psi|^2 = (\psi\triangle\overline{\psi} - \overline{\psi}\triangle\psi) = \nabla\cdot(\psi\nabla\overline{\psi} - \overline{\psi}\nabla\psi). \tag{2.9}$$

By integration over $\mathbb{R}^3$ we obtain the desired result

$$\frac{d}{dt}M(\psi(t)) = \frac{d}{dt}\|\psi(t)\|^2_{L^2(\mathbb{R}^3)} = 0 \tag{2.10}$$

and the proof of the Lemma is completed. □

We should note here that the specific power of 8/3 in the Hartree-type nonlinearities of (1.1) does not allow the establishment of the energy conservation law (due mainly to the presence of the convolution). The latter would cause difficulties in proving, for example, global existence for solution of (1.1) at the H^1-level.

The arguments of Theorem 1.1 rely primarily on the Strichartz estimate (2.4) in Lemma 2.2 and on the Hölder inequality.

Let us denote by

$$N(\psi) = (|x|^{-1} * |\psi|^{\frac{8}{3}})\psi \tag{2.11}$$

the nonlinear term in the Hartree equation (1.1). Consider the form

$$V(\psi_1,\psi_2)(t,x) = |x|^{-1} * |\psi_1(t,x)|^{\alpha}|\psi_2(t,x)|^{\frac{8}{3}-\alpha}, \quad 0 \le \alpha \le \frac{8}{3}. \tag{2.12}$$

Then

$$N(\psi) = V(\psi,\psi)\psi$$

and we shall need the following estimates.

Lemma 2.5. *For any* $3 < p < \infty$ *and* r_1, r_2 *satisfying* $\frac{1}{p} = \frac{\alpha}{r_1} + \frac{8/3-\alpha}{r_2} - \frac{2}{3}$, $0 \le \alpha \le \frac{8}{3}$ *we have*

$$\|V(\psi_1,\psi_2)(t,\cdot)\|_{L^p(\mathbb{R}^3)} \le C\|\psi_1(t,\cdot)\|^{\alpha}_{L^{r_1}(\mathbb{R}^3)}\|\psi_2(t,\cdot)\|^{\frac{8}{3}-\alpha}_{L^{r_2}(\mathbb{R}^3)}. \tag{2.13}$$

Proof. It is sufficient to apply the Hardy-Littlewood-Sobolev inequality Lemma 2.3 and Hölder inequality to get

$$\||\cdot|^{-1}*|\psi_1(t,x)|^{\alpha}|\psi_2(t,x)|^{\frac{8}{3}-\alpha}\|_{L^p}\le C\|\psi_1(t,\cdot)\|^{\alpha}_{L^{r_1}}\|\psi_2(t,\cdot)\|^{\frac{8}{3}-\alpha}_{L^{r_2}},$$

provided

$$3<p<\infty,\quad \frac{1}{p}=\frac{\alpha}{r_1}+\frac{8/3-\alpha}{r_2}-\frac{2}{3}.$$

□

Lemma 2.6. *For any $\tilde r$ satisfying $2\le\tilde r\le 6$ and for any r_1,r_2,r_3, $2\le r_j\le 6, j=1,2,3,$ satisfying*

$$\frac{1}{\tilde r'}=\frac{\alpha}{r_1}+\frac{8/3-\alpha}{r_2}+\frac{1}{r_3}-\frac{2}{3},\tag{2.14}$$

we have

$$\begin{aligned}&\|V(\psi_1,\psi_2)(t,\cdot)\psi_3(t,\cdot)\|_{L^{\tilde r'}(\mathbb{R}^3)}\\ \le C\|\psi_1(t,\cdot)\|^{\alpha}_{L^{r_1}(\mathbb{R}^3)}&\|\psi_2(t,\cdot)\|^{\frac{8}{3}-\alpha}_{L^{r_2}(\mathbb{R}^3)}\|\psi_3(t,\cdot)\|_{L^{r_3}(\mathbb{R}^3)}.\end{aligned}\tag{2.15}$$

Proof. The Hölder inequality implies

$$\|V(\psi_1,\psi_2)(t,\cdot)\psi_3(t,\cdot)\|_{L^{\tilde r'}}\le C\|V(\psi_1,\psi_2)(t,\cdot)\|_{L^p}\|\psi_3(t,\cdot)\|_{L^{r_3}},$$

where

$$\frac{1}{\tilde r'}=\frac{1}{p}+\frac{1}{r_3}.$$

Applying further the estimate of Lemma 2.5, we complete the proof of the Lemma. □

The corresponding space - time estimate follows easily from the above estimate.

Lemma 2.7. *Let $0<T\le\infty$ and consider the Schrödinger-admissible pair $(q,r)=(14/3,14/5)$. Then, we have the estimate*

$$\|N(\psi)\|_{L^{\frac{14}{11}}([0,T],L^{\frac{14}{9}}(\mathbb{R}^3))}\le C\|\psi\|^{\frac{11}{3}}_{L^{\frac{14}{3}}([0,T],L^{\frac{14}{5}}(\mathbb{R}^3))}.\tag{2.16}$$

Proof. Applying Hölder inequality in time to (2.16) in Lemma 2.6 we obtain

$$\begin{aligned}&\|V(\psi_1,\psi_2)(t,\cdot)\psi_3(t,\cdot)\|_{L^{\tilde q'}([0,T],L^{\tilde r'})}\\ \le C\|\psi_1(t,\cdot)\|^{\alpha}_{L^{q_1}([0,T],L^{r_1})}&\|\psi_2(t,\cdot)\|^{\frac{8}{3}-\alpha}_{L^{q_2}([0,T],L^{r_2})}\|\psi_3(t,\cdot)\|_{L^{q_3}([0,T],L^{r_3})},\end{aligned}$$

where $0\le\alpha\le 8/3$ and

$$\frac{1}{\tilde r'}=\frac{\alpha}{r_1}+\frac{8/3-\alpha}{r_2}+\frac{1}{r_3}-\frac{2}{3},\tag{2.17}$$

$$\frac{1}{\tilde q'}=\frac{\alpha}{q_1}+\frac{8/3-\alpha}{q_2}+\frac{1}{q_3}.\tag{2.18}$$

Now we shall choose the couples (q_j, r_j) so that

$$\frac{1}{q_j} = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{r_j}\right), \quad j = 1, 2, 3 \tag{2.19}$$

and the relations (2.17), (2.18) are satisfied. Indeed, the simplest choice is

$$r = \tilde{r} = r_1 = r_2 = r_3, \quad q = \tilde{q} = q_1 = q_2 = q_3.$$

Then (2.17) reads as

$$\frac{1}{r'} = \frac{11}{3r} - \frac{2}{3}, \tag{2.20}$$

while (2.18) becomes

$$\frac{1}{q'} = \frac{11}{3q} \tag{2.21}$$

which together give the couple $(q, r) = (14/3, 14/5)$ and the proof is completed. □

Lemma 2.8. *Let $0 < T \leq \infty$ and let (q, r) be a Schrödinger-admissible pair. Then there exists a constant $C > 0$, independent of T such that*

$$\left\| \int_0^{\cdot} U(\cdot - s)[N(\psi)(s) - N(\chi)(s)]ds \right\|_{L^q([0,T],L^r)} \tag{2.22}$$

$$\leq C \left(\|\psi\|^{\frac{8}{3}}_{L^{\frac{14}{3}}([0,T],L^{\frac{14}{5}})} + \|\chi\|^{\frac{8}{3}}_{L^{\frac{14}{3}}([0,T],L^{\frac{14}{5}})} \right) \|\psi - \chi\|_{L^{\frac{14}{3}}([0,T],L^{\frac{14}{5}})},$$

for every $\psi, \chi \in L^{\frac{14}{3}}([0,T], L^{\frac{14}{5}}(\mathbb{R}^3))$.

Proof. To prove the Lemma, we shall use the estimates in Lemma 2.1. Then the estimate (2.22) follows directly from (2.15), (2.16), (2.2) and Hölder inequality. □

Proof of Theorem 1.1 . The proof of (ii) follows from Lemma 2.4.

We shall prove the existence of solution to (1.1) by a fix point argument. Let $\psi_0 \in L^2(\mathbb{R}^3)$ with $\|\psi_0\|_{L^2(\mathbb{R}^3)} < \varepsilon$, where $\varepsilon > 0$ is sufficiently small. Let $R > 0$ and consider the ball

$$B_{2R}(T) = \{\psi \in C^0([0,T], L^2)) \cap L^{\frac{14}{3}}([0,T], L^{\frac{14}{5}}); \|\psi\|_{L^{\frac{14}{3}}([0,T],L^{\frac{14}{5}})} \leq 2R\},$$

endowed with the metric

$$d(\psi, \chi) = \|\psi - \chi\|_{L^{\frac{14}{3}}([0,T],L^{\frac{14}{5}})}.$$

Since the space $L^{\frac{14}{3}}([0,T], L^{\frac{14}{5}})$ is reflexive, the ball B_{2R} is weakly compact, implying B_{2R} is a complete metric space.

Consider the map $\Phi[\psi](t)$, defined by the right-hand side of the Duhamel's integral representation (1.13). Then, for $\psi \in B_{2R}$, using (2.1), (2.2) and (2.16), we can write

$$\begin{aligned} \|\psi\|_{L^{\frac{14}{3}}([0,T],L^{\frac{14}{5}})} &\leq \|U(\cdot)\psi_0\|_{L^{\frac{14}{3}}([0,T],L^{\frac{14}{5}})} + C_1\|\psi\|^{\frac{11}{3}}_{L^{\frac{14}{3}}([0,T],L^{\frac{14}{5}})} \\ &\leq C_0\varepsilon + C_1\|\psi\|^{\frac{11}{3}}_{L^{\frac{14}{3}}([0,T],L^{\frac{14}{5}})}. \end{aligned} \tag{2.23}$$

Now, write $R = C_0\varepsilon$ and choose $\varepsilon > 0$ so small that there exists a positive number y, satisfying $C_1 y^{\frac{11}{3}} - y + R > 0$, $0 < y \leq 2R$. For that purpose, it is sufficient to take

$$C_1(2R)^{\frac{8}{3}} < \frac{1}{2},$$

or equivalently

$$\varepsilon < \frac{1}{2C_0(2C_1)^{\frac{3}{8}}},$$

implying that $\Phi[\psi] \in C^0([0,T], L^2)) \cap L^{\frac{14}{3}}([0,T], L^{\frac{14}{5}})$.

On the other hand, from (2.22) it follows that

$$\begin{aligned} &\|N(\psi) - N(\chi)\|_{L^{\frac{14}{11}}([0,T],L^{\frac{14}{9}}} \\ &\leq C_2 \left(\|\psi\|^{\frac{8}{3}}_{L^{\frac{14}{3}}([0,T],L^{\frac{14}{5}})} + \|\chi\|^{\frac{8}{3}}_{L^{\frac{14}{3}}([0,T],L^{\frac{14}{5}})} \right) \|\psi - \chi\|_{L^{\frac{14}{3}}([0,T],L^{\frac{14}{5}})} \\ &\leq C_2 2(2R)^{\frac{8}{3}} \|\psi - \chi\|_{L^{\frac{14}{3}}([0,T],L^{\frac{14}{5}})}, \end{aligned} \tag{2.24}$$

for every $\psi, \chi \in B_{2R}$. If we choose R such that

$$C_2 2(2R)^{\frac{8}{3}} \leq \frac{1}{2},$$

or equivalently

$$\varepsilon \leq \frac{1}{2C_0(4C_2)^{\frac{3}{8}}},$$

we finally obtain that the map $\Phi[\psi](t)$ is a strict contraction on the ball B_{2R}. Thus $\Phi[\psi]$ has a fixed point ψ, which is the unique solution of (1.1) in B_{2R}. So far, we have proved the statement of Theorem 1.1, as well as the part (i).

Notice, that the Strichartz estimate (2.4) implies a similar inequality, namely

$$\|\psi\|_{L^{\frac{14}{3}}([0,T],L^{\frac{14}{5}})} \leq C_0\|\psi_0\|_{L^2} + C_1\|\psi\|^{\frac{11}{3}}_{L^{\frac{14}{3}}([0,T],L^{\frac{14}{5}})} \tag{2.25}$$

where we can take both the constants in (2.23) and (2.25) to be the same.

The second comment on the above proof is that from (2.23) and (2.25), the size of the ball B_{2R} depends directly on the size of the norm $\|U(t)\psi_0\|_{L^{\frac{14}{3}}([0,T],L^{\frac{14}{5}}(\mathbb{R}^3))}$ and it can be done small by taking either $\|\psi_0\|_{L^2(\mathbb{R}^3)}$ small or the interval $[0,T]$ small.

Let us denote by T^* the supremum of all $T > 0$ for which there exists a solution of (1.1) in $C^0([0,T], L^2(\mathbb{R}^3)) \cap L^{\frac{14}{3}}([0,T], L^{\frac{14}{5}}(\mathbb{R}^3))$. To prove (iii), observe that if ψ_0 is sufficiently small, then (2.23) holds regardless of the value of T. Thus we may accomplish the fixed point procedure in the ball $B_{2R}(\infty)$, providing $T^* = \infty$.

Further, we claim that if $T^* < \infty$, then $\|\psi\|_{L^q([0,T^*),L^r(\mathbb{R}^3))} = \infty$ for every $r > 14/5$. Indeed, on the contrary, let us assume that $T^* < \infty$ and $\|\psi\|_{L^{\frac{14}{3}}([0,T^*),L^{\frac{14}{5}})} < \infty$. For any $t \in [0,T^*)$ let

$\tau \in [0, T^* - t)$. Using Duhamel's formula (1.13), we can write

$$\begin{aligned}\psi(t+\tau) &= U(t+\tau)\psi_0 - i\int_0^{t+\tau} U(t+\tau-s)N(\psi)(s)ds \\ &= U(\tau)U(t)\psi_0 - i\int_0^{t+\tau} U(t+\tau-s)N(\psi)(s)ds \\ &= U(\tau)\psi(t) + iU(\tau)\int_0^t U(t-s)N(\psi)(s)ds \\ &\quad -i\int_0^{t+\tau} U(t+\tau-s)N(\psi)(s)ds \\ &= U(\tau)\psi(t) - i\int_t^{t+\tau} U(t+\tau-s)N(\psi)(s)ds. \end{aligned} \tag{2.26}$$

From (2.26) and the estimate (2.22) in Lemma 2.8 we obtain

$$\|U(\cdot)\psi(t)\|_{L^{\frac{14}{3}}([0,T^*-t),L^{\frac{14}{5}})} \le C\Big(\|\psi(t)\|_{L^{\frac{14}{3}}([t,T^*),L^{\frac{14}{5}})} + \|\psi\|^{\frac{11}{3}}_{L^{\frac{14}{3}}([t,T^*),L^{\frac{14}{5}})}\Big). \tag{2.27}$$

Observing now that $\|U(\cdot)\psi\|_{L^{\frac{14}{3}}([0,T],L^{\frac{14}{5}})} \to 0$ as $T \to 0$ and taking t close enough to T^*, it follows that $\|U(\cdot)\psi(t)\|_{L^{\frac{14}{3}}([0,T^*-t),L^{\frac{14}{5}})}$ can be made small enough and the assumptions in (iii) are fulfilled. Therefore, ψ can be extended after T^*, which contradicts the maximality. Let (q,r) be a Schrödinger-admissible pair with $r \ge \frac{14}{5}$. Then, from Hölder inequality for $T < T^*$, we can write

$$\|\psi\|_{L^{\frac{14}{3}}([0,T],L^{\frac{14}{5}})} \le \|\psi\|^{1-\alpha}_{L^\infty([0,T],L^2)}\|\psi\|^{\alpha}_{L^q([0,T],L^r)}, \quad \alpha \in (0,1). \tag{2.28}$$

Letting $T \to T^*$, we obtain that $\|\psi\|_{L^q([0,T],L^r)} = \infty$, which proves the statement (iv).

To prove (v), consider a sequence $\psi_0^k \in L^2(\mathbb{R}^3)$, such that $\psi_0^k \to \psi_0 \in L^2(\mathbb{R}^3)$ as $k \to \infty$. Thus, for k large enough, $\|\psi_0^k\|_{L^2(\mathbb{R}^3)} < \varepsilon$. We can use the Duhamel's formula (1.13) to construct a sequence of solutions $\psi^k \in L^{\frac{14}{3}}([0,T], L^{\frac{14}{5}}(\mathbb{R}^3))$ to (1.1) with initial datum ψ_0^k. Applying the proof of (iii), we obtain that $\psi^k \to \psi$ in $C^0([0,T], L^2(\mathbb{R}^3)) \cap L^{14/3}([0,T], L^{14/5}(\mathbb{R}^3))$ as $k \to \infty$, and in fact in every $L^q([0,T], L^r(\mathbb{R}^3)))$ for (q,r) be an admissible pair. Thus, the proof of the Theorem is completed.

3 Small data scattering theory in $L^2(\mathbb{R}^3)$

In this section we shall prove Theorem 1.2. The arguments for proving the existence of the wave operator are standard and follows the exposition in [12,15]. We shall prove only the $(+)$ case since the $(-)$ case can be proved similarly. Let $\psi_0 \in L^2(\mathbb{R}^3)$ with $\|\psi_0\|_{L^2} < \varepsilon$ and $\psi \in B_{2R}$, where the ball B_{2R} is defined in the previous section. Then, for $t > t_0$, using Duhamel's integral formula (1.13) we have

$$U(-t)\psi(t) = U(-t_0)\psi(t_0) - i\int_{t_0}^t U(-s)N(\psi)(s)ds. \tag{3.1}$$

Therefore, using the estimate (2.16), we have

$$\begin{aligned}\|U(-t)\psi(t) - U(-t_0)\psi(t_0)\|_{L^2} &= \left\| \int_{t_0}^{t} U(-s)N(\psi)(s)ds \right\|_{L^2} \\ &\leq C \|\psi\|^{\frac{11}{3}}_{L^{\frac{14}{3}}([t_0,t],L^{\frac{14}{5}})} \to 0, \end{aligned} \tag{3.2}$$

as $t_0 \to \infty$. Since $U(-t_0)\psi(t_0) \in L^2$, then the proof of part (ii) is completed.

To prove (i), assume that $\psi_+ \in L^2(\mathbb{R}^3)$, $\psi \in B_{2R}$ and consider the map

$$\Phi_+[\psi](t) = U(t)\psi_+ + i \int_t^{\infty} U(t-s)N(\psi)(s)ds, \quad t > T, \tag{3.3}$$

for some $T = T(\psi_+)$ large enough. Then, using the same arguments as in the proof of part (iii) of Theorem 1.1, we find that Φ_+ is a contraction on B_{2R} and has a unique fixed point if $\|\psi_+\|_{L^2} < \varepsilon$. Using the global well-posedness result established in Theorem 1.1 for small data, one can then extend this solution uniquely for any $t \in [0, \infty)$, and in particular ψ will take some value $\psi_0 = \psi(0) \in L^2$ at time $t = 0$. This gives existence of the wave operator Ω_+, defined by

$$\psi_0 = \Omega_+\phi_+ = \psi_+ + i \int_0^{\infty} U(-s)N(\psi)(s)ds. \tag{3.4}$$

This proves part (i) of the Theorem 1.2.

To prove (iii), we shall use the following observations. Since, the wave operators $\Omega_\pm$ are isometric in the space B_ε, it is clear that the scattering operator $S : \phi_- \mapsto \phi_+$ is well defined as a map from B_ε onto itself and isometric in the L^2 norm, i.e. $\|S\psi\|_{L^2} = \|\psi\|_{L^2}$. This completes the proof of the Theorem 1.2.

It is clear that the finiteness of the $L_t^{14/3}L_x^{14/5}$-norm is sufficient to yield global well-posedness and scattering results for small data in L^2. As the $L_{t,x}^{2(n+2)/n}$-norm for the Schrödinger equation with pure power nonlinearity (1.11), this is the best possible choice of admissible indices for (1.1), which ensures the space-time integrability via the Strichartz estimates and the contraction property of the nonlinear map. Moreover, the above results does not depend on the repulsive character of the problem and can be proved in a similar way for the focussing analogue of (1.1). Finally, we note that the global existence, well-posedness and scattering problems for (1.1) with an arbitrary initial data require spaces, strictly smaller than the L^2 one. For example, in another work, following the approach of Hayashi, Naumkin and Ozawa [9], we shall consider the above problems for data in the pseudoconformal space $\Sigma_s = \{\psi_0 \in H^s; |x|^s\psi_0 \in L^2\}$ for some $0 < s < 1$, depending on the behavior of the leading term of the solution ψ for large time.

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