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On Tikhonov Functionals Penalized by Bregman Distances

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ABSTRACT

We investigate Tikhonov regularization methods for linear and nonlinear ill-posed problems in Banach spaces, where the penalty term is described by Bregman distances. We prove convergence and stability results. Moreover, using appropriate source conditions, we are able to derive rates of convergence in terms of Bregman distances. We also analyze an iterated Tikhonov method for nonlinear problems, where the penalization is given by an appropriate convex functional.

RESUMEN

Investigamos métodos de regularización de Tikhonov para problemas no lineales mal-puestos en espacios de Banach, donde el término de penalización es descrito por distancias de Bregman. Probamos resultados de convergencia y estabilidad. Además, usando condiciones apropriadas, somos capazes de obtener tasas de convergencia en términos de las distancias de Bergman. También analizamos un método de iterados de Tikhnov para problemas no lineales donde la penalización es dada por un funcional convexo apropriado.

Key words and phrases: *Tikhonov functionals, Bregman distances, Total variation regularization.*

Math. Subj. Class.: *65J20, 47J06, 47J25.*

1 Introduction

In this paper we study non-quadratic regularization methods for solving ill-posed operator equations of the form

$$F(u) = y \ , \tag{1}$$

where $F : \mathscr{D}(F) \subset \mathcal{U} \to \mathcal{H}$ is an operator between infinite dimensional Banach spaces. Both linear and nonlinear problems are considered.

Tikhonov method is widely used to approximate solutions of inverse problems modeled by operator equations in Hilbert spaces [11, 5]. In this article we investigate a Tikhonov methods, which consist of the minimization of functionals of the type

$$J_\alpha^\delta(u) = \frac{1}{2}\|F(u) - y^\delta\|^2 + \alpha h(u) \ , \tag{2}$$

where $\alpha \in \mathbb{R}_+$ is called regularization parameter, $h(\cdot)$ is a proper convex functional, and the noisy data y^δ satisfy

$$\|y - y^\delta\| < \delta \ . \tag{3}$$

The method presented above represents a generalization of the classical Tikhonov regularization. Therefore, the following questions arise:

- For $\alpha > 0$, does the solution (2) exist? Does the solution depends continuously on the data y^δ?
- Is the method convergent? (i.e., if the data y is exact and $\alpha \to 0$, do the minimizers of (2) converge to a solution of (1)?)
- Is the method stable in the sense that: if $\alpha = \alpha(\delta)$ is chosen appropriately, do the minimizers of (2) converge to a solution of (1) as $\delta \to 0$?
- What is the rate of convergence? How should the parameter $\alpha = \alpha(\delta)$ be chosen in order to get optimal convergence rates?

The first point above is answered in [6]. Throughout this article we assume the following assumptions.

Assumption 1.1.

(A1) Given the Banach spaces $\mathcal{U}$ and $\mathcal{H}$ one associates the topologies $\tau_{\mathcal{U}}$ and $\tau_{\mathcal{H}}$, respectively, which are weaker than the norm topologies;

(A2) The topological duals of $\mathcal{U}$ and $\mathcal{H}$ are denoted by $\mathcal{U}^*$ and $\mathcal{H}^*$, respectively;

(A3) The norm $\|\cdot\|$ is sequentially lower semi-continuous with respect to $\tau_{\mathcal{H}}$, i.e., for $u_k \to u$ with respect to the $\tau_{\mathcal{U}}$ topology, $h(u) \le \liminf_k h(u_k)$;

(A4) $\mathscr{D}(F)$ *has non-empty interior with respect to the norm topology and is* $\tau_{\mathcal{U}}$*-closed. Moreover,* $\mathscr{D}(F) \cap dom\ h \neq \emptyset$*;*

(A5) $F : \mathscr{D}(F) \subseteq \mathcal{U} \to \mathcal{H}$ *is continuous from* $(\mathcal{U}, \tau_{\mathcal{U}})$ *to* $(\mathcal{H}, \tau_{\mathcal{H}})$*;*

(A6) The functional $h : [0, +\infty] \to \mathcal{H}$ *is proper, convex, bounded from below and* $\tau_{\mathcal{U}}$ *lower semi-continuous;*

(A7) For every $M > 0$ *,* $\alpha > 0$*, the sets*

$$\mathcal{M}_\alpha(M) = \left\{ u \in \mathcal{U} \;\;|\;\; J^\delta_\alpha(u) \leq M \right\}$$

are $\tau_{\mathcal{U}}$ *compact, i.e. every sequence* (u_k) *in* $\mathcal{M}_\alpha(M)$ *has a subsequence, which is convergent in* $\mathcal{U}$ *with respect to the* $\tau_{\mathcal{U}}$ *topology.*

The goal of this paper is to answer the last three questions posed above. We obtain convergence rates and error estimates with respect to the generalized *Bregman distances*, originally introduced in [3]. Even though this tool does not satisfy symmetry requirement nor the triangular inequality, it is the main ingredient to this work.

This paper is organized as follow: In section 2 we consider the linear case and give quantitative estimates for the minimizers of (2), for exact and for noisy data. In section 3 contains similar results as the section 2 for nonlinear problems. In section 4 we briefly discuss a iterative method for the nonlinear case, the main results contains convergence analysis.

2 Convergence Analysis for Linear Problems

In this section we consider only the linear case. Equation (1) will be denoted by $Fu = y$, and the operator is defined from a Banach space to a Hilbert space. The main results of this section were proposed originally in [4, 8].

2.1 Rates of convergence for source condition of type I

Error estimates for the solution error can be obtained only under additional smoothness assumption on the data, the so called *source conditions*. At a first moment we assume that $y \in \mathscr{R}(F)$ and let $\overline{u}$ be an h-minimizing solution by definition A.2. We assume that there exist at least one element ξ in $\partial h(\overline{u})$ which belongs to the range of adjoint of the operator F. Note that $\mathscr{R}(F^*) \subseteq \mathcal{U}^*$ and $\partial h(\overline{u}) \subseteq \mathcal{U}^*$. Summarizing, we have

$$\xi \in \mathscr{R}(F^*) \cap \partial h(\overline{u}) \neq \varnothing \,, \tag{4}$$

where $\overline{u}$ is such that

$$F\overline{u} = y \,. \tag{5}$$

We can rewrite the source condition (4) as following: there exists an element $\omega \in \mathcal{H}$ such that $\xi = F^*\omega$. Note that under this assumption we can define the dual pairing for $(\psi, u) \in \mathcal{U}^* \times \mathcal{U}$, where $\psi \in \mathscr{R}(F^*)$ as

$$\langle \psi \ , \ u \rangle = \langle F^*\nu \ , \ u \rangle := \langle \nu \ , \ Fu \rangle_{\mathcal{H}} \ ,$$

for some $\nu \in \mathcal{H}$.

Theorem 2.1 (Stability). *Let* (3) *hold and let* $\overline{u}$ *be an h-minimizing solution of* (1) *such that the source condition (4) and* (5) *are satisfied. Then, for each minimizer* u_α^δ *of* (2) *the estimate*

$$D_h^{F^*\omega}\left(u_\alpha^\delta, \overline{u}\right) \leq \frac{1}{2\alpha}\left(\alpha \|\omega\| + \delta\right)^2 \tag{6}$$

holds for $\alpha > 0$. *In particular, if* $\alpha \sim \delta$, *then* $D_h^{F^*\omega}\left(u_\alpha^\delta, \overline{u}\right) = \mathcal{O}(\delta)$.

Proof. We note that $\left\|F\overline{u} - y^\delta\right\|^2 \leq \delta^2$, by (5) and (3). Since u_α^δ is a minimizer of the regularized problem (2), we have

$$\frac{1}{2}\left\|Fu_\alpha^\delta - y^\delta\right\|^2 + \alpha h\left(u_\alpha^\delta\right) \ \leq \ \frac{\delta^2}{2} + \alpha h\left(\overline{u}\right) \ .$$

Let $D_h^{F^*\omega}\left(u_\alpha^\delta, \overline{u}\right)$ the Bregman distance between u_α^δ and $\overline{u}$, so the above inequality becomes

$$\frac{1}{2}\left\|Fu_\alpha^\delta - y^\delta\right\|^2 + \alpha\left(D_h^{F^*\omega}\left(u_\alpha^\delta, \overline{u}\right) + \left\langle F^*\omega \ , \ u_\alpha^\delta - \overline{u}\right\rangle\right) \ \leq \ \frac{\delta^2}{2} \, .$$

Hence, using (3) and Cauchy-Schwarz inequality we can derive the estimate

$$\frac{1}{2}\left\|Fu_\alpha^\delta - y^\delta\right\|^2 + \left\langle \alpha\omega \ , \ Fu_\alpha^\delta - y^\delta\right\rangle_{\mathcal{H}} + \alpha D_h^{F^*\omega}\left(u_\alpha^\delta, \overline{u}\right) \ \leq \ \frac{\delta^2}{2} + \alpha\|\omega\|\,\delta \, .$$

Using the the equality $\|a+b\|^2 = \|a\|^2 + 2\langle a \ , \ b\rangle + \|b\|^2$, it is easy to see that

$$\frac{1}{2}\left\|Fu_\alpha^\delta - y^\delta + \alpha\omega\right\|^2 + \alpha D_h^{F^*\omega}\left(u_\alpha^\delta, \overline{u}\right) \ \leq \ \frac{\alpha^2}{2}\|\omega\|^2 + \alpha\delta\|\omega\| + \frac{\delta^2}{2} \ ,$$

which yields (6) for $\alpha > 0$. ☐

Theorem 2.2 (Convergence). *If* $\overline{u}$ *is an h-minimizing solution of* (1) *such that the source condition (4) and* (5) *are satisfied, then for each minimizer* u_α *of* (2) *with exact data, the estimate*

$$D_h^{F^*\omega}\left(u_\alpha, \overline{u}\right) \leq \frac{\alpha}{2}\|\omega\|^2$$

holds true.

Proof. The proof is analogous to the proof of theorem 2.1, taking $\delta = 0$. ☐

2.2 Rates of convergence for source condition of type II

In this section we use another source condition, which is stronger than the one used in previous subsection. This condition corresponds the existence of some element $\xi \in \partial h(\overline{u}) \subset \mathcal{U}^*$ in the range of the operator F^*F, i.e.

$$\xi \in \mathscr{R}(F^*F) \cap \partial h(\overline{u}) \neq \varnothing\,, \tag{7}$$

where $\overline{u}$ is such that

$$F^*F\overline{u} = F^*y\,. \tag{8}$$

Note that in (8) we do not require $y \in \mathscr{R}(F)$. Moreover, the definition A.2 is given in context of least-squares solution. The condition (7) is equivalent to the existence of $\omega \in \mathcal{U} \setminus \{0\}$ such that $\xi = F^*F\omega$, where F^* is the adjoint operator of F and $F^*F : \mathcal{U} \to \mathcal{U}^*$.

Theorem 2.3 (Stability)**.** *Let* (3) *hold and let* $\overline{u}$ *be an h-minimizing solution of* (1) *such that the source condition* (7) *as well as* (8) *are satisfied. Then the following inequalities hold for any* $\alpha > 0$*:*

$$D_h^{F^*F\omega}\left(u_\alpha^\delta, \overline{u}\right) \le D_h^{F^*F\omega}\left(\overline{u} - \alpha\omega, \overline{u}\right) + \frac{\delta^2}{\alpha} + \frac{\delta}{\alpha}\sqrt{\delta^2 + 2\alpha D_h^{F^*F\omega}\left(\overline{u} - \alpha\omega, \overline{u}\right)}, \tag{9}$$

$$\left\|Fu_\alpha^\delta - F\overline{u}\right\| \le \alpha\left\|F\omega\right\| + \delta + \sqrt{\delta^2 + 2\alpha D_h^{F^*F\omega}\left(\overline{u} - \alpha\omega, \overline{u}\right)}\,. \tag{10}$$

Proof. Since u_α^δ is a minimizer of (2), it follows from algebraic manipulation and from the definition of Bregman distance that

$$\begin{aligned} 0 \;&\ge\; \frac{1}{2}\left[\left\|Fu_\alpha^\delta - y^\delta\right\|^2 - \left\|Fu - y^\delta\right\|^2\right] + \alpha h\left(u_\alpha^\delta\right) - \alpha h(u) \\ &=\; \frac{1}{2}\left[\left\|Fu_\alpha^\delta\right\|^2 - \left\|Fu\right\|^2\right] - \left\langle F\left(u_\alpha^\delta - u\right)\,,\, y^\delta\right\rangle_{\mathcal{H}} - \alpha D_h^{F^*F\omega}(u, \overline{u}) \\ &\quad + \alpha\left\langle F\omega\,,\, F\left(u_\alpha^\delta - u\right)\right\rangle_{\mathcal{H}} + \alpha D_h^{F^*F\omega}\left(u_\alpha^\delta, \overline{u}\right)\,. \end{aligned} \tag{11}$$

Notice that

$$\begin{aligned} \left\|Fu_\alpha^\delta\right\|^2 - \left\|Fu\right\|^2 \;&=\; \left\|F\left(u_\alpha^\delta - \overline{u} + \alpha\omega\right)\right\|^2 - \left\|F\left(u - \overline{u} + \alpha\omega\right)\right\|^2 \\ &\quad + 2\left\langle Fu_\alpha^\delta - Fu\,,\, F\overline{u} - \alpha F\omega\right\rangle_{\mathcal{H}}\,. \end{aligned}$$

Moreover, by (8), we have $\left\langle F\left(u_\alpha^\delta - u\right)\,,\, y^\delta - F\overline{u}\right\rangle_{\mathcal{H}} = \left\langle F\left(u_\alpha^\delta - u\right)\,,\, y^\delta - y\right\rangle_{\mathcal{H}}$. Therefore, it follows from (11) that

$$\begin{aligned} &\frac{1}{2}\left\|F\left(u_\alpha^\delta - \overline{u} + \alpha\omega\right)\right\|^2 + \alpha D_h^{F^*F\omega}\left(u_\alpha^\delta, \overline{u}\right) \\ \le\;& \left\langle F\left(u_\alpha^\delta - u\right)\,,\, y^\delta - y\right\rangle_{\mathcal{H}} + \alpha D_h^{F^*F\omega}(u, \overline{u}) + \frac{1}{2}\left\|F\left(u - \overline{u} + \alpha\omega\right)\right\|^2 \end{aligned}$$

for every $u \in \mathcal{U}$, $\alpha \ge 0$ and $\delta \ge 0$.

Replacing u by $\overline{u}-\alpha\omega$ in the last inequality, using (3), relations $\langle a\ ,\ b\rangle \le |\langle a\ ,\ b\rangle| \le \|a\|\,\|b\|$, and defining $\gamma = \left\|F\left(u_\alpha^\delta - \overline{u} + \alpha\omega\right)\right\|$ we obtain

$$\frac{1}{2}\gamma^2 + \alpha D_h^{F^*F\omega}\left(u_\alpha^\delta,\overline{u}\right) \;\le\; \delta\gamma + \alpha D_h^{F^*F\omega}\left(\overline{u}-\alpha\omega,\overline{u}\right)\,.$$

We estimate separately each term on the left hand side by right hand side. One of the estimates is an inequality in the form of a polynomial of the second degree for γ, which gives us the inequality

$$\gamma \le \delta + \sqrt{\delta^2 + 2\alpha D_h^{F^*F\omega}\left(\overline{u}-\alpha\omega,\overline{u}\right)}\,.$$

This inequality together with the other estimate, gives us (9). Now, (10) follows from the fact that $\left\|F\left(u_\alpha^\delta - \overline{u}\right)\right\| \le \gamma + \alpha\left\|F\omega\right\|$. ◻

Theorem 2.4 (Convergence)**.** *Let $\alpha \ge 0$ be given. If $\overline{u}$ is a h-minimizing solution of* (1) *satisfying the source condition* (7) *as well as* (8)*, then the following inequalities hold true:*

$$D_h^{F^*F\omega}\left(u_\alpha,\overline{u}\right) \le D_h^{F^*F\omega}\left(\overline{u}-\alpha\omega,\overline{u}\right)\,,$$

$$\left\|Fu_\alpha - F\overline{u}\right\| \le \alpha\left\|F\omega\right\| + \sqrt{2\alpha D_h^{F^*F\omega}\left(\overline{u}-\alpha\omega,\overline{u}\right)}\,.$$

Proof. The proof is analogous to the proof of theorem 2.3, taking $\delta = 0$. Notice that here α can be taken equal to zero. ◻

Corollary 2.5. *Let the assumptions of the theorem 2.3 hold true. Further, assume that h is twice differentiable in a neighborhood U of $\overline{u}$ and there exists a number $M > 0$ such that for any $v \in \mathcal{U}$ and $u \in U$ the inequality*

$$\langle h''(u)v\ ,\ v\rangle \le M\left\|v\right\|^2 \tag{12}$$

hold true. Then, for the parameter choice $\alpha \sim \delta^{\frac{2}{3}}$ we have $D_h^{\xi}\left(u_\alpha^\delta,\overline{u}\right) = \mathcal{O}\left(\delta^{\frac{4}{3}}\right)$. Moreover, for exact data we have $D_h^{\xi}\left(u_\alpha,\overline{u}\right) = \mathcal{O}\left(\alpha^2\right)$.

Proof. Using Taylor's expansion at the point $\overline{u}$ we obtain

$$h\left(u\right) \;=\; h\left(\overline{u}\right) + \langle h'(\overline{u})\ ,\ u-\overline{u}\rangle + \frac{1}{2}\langle h''(\mu)(u-\overline{u})\ ,\ u-\overline{u}\rangle$$

for some $\mu \in [u,\overline{u}]$. Let $u = \overline{u}-\alpha\omega$ in the above equality. For sufficiently small α, it follows from assumption (12) and the definition of the Bregman distance, with $\xi = h'(\overline{u})$, that

$$\begin{aligned} D_h^{\xi}\left(\overline{u}-\alpha\omega,\overline{u}\right) &= \frac{1}{2}\langle h''(\mu)(-\alpha\omega)\ ,\ -\alpha\omega\rangle \\ &\le \alpha^2\frac{M}{2}\left\|\omega\right\|_{\mathcal{U}}^2\,. \end{aligned}$$

Note that $D_h^{\xi}\left(\overline{u}-\alpha\omega,\overline{u}\right) = \mathcal{O}\left(\alpha^2\right)$, so the desired rates of convergence follow from theorems 2.3 and 2.4. ◻

3 Convergence Analysis for Nonlinear Problems

This section points out the convergence analysis for the nonlinear problems. We need to assume a nonlinear condition. In contrast with other classical conditions, the following analysis covers the case when both $\mathcal{U}$ and $\mathcal{H}$ are Banach spaces.

Assumption 3.1. *Assume that an h-minimizing solution* $\overline{u}$ *of* (1) *exists and that the operator* $F : \mathscr{D}(F) \subseteq \mathcal{U} \to \mathcal{H}$ *is Gâteaux differentiable. Moreover, assume that there exists* $\rho > 0$ *such that, for every* $u \in \mathscr{D}(F) \cap \mathcal{B}_\rho(\overline{u})$

$$\|F(u) - F(\overline{u}) - F'(\overline{u})(u - \overline{u})\| \leq cD_h^\xi(u, \overline{u}) \ , \ c > 0 \tag{13}$$

and $\xi \in \partial h(\overline{u})$.

This assumption was proposed originally in [9].

3.1 Rates of convergence for source condition of type I

For nonlinear operators we cannot define an adjoint operator. Therefore the assumptions are done with respect to the linearization of the operator F. In comparison with the source condition (4) introduced on previous section, we assume that

$$\xi \in \mathscr{R}(F'(\overline{u})^*) \cap \partial h(\overline{u}) \neq \varnothing \tag{14}$$

where $\overline{u}$ solves

$$F(\overline{u}) = y\,. \tag{15}$$

The derivative of operator F is defined between the Banach space $\mathcal{U}$ and $\mathscr{L}(\mathcal{U}, \mathcal{H})$, the space of the linear transformations from $\mathcal{U}$ to $\mathcal{H}$. When we apply the derivative at $\overline{u} \in \mathcal{U}$ we have a linear operator $F'(\overline{u}) : \mathcal{U} \to \mathcal{H}$ and so we can define its adjoint, $F'(\overline{u})^* : \mathcal{H}^* \to \mathcal{U}^*$.

The source condition (14) is stated as follows: There exists an element $\omega \in \mathcal{H}^*$ such that

$$\xi = F'(\overline{u})^* \omega \in \partial h(\overline{u}) \ . \tag{16}$$

Theorem 3.2 (Stability)**.** *Let the assumptions 1.1, 3.1 and relation* (3) *hold true. Moreover, assume that there exists* $\omega \in \mathcal{H}^*$ *such that* (16) *is satisfied and* $c\|\omega\|_{\mathcal{H}^*} < 1$*. Then, the following estimates hold:*

$$\left\|F\left(u_\alpha^\delta\right) - F(\overline{u})\right\| \leq 2\alpha\|\omega\|_{\mathcal{H}^*} + 2\left(\alpha^2\|\omega\|_{\mathcal{H}^*}^2 + \delta^2\right)^{\frac{1}{2}} ,$$

$$D_h^{F'(\overline{u})^*\omega}\left(u_\alpha^\delta, \overline{u}\right) \leq \frac{2}{1 - c\|\omega\|_{\mathcal{H}^*}}\left[\frac{\delta^2}{2\alpha} + \alpha\|\omega\|_{\mathcal{H}^*}^2 + \|\omega\|_{\mathcal{H}^*}\left(\alpha^2\|\omega\|_{\mathcal{H}^*}^2 + \delta^2\right)^{\frac{1}{2}}\right] .$$

In particular, if $\alpha \sim \delta$*, then* $\left\|F\left(u_\alpha^\delta\right) - F(\overline{u})\right\| = \mathcal{O}(\delta)$ *and* $D_h^{F'(\overline{u})^*\omega}\left(u_\alpha^\delta, \overline{u}\right) = \mathcal{O}(\delta)$.

Proof. Since u_α^δ is the minimizer of (2), it follows from the definition of the Bregman distance that

$$\frac{1}{2}\left\|F\left(u_\alpha^\delta\right)-y^\delta\right\|^2 \quad\leq\quad \frac{1}{2}\delta^2-\alpha\left(D_h^{F'(\overline{u})^*\omega}\left(u_\alpha^\delta,\overline{u}\right)+\left\langle F'\left(\overline{u}\right)^*\omega\ ,\ u_\alpha^\delta-\overline{u}\right\rangle\right).$$

By using (3) and (15) we obtain

$$\frac{1}{2}\left\|F\left(u_\alpha^\delta\right)-F\left(\overline{u}\right)\right\|^2 \quad\leq\quad \left\|F\left(u_\alpha^\delta\right)-y^\delta\right\|^2+\delta^2.$$

Now, using the last two inequalities above, the definition of Bregman distance, the nonlinearity condition and the assumption $\left(c\left\|\omega\right\|_{\mathcal{H}^*}-1\right)<0$, we obtain

$$\begin{aligned}
\frac{1}{4}\left\|F\left(u_\alpha^\delta\right)-F\left(\overline{u}\right)\right\|^2 &\leq \frac{1}{2}\left(\left\|F\left(u_\alpha^\delta\right)-y^\delta\right\|^2+\delta^2\right)\\
&\leq \delta^2-\alpha D_h^{F'(\overline{u})^*\omega}\left(u_\alpha^\delta,\overline{u}\right)+\alpha\left\langle\omega\ ,\ -F'\left(\overline{u}\right)\left(u_\alpha^\delta-\overline{u}\right)\right\rangle\\
&\leq \delta^2-\alpha D_h^{F'(\overline{u})^*\omega}\left(u_\alpha^\delta,\overline{u}\right)+\alpha\left\|\omega\right\|_{\mathcal{H}^*}\left\|F\left(u_\alpha^\delta\right)-F\left(\overline{u}\right)\right\|\\
&\quad+\alpha\left\|\omega\right\|_{\mathcal{H}^*}\left\|F\left(u_\alpha^\delta\right)-F\left(\overline{u}\right)-F'\left(\overline{u}\right)\left(u_\alpha^\delta-\overline{u}\right)\right\|\\
&= \delta^2+\alpha\left(c\left\|\omega\right\|_{\mathcal{H}^*}-1\right)D_h^{F'(\overline{u})^*\omega}\left(u_\alpha^\delta,\overline{u}\right)\\
&\quad+\alpha\left\|\omega\right\|_{\mathcal{H}^*}\left\|F\left(u_\alpha^\delta\right)-F\left(\overline{u}\right)\right\| \qquad (17)\\
&\leq \delta^2+\alpha\left\|\omega\right\|_{\mathcal{H}^*}\left\|F\left(u_\alpha^\delta\right)-F\left(\overline{u}\right)\right\| \qquad (18)
\end{aligned}$$

From (18) we obtain an inequality in the form of a polynomial of second degree for the variable $\gamma=\left\|F\left(u_\alpha^\delta\right)-F\left(\overline{u}\right)\right\|$. This gives us the first estimate stated by the theorem. For the second estimate we use (17) and the previous estimate for γ. □

Theorem 3.3 (Convergence)**.** *Let the assumptions 1.1 and 3.1 hold true. Moreover, assume the existence of $\omega\in\mathcal{H}^*$ such that* (16) *is satisfied and $c\left\|\omega\right\|_{\mathcal{H}^*}<1$. Then, the following estimates hold:*

$$\left\|F\left(u_\alpha\right)-F\left(\overline{u}\right)\right\|\leq 4\alpha\left\|\omega\right\|_{\mathcal{H}^*},$$

$$D_h^{F'(\overline{u})^*\omega}\left(u_\alpha,\overline{u}\right)\leq\frac{4\alpha\left\|\omega\right\|_{\mathcal{H}^*}^2}{1-c\left\|\omega\right\|_{\mathcal{H}^*}}.$$

Proof. The proof is analogous to the proof of theorem 3.2, taking $\delta=0$. □

3.2 Rates of convergence for source condition of type II

In this subsection we consider a source condition similar to the one in (7), namely we assume the existence of

$$\xi\in\mathscr{R}(F'\left(\overline{u}\right)^*F'\left(\overline{u}\right))\cap\partial h\left(\overline{u}\right)\neq\varnothing.$$

The assumption above is equivalent the existence of an element $\omega\in\mathcal{U}$ with

$$\xi=F'\left(\overline{u}\right)^*F'\left(\overline{u}\right)\omega\in\partial h\left(\overline{u}\right). \qquad (19)$$

Theorem 3.4 (Stability)**.** *Let the assumptions 1.1, 3.1 hold as well as estimate* (3)*. Moreover, let* $\mathcal{H}$ *be a Hilbert space and assume the existence of an h-minimizing solution* $\overline{u}$ *of* (1) *in the interior of* $\mathscr{D}(F)$*. Assume also the existence of* $\omega \in \mathcal{U}$ *such that* (19) *is satisfied and* $c\left\|F'(\overline{u})\omega\right\| < 1$*. Then, for* α *sufficiently small the following estimates hold:*

$$\left\|F\left(u_\alpha^\delta\right) - F\left(\overline{u}\right)\right\| \leq \alpha \left\|F'\left(\overline{u}\right)\omega\right\| + g(\alpha,\delta)\,,$$

$$D_h^\xi\left(u_\alpha^\delta,\overline{u}\right) \leq \frac{\alpha s + (cs)^2/2 + \delta g(\alpha,\delta) + cs\left(\delta + \alpha\left\|F'\left(\overline{u}\right)\omega\right\|\right)}{\alpha\left(1 - c\left\|F'\left(\overline{u}\right)\omega\right\|\right)}\,, \tag{20}$$

where $g(\alpha,\delta) = \delta + \sqrt{(\delta+cs)^2 + 2\alpha s\left(1 + c\left\|F'\left(\overline{u}\right)\omega\right\|\right)}$ *and* $s = D_h^\xi\left(\overline{u} - \alpha\omega, \overline{u}\right)$*.*

Proof. Since u_α^δ is the minimizer of (2), it follows that

$$\begin{aligned}
0 &\geq \frac{1}{2}\left\|F\left(u_\alpha^\delta\right) - y^\delta\right\|^2 - \frac{1}{2}\left\|F\left(u\right) - y^\delta\right\|^2 + \alpha\left(h\left(u_\alpha^\delta\right) - h\left(u\right)\right)\\
&= \frac{1}{2}\left\|F\left(u_\alpha^\delta\right)\right\|^2 - \frac{1}{2}\left\|F\left(u\right)\right\|^2 + \left\langle F\left(u\right) - F\left(u_\alpha^\delta\right)\ ,\ y^\delta\right\rangle_{\mathcal{H}}\\
&\quad + \alpha\left(h\left(u_\alpha^\delta\right) - h\left(u\right)\right)\\
&= \Phi\left(u_\alpha^\delta\right) - \Phi\left(u\right)\,.
\end{aligned} \tag{21}$$

where $\Phi\left(u\right) = \frac{1}{2}\left\|F\left(u\right) - q\right\|^2 + \alpha D_h^\xi\left(u,\overline{u}\right) - \left\langle F\left(u\right)\ ,\ y^\delta - q\right\rangle_{\mathcal{H}} + \alpha\left\langle \xi\ ,\ u\right\rangle$, $q = F\left(\overline{u}\right) - \alpha F'\left(\overline{u}\right)\omega$ and ξ is given by source condition (19).

From (21) we have $\Phi\left(u_\alpha^\delta\right) \leq \Phi\left(u\right)$. By the definition of $\Phi\left(\cdot\right)$, taking $u = \overline{u} - \alpha\omega$ and setting $v = F\left(u_\alpha^\delta\right) - F\left(\overline{u}\right) + \alpha F'\left(\overline{u}\right)\omega$ we obtain

$$\frac{1}{2}\left\|v\right\|^2 + \alpha D_h^\xi\left(u_\alpha^\delta,\overline{u}\right) \quad\leq\quad \alpha s + T_1 + T_2 + T_3\,, \tag{22}$$

where s is given in the theorem, and

$$T_1 = \frac{1}{2}\left\|F\left(\overline{u} - \alpha\omega\right) - F\left(\overline{u}\right) + \alpha F'\left(\overline{u}\right)\omega\right\|^2\,,$$

$$T_2 = \left|\left\langle F\left(u_\alpha^\delta\right) - F\left(\overline{u} - \alpha\omega\right)\ ,\ y^\delta - y\right\rangle_{\mathcal{H}}\right|\,,$$

$$T_3 = \alpha\left\langle F'\left(\overline{u}\right)\omega\ ,\ F\left(u_\alpha^\delta\right) - F\left(\overline{u} - \alpha\omega\right) - F'\left(\overline{u}\right)\left(u_\alpha^\delta - \left(\overline{u} - \alpha\omega\right)\right)\right\rangle_{\mathcal{H}}\,.$$

The next step is to estimate the constants T_j, $j = 1,2,3$ above. We use the nonlinear condition (13), Cauchy-Schwarz, and some algebraic manipulation to obtain $T_1 \leq \frac{c^2 s^2}{2}$,

$$\begin{aligned}
T_2 &\leq \left|\left\langle v\ ,\ y^\delta - y\right\rangle_{\mathcal{H}}\right| + \left|\left\langle F\left(\overline{u} - \alpha\omega\right) - F\left(\overline{u}\right) + \alpha F'\left(\overline{u}\right)\omega -\ ,\ y^\delta - y\right\rangle_{\mathcal{H}}\right|\\
&\leq \left\|v\right\|\left\|y^\delta - y\right\| + cD_h^\xi\left(\overline{u} - \alpha\omega,\overline{u}\right)\left\|y^\delta - y\right\|\\
&\leq \delta\left\|v\right\| + \delta cs\,,
\end{aligned}$$

and

$$\begin{aligned}
T_3 &= \alpha \left\langle F'(\overline{u})\,\omega \;,\; F\left(u_\alpha^\delta\right) - F(\overline{u}) - F'(\overline{u})\left(u_\alpha^\delta - \overline{u}\right)\right\rangle_{\mathcal{H}} \\
&\quad + \alpha \left\langle F'(\overline{u})\,\omega \;,\; -\left(F(\overline{u}-\alpha\omega) - F(\overline{u}) + \alpha F'(\overline{u})\,\omega\right)\right\rangle_{\mathcal{H}} \\
&\leq \alpha \left\|F'(\overline{u})\,\omega\right\| \left\|F\left(u_\alpha^\delta\right) - F(\overline{u}) - F'(\overline{u})\left(u_\alpha^\delta - \overline{u}\right)\right\| \\
&\quad + \alpha \left\|F'(\overline{u})\,\omega\right\| \left\|F(\overline{u}-\alpha\omega) - F(\overline{u}) + \alpha F'(\overline{u})\,\omega\right\| \\
&\leq \alpha \left\|F'(\overline{u})\,\omega\right\| c D_h^\xi\left(u_\alpha^\delta, \overline{u}\right) + \alpha \left\|F'(\overline{u})\,\omega\right\| c D_h^\xi\left(\overline{u}-\alpha\omega, \overline{u}\right) \\
&= \alpha c \left\|F'(\overline{u})\,\omega\right\| D_h^\xi\left(u_\alpha^\delta, \overline{u}\right) + \alpha c s \left\|F'(\overline{u})\,\omega\right\| \,.
\end{aligned}$$

Using these estimates in (22), we obtain

$$\begin{aligned}
\|v\|^2 + 2\alpha D_h^\xi\left(u_\alpha^\delta, \overline{u}\right)\left[1 - c\left\|F'(\overline{u})\,\omega\right\|\right] \;\leq\;& 2\delta\|v\| + 2\alpha s + (cs)^2 \\
& + 2\delta c s + 2\alpha c s \left\|F'(\overline{u})\,\omega\right\| \,.
\end{aligned}$$

Analogously as in the proof of theorem 2.3, each term on the left hand side of the last inequality is estimated separately by the right hand side. This allows the derivation of an inequality described by a polynomial of second degree. From this inequality, the theorem follows. □

Theorem 3.5 (Convergence)**.** *Let assumptions 1.1, 3.1 hold and assume $\mathcal{H}$ to be a Hilbert space. Moreover, assume the existence of an h-minimizing solution $\overline{u}$ of* (1) *in the interior of $\mathcal{D}(F)$, and also the existence of $\omega \in \mathcal{U}$ such that* (19) *is satisfied, and $c\left\|F'(\overline{u})\,\omega\right\| < 1$. Then, for α sufficiently small the following estimates hold:*

$$\left\|F(u_\alpha) - F(\overline{u})\right\| \leq \alpha\left\|F'(\overline{u})\,\omega\right\| + \sqrt{(cs)^2 + 2\alpha s\left(1 + c\left\|F'(\overline{u})\,\omega\right\|\right)}\,,$$

$$D_h^\xi(u_\alpha, \overline{u}) \leq \frac{\alpha s + (cs)^2/2 + \alpha c s\left\|F'(\overline{u})\,\omega\right\|_{\mathcal{H}}}{\alpha\left(1 - c\left\|F'(\overline{u})\,\omega\right\|_{\mathcal{H}}\right)}\,, \tag{23}$$

where $s = D_h^\xi(\overline{u} - \alpha\omega, \overline{u})$.

Proof. The proof is analogous to the proof of theorem 3.4, taking $\delta = 0$. □

Corollary 3.6. *Let assumptions of the theorem 3.4 hold true. Moreover, assume that h is twice differentiable in a neighborhood U of $\overline{u}$, and that there exists a number $M > 0$ such that for all $u \in U$ and for all $v \in \mathcal{U}$, the inequality $\langle h''(u)v\,,\,v\rangle \leq M\|v\|^2$ holds. Then, for the choice of parameter $\alpha \sim \delta^{\frac{2}{3}}$ we have $D_h^\xi\left(u_\alpha^\delta, \overline{u}\right) = \mathcal{O}\left(\delta^{\frac{4}{3}}\right)$, while for exact data we obtain $D_h^\xi\left(u_\alpha^\delta, \overline{u}\right) = \mathcal{O}\left(\alpha^2\right)$.*

Proof. The proof is similar to the proof of corollary 2.5 and is based on theorems 3.4 and 3.5. □

4 An Iterated Tikhonov Method for Nonlinear Problems

On this section we investigate an iterative method based on Bregman distances for nonlinear problems. We consider the operator $F : \mathcal{U} \to \mathcal{H}$ defined between a Banach space and a Hilbert space, Fréchet differentiable with closed and convex domain $\mathscr{D}(F)$. The operator equation (1) is ill-posed in the sense of Hadamard, the solution does not need to be unique, so we define

$$\mathcal{S}(y) = \{u \in \mathscr{D}(F) \ \mid \ F(u) = y\} \ .$$

The method was originally proposed by Osher in [7], who generalized the ideas of the method ROF [10] (see [1] for further details).

The analyzed method generalizes the iterated Tikhonov method and is given by

$$u_{k+1} \in argmin \left\{ \frac{1}{2} \left\| F(u) - y^\delta \right\|^2 + \alpha_k D_h^{\xi_k}(u, u_k) \right\} , \tag{24}$$

where the subgradient required is updated by the rule

$$\xi_{k+1} = \xi_k - \frac{1}{\alpha_k} F'(u_{k+1})^* \left(F(u_{k+1}) - y^\delta \right) . \tag{25}$$

Algorithm 1 generalized Tikhonov with Bregman distance

Require: $u_0 \in \mathscr{D}(F) \cap dom\ h$, $\xi_0 \in \partial h(u_0)$

1: $k = 0$
2: $\alpha_k > 0$
3: **repeat**
4: $\quad u_{k+1} \in argmin \left\{ \frac{1}{2} \left\| F(u) - y^\delta \right\|^2 + \alpha_k D_h^{\xi_k}(u, u_k) \right\}$
5: $\quad \xi_{k+1} = \xi_k - \frac{1}{\alpha_k} F'(u_{k+1})^* \left(F(u_{k+1}) - y^\delta \right)$
6: $\quad k = k + 1$
7: $\quad \alpha_k > 0$
8: **until** convergence

Remark 4.1. *It is easy to see that the definition* (25) *is equivalent to*

$$\xi_{k+1} = \xi_0 - \sum_{j=0}^{k} \frac{1}{\alpha_j} F'(u_{j+1})^* \left(F(u_{j+1}) - y^\delta \right) . \tag{26}$$

We obtain monotonicity of residuals directly from the above definitions.

Lemma 4.2. *The iterates defined by algorithm 1 satisfy the estimate*

$$\left\| y^\delta - F(u_{k+1}) \right\| \leq \left\| y^\delta - F(u_k) \right\| .$$

Proof. Defining $J_\alpha^\delta(u) = \frac{1}{2} \left\| F(u) - y^\delta \right\|^2 + \alpha_k D_h^{\xi_k}(u, u_k)$, the lemma follows the fact that u_{k+1} is a minimizer of (24), i.e., $J_\alpha^\delta(u_{k+1}) \leq J_\alpha^\delta(u_k)$. □

Under a nonlinearity condition on F we prove a monotonicity result for the Bregman distance, i.e., $D_h^{\xi_{k+1}}(\overline{u}, u_{k+1}) \leq D_h^{\xi_k}(\overline{u}, u_k)$.

Lemma 4.3. *Let $y^\delta \in \mathcal{H}$ be the given data. If for some u_k and ξ_k, the iterate u_{k+1} in* (24) *satisfies*

$$\left\|y^\delta - F(u_{k+1}) - F'(u_{k+1})(\overline{u} - u_{k+1})\right\| \leq c\left\|y^\delta - F(u_{k+1})\right\| ,$$

for some $0 < c < 1$, then

$$D_h^{\xi_{k+1}}(\overline{u}, u_{k+1}) - D_h^{\xi_k}(\overline{u}, u_k) + D_h^{\xi_k}(u_{k+1}, u_k) \leq -\frac{1-c}{\alpha_k}\left\|y^\delta - F(u_{k+1})\right\|^2 . \tag{27}$$

Proof. This result follows from the equality (see [1] for details)

$$D_h^{\xi k+1}(\overline{u}, u_{k+1}) - D_h^{\xi_k}(\overline{u}, u_k) + D_h^{\xi_k}(u_{k+1}, u_k) = \langle \xi_{k+1} - \xi_k \ , \ u_{k+1} - \overline{u}\rangle .$$

Using (25) on the right hand side, summing $\pm(F(u_{k+1}) - y^\delta)$ on the second term (inside the inner product), using Cauchy-Schwarz and the assumptions, we conclude that estimate (27) holds. □

The subsequent results are obtained assuming that the nonlinear operator F is such that $\mathscr{D}(F) \subseteq L^2(\Omega)$ and $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, and assuming that the regularization convex functional is given by

$$h(u) = \frac{1}{2}\|u\|^2_{L^2(\Omega)} + |u|_{BV(\Omega)} . \tag{28}$$

Lemma 4.4. *If $h(\cdot)$ is the convex functional defined by* (28)*, then*

$$\frac{1}{2}\|v - u\|^2_{L^2(\Omega)} \leq D_h^\xi(v, u)$$

for every $u, v \in \mathscr{D}(F)$ and $\xi \in \partial h(u)$.

Proof. This proof is straightforward, once we establish some auxiliary properties concerning calculus of subgradients. For a complete proof we refer the reader to [2]. □

Assumption 4.5. *Let $F : \mathscr{D}(F) \subset L^2(\Omega) \to \mathcal{H}$ be a weakly sequentially closed nonlinear operator, $F'(\cdot)$ be locally bounded. Moreover, suppose that the nonlinearity condition*

$$\|F(v) - F(u) - F'(u)(v - u)\| \leq \eta\|u - v\|_{L^2(\Omega)}\|F(u) - F(v)\| \tag{29}$$

is satisfied for every u, $v \in \mathcal{B}_\rho(\overline{u}) \cap \mathscr{D}(F)$, where $\eta, \rho > 0$ and $\mathcal{B}_\rho(\overline{u})$ denotes the open ball around $\overline{u}$ of radius ρ in $L^2(\Omega)$ and $\overline{u} \in \mathcal{S}(y) \cap dom\ h$.

Remark 4.6. *We can rewrite the left side of the inequality given in (29) as*

$$\|F'(u)(v - u)\| \leq \left(1 + \eta\|u - v\|_{L^2(\Omega)}\right)\|F(u) - F(v)\| .$$

The next result gives the mean result about the sequence of iterates from algorithm 1 is well-defined.

Proposition 4.7. *Let assumption 4.5 hold, $k \in \mathbb{N}$ and u_k, ξ_k be a pair of iterates according to algorithm 1. Then, there exists a minimizer u_{k+1} for (24) and ξ_{k+1} given by (25) satisfies $\xi_{k+1} \in \partial h(u_{k+1})$.*

Proof. If there exists an u such that $J^{\delta}_{\alpha_k}(u)$ is finite, then there is a sequence $(u_j) \in \mathscr{D}(F) \cap BV(\Omega)$ such that $\lim_j J^{\delta}_{\alpha_k}(u_j) \to \beta$, where $\beta = \inf\left\{J^{\delta}_{\alpha_k}(u) \mid u \in \mathscr{D}(F)\right\}$. In particular, $D^{\xi_k}_h(u_j, u_k) \leq \dfrac{M}{\alpha_k}$. By definition of the Bregman distance, together with (28) and observing that $\dfrac{1}{2}\|u_j\|^2_{L^2(\Omega)} - \langle \xi_k\ , u_j\rangle = \dfrac{1}{2}\|u_j - \xi_k\|^2_{L^2(\Omega)} - \dfrac{1}{2}\|\xi_k\|^2_{L^2(\Omega)}$, we obtain $|u_j|_{BV(\Omega)} \leq \tilde{M}_k$, where $\tilde{M}_k \geq 0$ depends on the current iterates. Thus, the existence of a minimizer follows from compactness arguments.

It remains to prove that $\xi_{k+1} \in \partial h(u_{k+1})$. This result follows from the inequality $\phi_2(v) \geq \phi_2(u_{k+1}) + \langle -\phi_1'(u_{k+1})\ , v - u_{k+1}\rangle$, where $\phi_1(u) = \dfrac{1}{2}\left\|y^{\delta} - F(u)\right\|^2_{L^2(\Omega)}$ and $\phi_2(u) = \alpha_k D^{\xi_k}_h(u, u_k)$ (see [1, 2] for details). □

4.1 Main results

The main results of this section give sufficient conditions to guarantee existence of a convergence subsequence in algorithm 1, (for both exact and noisy data). In particular, for noisy data, we introduce a stopping rule based on the discrepancy principle. For a complete proof we refer the reader to [2].

Theorem 4.8 (Convergence)**.** *Let the assumption 4.5 hold, $\gamma < \min\left\{\frac{1}{\eta}, \frac{\rho}{2}\right\}$ for η, ρ as in (29), $0 < \alpha_k < \bar{\alpha}$, $h(\overline{u}) < \infty$. Moreover, assume that the starting values u_0, $\xi_0 \in L^2(\Omega)$ satisfy $D^{\xi_0}_h(\overline{u}, u_0) < \frac{\gamma^2}{8}$ for some $\overline{u} \in \mathcal{S}(y)$. Then, for exact data, the sequence (u_k) has a subsequence converging to some $u \in \mathcal{S}(y)$ in the weak-$*$ topology of $BV(\Omega)$. Moreover, if $\mathcal{S}(y) \cap \overline{\mathcal{B}_\rho(\overline{u})} = \{\overline{u}\}$, then $u_k \overset{*}{\rightharpoonup} \overline{u}$ in $BV(\Omega)$.*

Proof. **Step 1:** First we rewrite the assumption in the form $2\sqrt{2D^{\xi_0}_h(\overline{u}, u_0)} < \gamma$. Assuming that the same condition holds for a pair of iterates u_k, ξ_k we proof by induction that it also holds for the index $k+1$.

Let u_{k+1} be the minimizer of $J_{\alpha_k}(\cdot)$, so $J_{\alpha_k}(u_{k+1}) \leq J_{\alpha_k}(\overline{u})$. Thus we rewrite the inequality, then apply lemma 4.4 twice, and conclude that $\|u_{k+1} - \overline{u}\|_{L^2(\Omega)} < \gamma$. Hence, assumption 4.5 is satisfied and the lemma 4.3 hold for all iterates.

Step 2: In this step we proof that $\sum_{i=0}^{\infty} \dfrac{1}{\alpha_i}\|y - F(u_{i+1})\|^2 < \infty$.
As in the previous step, it follows from Lemma 4.3 that (27) holds for every k. Adding up the first k terms and using the assumptions on the starting values, we conclude that

$$D^{\xi_k}_h(\overline{u}, u_k) + \sum_{i=0}^{k-1} D^{\xi_i}_h(u_{i+1}, u_i) + \sum_{i=0}^{k-1} \frac{1-\eta\gamma}{\alpha_i}\|y - F(u_{i+1})\|^2 \leq \frac{\gamma^2}{8}\,.$$

Since all terms on the left hand side are positive, step 2 follows from the third term taking the limit as k tends to infinity. Note that this series is convergent, by the convergence criterion for series follows $F(u_k) \to y$.

Step 3: We show the uniform limitation of the sequence $(h(u_k))$. Applying the Bregman distance (it is always grater than zero) we have $h(u_k) \leq h(\overline{u}) - \langle \xi_k , \overline{u} - u_k \rangle$. Thus, by remark 4.1, $\|u_{k+1} - \overline{u}\|_{L^2(\Omega)} < \gamma$ and the Cauchy-Schwarz inequality, we obtain

$$h(u_k) \leq h(\overline{u}) + \gamma \|\xi_0\|_{L^2(\Omega)} + \sum_{i=0}^{k-1} \frac{1}{\alpha_i} \|F(u_{i+1}) - y\| \, \|F'(u_{i+1})(\overline{u} - u_k)\| \, .$$

In order to estimate the term inside the sum, note that for $0 \leq i \leq k-1$, the estimate $\|F'(u_{i+1})(\overline{u} - u_k)\| \leq \|F'(u_{i+1})(\overline{u} - u_{i+1})\| + \|F'(u_{i+1})(u_k - u_{i+1})\|$ holds. Now, using remark 4.6 twice, we find the bound $(3 + 5\eta\gamma)\|F(u_{i+1}) - y\|$ for the previous estimate. Substituting this estimate in the sum above and using step 2, the desired boundedness of the sequence $(h(u_k))$ follows.

Step 4: We know that $|h(u_k)| = h(u_k) \leq N$, for some $N > 0$ (see (28)). The remaining assertions of the theorem follow from standard compactness results (Banach-Alaoglu theorem). We use the closed graph theorem to ensure that the limit of the obtained sequence belongs to $\mathcal{S}(y)$. □

In the case of noisy data we use a generalized discrepancy principle as stopping rule. The stopping index is defined as the smallest integer k^* satisfying

$$\|F(u_{k^*}) - y^\delta\| \leq \tau\delta \tag{30}$$

where $\tau > 1$ still has to be chosen.

Theorem 4.9 (Stability)**.** *Let assumption 4.5 hold,* $\gamma < \min\left\{\frac{1}{\eta}, \frac{\rho}{2}\right\}$ *for* η *and* ρ *as in* (29)*,* $0 < \underline{\alpha} \leq \alpha_k \leq \overline{\alpha}$*,* $h(\overline{u}) < \infty$ *and the starting values* u_0*,* $\xi_0 \in L^2(\Omega)$ *satisfy* $D_h^{\xi_0}(\overline{u}, u_0) < \frac{\gamma^2}{8}$ *for an* $\overline{u} \in \mathcal{S}(y)$*. Moreover, let* $\delta_m > 0$ *be a sequence such that* $\delta_m \to 0$*, and let the corresponding stopping indices* k_m^* *be chosen according to* (30) *with* $\tau > (1+\eta\gamma)/(1-\eta\gamma)$*. Then for every* δ_m *the stopping index is finite and the sequence* $\left(u_{k_m^*}\right)$ *has a subsequence converging to an* $u \in \mathcal{S}(y)$ *in the weak-∗ topology of* $BV(\Omega)$*. Moreover, if* $\mathcal{S}(y) \cap \overline{\mathcal{B}_\rho(\overline{u})} = \{\overline{u}\}$*, then* $u_{k_m^*} \overset{*}{\rightharpoonup} \overline{u}$ *in* $BV(\Omega)$*.*

Proof. **Step 1:** This step is analogous to step 1 in the proof of theorem 4.8. For each k such that $k < k^*-1$, we have $\left\|F(u_k) - y^\delta\right\| > \tau\delta$. By induction one can prove that $\|u_{k+1} - \overline{u}\|_{L^2(\Omega)} < \gamma$, and that the nonlinear condition (27) holds. Therefore, lemma 4.3 holds for $c = \frac{1}{\tau}(1+\eta\gamma) + \eta\gamma$.
Step 2: We show that the stopping index k^* is finite. Analogous to step 2 in the proof of theorem 4.8, we sum up the first $k^* - 1$ terms of (27), obtaining

$$\sum_{i=0}^{k^*-2} \frac{1}{\alpha_i} \left\|y^\delta - F(u_{i+1})\right\|^2 < \frac{\gamma^2}{8(1-c)} \, . \tag{31}$$

Since for every $k < k^* - 1$ the inequality $\left\|F(u_k) - y^\delta\right\| > \tau\delta$ holds, we use this inequality on the left hand side of the above estimate and conclude that

$$k^* < \left(\frac{\gamma}{\tau\delta}\right)^2 \frac{\overline{\alpha}}{8(1-c)} + 1 \, .$$

Step 3: In order to prove the convergence of the series in (31), notice that the right hand side of (31) does not depend on k^*.

Step 4: Analogous to step 3 in the proof of theorem 4.8, we use the Bregman distance, and remark 4.1 to conclude that

$$\begin{aligned} h\left(u_{k^*}\right) \leq\ & h\left(\overline{u}\right)+\left|\left\langle \xi_0\ ,\ \overline{u}-u_{k^*}\right\rangle\right| \\ & +\sum_{i=0}^{k^*-2}\frac{1}{\alpha_i}\left|\left\langle F\left(u_{i+1}\right)-y^{\delta}\ ,\ F'\left(u_{i+1}\right)\left(\overline{u}-u_{k^*}\right)\right\rangle_{\mathcal{H}}\right| \\ & +\frac{1}{\alpha_{k^*-1}}\left|\left\langle F\left(u_{k^*}\right)-y^{\delta}\ ,\ F'\left(u_{k^*}\right)\left(\overline{u}-u_{k^*}\right)\right\rangle_{\mathcal{H}}\right| . \end{aligned}$$

In the sequel we estimate the three terms on the right hand side of this inequality. For the first of them we have $\left\|\overline{u}-u_{k^*}\right\|_{L^2(\Omega)}<\rho$. Indeed, on step k^*-1 we have u_{k^*} as minimizer of $J_{\alpha_k}^{\delta}(\cdot)$, thus $J_{\alpha_k}^{\delta}\left(u_{k^*}\right)\leq J_{\alpha_k}^{\delta}\left(\overline{u}\right)$. Rearranging the terms and discarding some positive terms, it follows that $D_h^{\xi_{k^*-1}}\left(u_{k^*},u_{k^*-1}\right)<\dfrac{\delta^2}{2\alpha_{k^*-1}}+\dfrac{\gamma^2}{8}$. Finally, we apply lemma 4.4 with $\delta<\overline{\delta}=\sqrt{3/4\gamma^2\underline{\alpha}}$. To estimate the last two terms we use Cauchy-Schwarz, assumption 4.5, lemma 4.2 and 4.3, remark 4.6 together with steps 1, 2 and 3 above. Summarizing, we obtain

$$h\left(u_{k^*}\right) < h\left(\overline{u}\right)+\rho\left\|\xi_0\right\|_{L^2(\Omega)}+\frac{\left(c+3+4\eta\rho\right)}{\left(1-c\right)}\frac{\gamma^2}{8}+\frac{\tau\delta^2\left(1+\eta\rho\right)\left(1+\tau\right)}{\underline{\alpha}} .$$

Step 5: This step is very similar to step 4 in the proof of theorem 4.8. We just need to show that $F\left(u_{k_m^*}\right)\to y$. This convergence follows from the estimate

$$\begin{aligned} \left\|F\left(u_{k_m^*}\right)-y\right\| & \leq \left\|F\left(u_{k_m^*}\right)-y^{\delta_m}\right\|+\left\|y^{\delta_m}-y\right\| \\ & \leq \left(1+\tau\right)\delta_m \end{aligned}$$

when δ_m goes to zero. □

A Definitions

Definition A.1. *Given $h(\cdot)$ a convex functional, one can define the Bregman distance with respect to h between the elements $v,u\in$ dom h as*

$$D_h\left(v,u\right)=\left\{D_h^{\xi}\left(v,u\right)\mid \xi\in\partial h\left(u\right)\right\} ,$$

where $\partial h(u)$ denotes the subdifferential of h at u and

$$D_h^{\xi}\left(v,u\right)=h\left(v\right)-h\left(u\right)-\left\langle \xi\ ,\ v-u\right\rangle .$$

We remark that $\langle\ ,\ \rangle$ denotes the standard dual pairing (duality product) with respect to $\mathcal{U}^*\times\mathcal{U}$.

Another important definition is the generalized solution, we introduce the notion of the *h-minimizing solution* bellow.

Definition A.2. *An element* $\overline{u} \in dom\ h \cap \mathscr{D}(F)$ *is called an h-minimizing solution of* (1) *if it minimizes the functional h among every possible solutions, that is,*

$$\overline{u} = argmin \{h(u) \ \mid \ F(u) = y\} \ .$$

Whenever we need, we can choose the least-square solution instead the standard solution $F(u) = y$.

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