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Boundary Stabilization of the Transmission Problem for the Bernoulli-Euler Plate Equation

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ABSTRACT

In this paper we consider a boundary stabilization problem for the transmission Bernoulli-Euler plate equation. We prove uniform exponential energy decay under natural conditions.

RESUMEN

En este artículo consideramos un problema de estabilización en la frontera para la ecuación de Bernoulli-Euler Plate. Nosotros probamos decaimiento exponencial uniforme de la energia sobre condiciones naturales

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1 Introduction and Statement of Results

Let $\Omega_1 \subset \Omega \subset \mathbf{R}^n$, $n \geq 2$, be strictly convex, bounded domains with smooth boundaries $\Gamma_1 = \partial\Omega_1$, $\Gamma = \partial\Omega$, $\Gamma_1 \cap \Gamma = \emptyset$. Then $\mathcal{O} = \Omega \backslash \Omega_1$ is a bounded, connected domain with boundary $\partial\mathcal{O} = \Gamma_1 \cup \Gamma$. We are going to study the following mixed boundary value problem

$$\begin{cases} (\partial_t^2 + c^2\Delta^2)u_1(x,t) = 0 \quad \text{in} \quad \Omega_1 \times (0,+\infty), \\ (\partial_t^2 + \Delta^2)u_2(x,t) = 0 \quad \text{in} \quad \mathcal{O} \times (0,+\infty), \\ u_1|_{\Gamma_1} = u_2|_{\Gamma_1},\ \partial_\nu u_1|_{\Gamma_1} = \partial_\nu u_2|_{\Gamma_1},\ c\Delta u_1|_{\Gamma_1} = \Delta u_2|_{\Gamma_1},\ c\partial_\nu \Delta u_1|_{\Gamma_1} = \partial_\nu \Delta u_2|_{\Gamma_1}, \\ u_2|_{\Gamma} = 0,\ \Delta u_2|_{\Gamma} = -a\partial_\nu\partial_t u_2|_{\Gamma}, \\ u_1(x,0) = u_1^0(x),\ \partial_t u(x,0) = u_1^1(x) \quad \text{in} \quad \Omega_1, \\ u_2(x,0) = u_2^0(x),\ \partial_t u(x,0) = u_2^1(x) \quad \text{in} \quad \mathcal{O}, \end{cases} \tag{1.1}$$

where $c > 1$ is a constant, ν denotes the inner unit normal to the boundary and a is a non-negative function on Γ. We suppose that there exists a constant $a_0 > 0$ such that

$$a \geq a_0 \quad \text{on} \quad \Gamma. \tag{1.2}$$

The controllability of the dynamical system determined by (1.1) without transmission (i.e. when $\Omega_1 = \emptyset$) has been investigated by Krabs, Leugering and Seidman [10], Leugering [9], Lasiecka and Triggiani [5], Ammari and Khenissi [1]. With transmission the exact controllability has been established by Liu and Williams [11] in the case when the control is active on the part of boundary whereas the controlled part of the boundary is supposed to satisfy the Lions geometric condition.

In this paper we prove that, under (1.2), the solutions of (1.1) are exponentially stable in the energy space. The energy of a solution

$$u = \begin{cases} u_1 & \text{in} \quad \Omega_1, \\ u_2 & \text{in} \quad \mathcal{O}, \end{cases}$$

of (1.1) at the time instant t is defined by

$$E(t) = \frac{1}{2}\int_\Omega \left(|\partial_t u(x,t)|^2 + \alpha^2(x)\ |\Delta u(x,t)|^2\right)\alpha(x)^{-1}\,dx,$$

where

$$\alpha(x) = \begin{cases} c & \text{in} \quad \Omega_1, \\ 1 & \text{in} \quad \mathcal{O}. \end{cases}$$

The solution of (1.1) satisfies the energy identity

$$E(t_2) - E(t_1) = -\int_{t_1}^{t_2}\int_\Gamma a\,|\partial_\nu\partial_t u(x,t)|^2\,d\Gamma\,dt$$

for all $t_2 > t_1 \geq 0$, and therefore the energy is a nonincreasing function of the time variable t. Introduce the Hilbert space $\mathcal{H} = V \times H$, where $H = L^2(\Omega, \alpha(x)^{-1}dx)$ and the space V is definded as follows. On the Hilbert space H consider the operator G defined by

$$G\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -\,c\Delta u_1 \\ -\,\Delta u_2 \end{pmatrix}, \quad \forall \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathcal{D}(G),$$

with domain

$$\mathcal{D}(G) = \Big\{(u_1, u_2) \in H = L^2(\Omega_1, c^{-1}dx) \oplus L^2(\mathcal{O}) : u_1 \in H^2(\Omega_1), u_2 \in H^2(\mathcal{O}),$$

$$u_2|_\Gamma = 0,\, u_1|_{\Gamma_1} = u_2|_{\Gamma_1},\, \partial_\nu u_1|_{\Gamma_1} = \partial_\nu u_2|_{\Gamma_1}\Big\}.$$

The operator G is a strictly positive self-adjoint one with a compact resolvent. Set $V = \mathcal{D}(G)$ with norm $\|f\|_V := \|Gf\|_H$. The solutions to (1.1) can be expressed by means of a semigroup on $\mathcal{H}$ as follows

$$\begin{pmatrix} u \\ u_t \end{pmatrix} = e^{it\mathcal{A}} \begin{pmatrix} u^0 \\ u^1 \end{pmatrix},$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ v \end{pmatrix} = -i \begin{pmatrix} v \\ -\,\alpha^2\,\Delta^2 u \end{pmatrix}, \quad \forall \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}(\mathcal{A}),$$

with domain

$$\mathcal{D}(\mathcal{A}) = \Big\{(u, v) \in \mathcal{H} :\, (v, \Delta^2 u) \in \mathcal{H},\, u|_\Gamma = 0, \Delta u|_\Gamma = -a\,\partial_\nu v|_\Gamma,\, u_1|_{\Gamma_1} = u_2|_{\Gamma_1},$$

$$\partial_\nu u_1|_{\Gamma_1} = \partial_\nu u_2|_{\Gamma_1},\, c\,\Delta u_1|_{\Gamma_1} = \Delta u_2|_{\Gamma_1},\, c\,\partial_\nu \Delta u_1|_{\Gamma_1} = \partial_\nu \Delta u_2|_{\Gamma_1}\Big\}.$$

Using Green's formula it is easy to see that

$$\mathrm{Im}\left\langle \mathcal{A}\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right\rangle_{\mathcal{H}} = \int_\Gamma a\, |\partial_\nu v|^2\, d\Gamma \geq 0, \qquad \forall \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}(\mathcal{A}), \tag{1.3}$$

which in turn implies that $\mathcal{A}$ generates a continuous semigroup (e.g. see Theorems 4.3 and 4.6 from [12, p.14-15]). Let $\rho(\mathcal{A})$ denote the resolvent set of $\mathcal{A}$. Since the resolvent of $\mathcal{A}$ is a compact operator on $\mathcal{H}$, $\mathbf{C} \setminus \rho(\mathcal{A})$ is a discrete set of eigenvalues of $\mathcal{A}$. It follows from (1.3) that

$$\mathbf{C} \setminus \rho(\mathcal{A}) \subset \{z \in \mathbf{C} : \mathrm{Im}\, z \geq 0\}.$$

Moreover, using the Carleman estimates of [3] one can conclude that, under the condition (1.2), the operator $\mathcal{A}$ has no eigenvalues on the real axis. Our main result is the following

Theorem 1.1 *Assume (1.2) fulfilled. Then there exist constants $C, \gamma > 0$ such that*

$$E(t) \leq C\, e^{-\gamma t}\, E(0). \tag{1.4}$$

When $\Omega_1 = \emptyset$ the estimate (1.4) follows from combining the results of [1] and [2] under the more natural assumptions that (1.2) holds only on some non-empty part Γ_0 of Γ and that every generalized ray in Ω hits Γ_0 at a non-diffractive point (see [2] for the definition and more details). Therefore in the general case of transmission, the estimate (1.4) should hold true under less restrictive conditions, but to our best knoweledge no such results have been proved so far. One of the reasons for this is the fact that the classical flow for the transmission problem is quite complex. Indeed, when a ray in $\mathcal{O}$ hits Γ_1 it splits into two rays - a reflected one staying in $\mathcal{O}$ and another one entering into Ω_1. The picture is similar when a ray in Ω_1 hits Γ_1. In particular, there are rays which never reach the boundary Γ where the dissipation is active. Note that it is crucial for (1.4) to hold that $c > 1$. Roughly speaking, what happens in this case is that the rays staying inside Ω_1 carry a negligible amount of energy. Note that when $c < 1$ the estimate (1.4) does not hold anymore. Indeed, in this case one can use the quasi-modes constructed in [13] (which are due to the existence of the so-called interior totally reflected rays and which are concentrated on Γ_1) to get quasi-modes for our problem, too. Consequently, we conclude that when $c < 1$ there exist infinitely many eigenvalues $\{\lambda_j\}$ of $\mathcal{A}$ such that $0 < \operatorname{Im} \lambda_j \le C_N |\lambda_j|^{-N}$, $\forall N \gg 1$, which is an obstruction to (1.4).

To prove Theorem 1.1 it suffices to show that $(\mathcal{A} - z)^{-1} : \mathcal{H} \to \mathcal{H}$ is $O(1)$ for $z \in \mathbf{R}$, $|z| \gg 1$, which implies that $(\mathcal{A} - z)^{-1}$ is analytic in $\operatorname{Im} z \le C$, $C > 0$. This is carried out in Section 3 (see Theorem 3.1) using the a priori estimates established in Section 2. The advantage of this approach is that the problem of proving (1.4) is reduced to obtaining a priori estimates for the solutions to the corresponding stationary problem (see Theorem 2.1). This in turn is easier than studying (1.1) directly because we can make use of the estimates for the stationary transmission problem obtained in [4] (under the assumptions that Ω_1 is strictly convex and $c > 1$).

2 A Priori Estimates

Let $\lambda \gg 1$ and consider the problem

$$\begin{cases} (-c^2\Delta^2 + \lambda^4)u_1 = v_1 \quad \text{in} \quad \Omega_1, \\ (-\Delta^2 + \lambda^4)u_2 = v_2 \quad \text{in} \quad \mathcal{O}, \\ u_1|_{\Gamma_1} = u_2|_{\Gamma_1},\, \partial_\nu u_1|_{\Gamma_1} = \partial_\nu u_2|_{\Gamma_1},\, c\Delta u_1|_{\Gamma_1} = \Delta u_2|_{\Gamma_1},\, c\partial_\nu \Delta u_1|_{\Gamma_1} = \partial_\nu \Delta u_2|_{\Gamma_1}, \\ u_2|_\Gamma = 0,\, \Delta u_2|_\Gamma = a\left(-i\lambda^2 \partial_\nu u_2|_\Gamma + F\right), \end{cases} \tag{2.1}$$

with functions $v_1 \in L^2(\Omega_1)$, $v_2 \in L^2(\mathcal{O})$ and $F \in L^2(\Gamma)$. In what follows in this section all Sobolev spaces H^s, $s \ge 0$, will be equipped with the semi-classical norm (with a small parameter λ^{-1}). This means that the semi-classical norm in H^s is equivalent to the classical one times a factor λ^{-s}.

Theorem 2.1 *Assume (1.2) fulfilled. Then there exist constants $C, \lambda_0 > 0$ so that for $\lambda \ge \lambda_0$ the solution to (2.1) satisfies the estimate*

$$\|u_1\|_{H^3(\Omega_1)} + \|u_2\|_{H^3(\mathcal{O})} \le C\lambda^{-2} \|v_1\|_{L^2(\Omega_1)} + C\lambda^{-2} \|v_2\|_{L^2(\mathcal{O})} + C\lambda^{-2} \|F\|_{L^2(\Gamma)} . \tag{2.2}$$

Proof. We will first prove the following

Lemma 2.2 *Assume (1.2) fulfilled. Then for every $\lambda, \mu > 0$ we have the estimate*

$$\|\partial_\nu u_2|_\Gamma\|_{L^2(\Gamma)} \le C\mu \|u_1\|_{L^2(\Omega_1)} + C\mu \|u_2\|_{L^2(\mathcal{O})}$$
$$+C\lambda^{-2}\mu^{-1} \|v_1\|_{L^2(\Omega_1)} + C\lambda^{-2}\mu^{-1} \|v_2\|_{L^2(\mathcal{O})} + C\lambda^{-2} \|F\|_{L^2(\Gamma)} . \tag{2.3}$$

Proof. Applying Green's formula in Ω_1 and $\mathcal{O}$, we get respectively

$$\mathrm{Im} \left\langle c^{-1}v_1, u_1\right\rangle_{L^2(\Omega_1)} = \mathrm{Im} \left\langle c\Delta u_1|_{\Gamma_1}, \partial_\nu u_1|_{\Gamma_1}\right\rangle_{L^2(\Gamma_1)} - \mathrm{Im} \left\langle c\partial_\nu \Delta u_1|_{\Gamma_1}, u_1|_{\Gamma_1}\right\rangle_{L^2(\Gamma_1)} , \tag{2.4}$$

$$\mathrm{Im} \left\langle v_2, u_2\right\rangle_{L^2(\mathcal{O})} = -\mathrm{Im} \left\langle \Delta u_2|_{\Gamma_1}, \partial_\nu u_2|_{\Gamma_1}\right\rangle_{L^2(\Gamma_1)} + \mathrm{Im} \left\langle \partial_\nu \Delta u_2|_{\Gamma_1}, u_2|_{\Gamma_1}\right\rangle_{L^2(\Gamma_1)}$$
$$+\mathrm{Im} \left\langle \Delta u_2|_\Gamma, \partial_\nu u_2|_\Gamma\right\rangle_{L^2(\Gamma)} . \tag{2.5}$$

Summing up (2.4) and (2.5) and using the boundary conditions, we obtain the identity

$$\mathrm{Im} \left\langle c^{-1}v_1, u_1\right\rangle_{L^2(\Omega_1)} + \mathrm{Im} \left\langle v_2, u_2\right\rangle_{L^2(\mathcal{O})}$$
$$= -\lambda^2 \left\langle a\partial_\nu u_2|_\Gamma, \partial_\nu u_2|_\Gamma\right\rangle_{L^2(\Gamma)} + \left\langle F, a\partial_\nu u_2|_\Gamma\right\rangle_{L^2(\Gamma)} . \tag{2.6}$$

By (1.2) and (2.6), we conclude

$$a_0\lambda^2 \|\partial_\nu u_2|_\Gamma\|^2_{L^2(\Gamma)} \le \lambda^2\mu^2 \|u_1\|^2_{L^2(\Omega_1)} + \lambda^2\mu^2 \|u_2\|^2_{L^2(\mathcal{O})}$$
$$+\lambda^{-2}\mu^{-2} \|v_1\|^2_{L^2(\Omega_1)} + \lambda^{-2}\mu^{-2} \|v_2\|^2_{L^2(\mathcal{O})} + C\lambda^{-2} \|F\|^2_{L^2(\Gamma)} , \tag{2.7}$$

which clearly implies (2.3). □

It is easy to see that (2.2) follows from combining Lemma 2.2 with the following

Proposition 2.3 *There exist constants $C, \lambda_0 > 0$ so that for $\lambda \ge \lambda_0$ the solution to (2.1) satisfies the estimate*

$$\|u_1\|_{H^3(\Omega_1)} + \|u_2\|_{H^3(\mathcal{O})} \le C\lambda^{-2} \|v_1\|_{L^2(\Omega_1)} + C\lambda^{-2} \|v_2\|_{L^2(\mathcal{O})}$$
$$+C \|\partial_\nu u_2|_\Gamma\|_{L^2(\Gamma)} + C\lambda^{-2} \|F\|_{L^2(\Gamma)} . \tag{2.8}$$

Proof. Clearly, the functions $\widetilde{u}_1 = (\lambda^2 - c\Delta)u_1$, $\widetilde{u}_2 = (\lambda^2 - \Delta)u_2$ satisfy the equation

$$\begin{cases} (c\Delta + \lambda^2)\widetilde{u}_1 = v_1 \quad \text{in} \quad \Omega_1, \\ (\Delta + \lambda^2)\widetilde{u}_2 = v_2 \quad \text{in} \quad \mathcal{O}, \\ \widetilde{u}_1|_{\Gamma_1} = \widetilde{u}_2|_{\Gamma_1},\, \partial_\nu \widetilde{u}_1|_{\Gamma_1} = \partial_\nu \widetilde{u}_2|_{\Gamma_1}. \end{cases} \tag{2.9}$$

Using the results of [4] we will prove the following

Proposition 2.4 *There exist constants $C, \lambda_0 > 0$ so that for $\lambda \geq \lambda_0$ the solution to (2.9) satisfies the estimate*

$$\begin{aligned}\|\widetilde{u}_1\|_{H^1(\Omega_1)} + \|\widetilde{u}_2\|_{H^1(\mathcal{O})} \leq C\lambda^{-1}\|v_1\|_{L^2(\Omega_1)} + C\lambda^{-1}\|v_2\|_{L^2(\mathcal{O})} \\ +C\|\widetilde{u}_2|_\Gamma\|_{L^2(\Gamma)} + C\lambda^{-1}\|\partial_\nu \widetilde{u}_2|_\Gamma\|_{L^2(\Gamma)}.\end{aligned} \tag{2.10}$$

Proof. Denote by $d(x)$ the distance between $x \in \mathcal{O}$ and Γ. Choose a function $\chi \in C^\infty(\mathbf{R}^n)$, $\chi = 1$ on Ω_1, $\chi = 0$ on $\mathbf{R}^n \setminus \Omega$, so that $\operatorname{supp}[\Delta, \chi] \subset \{x \in \mathcal{O} : d(x) \leq \delta\}$, where $0 < \delta \ll 1$. Choose also a function $\varphi(t) \in C_0^\infty(\mathbf{R})$, $0 \leq \varphi \leq 1$, $\varphi(t) = 1$ for $|t| \leq \delta$, $\varphi(t) = 0$ for $|t| \geq 2\delta$, $d\varphi(t)/dt \leq 0$ for $t \geq 0$, and set $\psi(x) = \varphi(d(x))$. Clearly, $\psi = 1$ on $\{x \in \mathcal{O} : d(x) \leq \delta\}$. We have

$$\begin{cases} (c\Delta + \lambda^2)\widetilde{u}_1 = v_1 \quad \text{in} \quad \Omega_1, \\ (\Delta + \lambda^2)\chi\widetilde{u}_2 = \widetilde{v}_2 = \chi v_2 + [\Delta, \chi]\widetilde{u}_2 \quad \text{in} \quad \mathbf{R}^n \setminus \Omega_1, \\ \widetilde{u}_1|_{\Gamma_1} = \chi\widetilde{u}_2|_{\Gamma_1},\ \partial_\nu \widetilde{u}_1|_{\Gamma_1} = \partial_\nu \chi\widetilde{u}_2|_{\Gamma_1}. \end{cases} \tag{2.11}$$

By the results of [4] (see (4.2)) we have

$$\begin{aligned}\|\widetilde{u}_1\|_{H^1(\Omega_1)} + \|\chi\widetilde{u}_2\|_{H^1(\mathbf{R}^n\setminus\Omega_1)} &\leq C\lambda^{-1}\|v_1\|_{L^2(\Omega_1)} + C\lambda^{-1}\|\widetilde{v}_2\|_{L^2(\mathbf{R}^n\setminus\Omega_1)} \\ &\leq C\lambda^{-1}\|v_1\|_{L^2(\Omega_1)} + C\lambda^{-1}\|v_2\|_{L^2(\mathcal{O})} + C\|\psi\widetilde{u}_2\|_{H^1(\mathcal{O})}.\end{aligned} \tag{2.12}$$

Since Γ is strictly concave viewed from the interior, we have the following (see Proposition 2.2 of [4])

Proposition 2.5 *There exist constants $C, \delta_0 > 0$ so that for $0 < \delta \leq \delta_0$ we have the estimate*

$$\begin{aligned}\|\psi\widetilde{u}_2\|_{H^1(\mathcal{O})} \leq C\lambda^{-1}\left\|(\Delta + \lambda^2)\widetilde{u}_2\right\|_{L^2(\mathcal{O})} + C\|\widetilde{u}_2|_\Gamma\|_{L^2(\Gamma)} \\ +C\lambda^{-1}\|\partial_\nu\widetilde{u}_2|_\Gamma\|_{L^2(\Gamma)} + O_\delta(\lambda^{-1/2})\|\widetilde{u}_2\|_{H^1(\mathcal{O})}.\end{aligned} \tag{2.13}$$

Combining (2.12) and (2.13), we get

$$\begin{aligned}&\|\widetilde{u}_1\|_{H^1(\Omega_1)} + \|\widetilde{u}_2\|_{H^1(\mathcal{O})} \\ &\leq C\lambda^{-1}\|v_1\|_{L^2(\Omega_1)} + C\lambda^{-1}\|v_2\|_{L^2(\mathcal{O})} + C\lambda^{-1/2}\|\widetilde{u}_2\|_{H^1(\mathcal{O})} \\ &\quad +C\|\widetilde{u}_2|_\Gamma\|_{L^2(\Gamma)} + C\lambda^{-1}\|\partial_\nu\widetilde{u}_2|_\Gamma\|_{L^2(\Gamma)},\end{aligned} \tag{2.14}$$

which clearly implies (2.10). $\square$

By Green's formula we have

$$\operatorname{Re}\left\langle c^{-1}\widetilde{u}_1, u_1\right\rangle_{L^2(\Omega_1)} = c^{-1}\lambda^2\|u_1\|^2_{L^2(\Omega_1)} + \|\nabla u_1\|^2_{L^2(\Omega_1)} + \operatorname{Re}\left\langle \partial_\nu u_1|_{\Gamma_1}, u_1|_{\Gamma_1}\right\rangle_{L^2(\Gamma_1)},$$

$$\operatorname{Re}\left\langle \widetilde{u}_2, u_2\right\rangle_{L^2(\mathcal{O})} = \lambda^2\|u_2\|^2_{L^2(\mathcal{O})} + \|\nabla u_2\|^2_{L^2(\mathcal{O})} - \operatorname{Re}\left\langle \partial_\nu u_2|_{\Gamma_1}, u_2|_{\Gamma_1}\right\rangle_{L^2(\Gamma_1)}$$

$$+\mathrm{Re}\,\left\langle \partial_\nu u_2|_\Gamma, u_2|_\Gamma\right\rangle_{L^2(\Gamma)}.$$

Summing up these identities and using that $u_2|_\Gamma = 0$, we get

$$\mathrm{Re}\,\left\langle c^{-1}\widetilde{u}_1, u_1\right\rangle_{L^2(\Omega_1)} + \mathrm{Re}\,\left\langle \widetilde{u}_2, u_2\right\rangle_{L^2(\mathcal{O})}$$

$$= c^{-1}\lambda^2 \left\|u_1\right\|^2_{L^2(\Omega_1)} + \left\|\nabla u_1\right\|^2_{L^2(\Omega_1)} + \lambda^2 \left\|u_2\right\|^2_{L^2(\mathcal{O})} + \left\|\nabla u_2\right\|^2_{L^2(\mathcal{O})}. \tag{2.15}$$

It is easy to see that (2.15) implies the estimate

$$\left\|u_1\right\|_{H^2(\Omega_1)} + \left\|u_2\right\|_{H^2(\mathcal{O})} \le C\lambda^{-2}\left\|\widetilde{u}_1\right\|_{L^2(\Omega_1)} + C\lambda^{-2}\left\|\widetilde{u}_2\right\|_{L^2(\mathcal{O})}. \tag{2.16}$$

Applying the same arguments to the functions ∇u_1 and ∇u_2 we also get

$$\left\|\lambda^{-1}\nabla u_1\right\|_{H^2(\Omega_1)} + \left\|\lambda^{-1}\nabla u_2\right\|_{H^2(\mathcal{O})} \le C\lambda^{-2}\left\|\widetilde{u}_1\right\|_{H^1(\Omega_1)} + C\lambda^{-2}\left\|\widetilde{u}_2\right\|_{H^1(\mathcal{O})}$$

$$+\varepsilon\lambda^{-1/2}\left\|(\lambda^{-1}\partial_\nu)^2 u_2|_\Gamma\right\|_{L^2(\Gamma)} + C\varepsilon^{-1}\lambda^{-1/2}\left\|\lambda^{-1}\partial_\nu u_2|_\Gamma\right\|_{L^2(\Gamma)}, \tag{2.17}$$

for any $\varepsilon > 0$. On the other hand, by the trace theorem we have

$$\left\|\lambda^{-1}\partial_\nu u_2|_\Gamma\right\|_{L^2(\Gamma)} \le C\lambda^{1/2}\left\|u_2\right\|_{H^2(\mathcal{O})}, \tag{2.18}$$

$$\left\|(\lambda^{-1}\partial_\nu)^2 u_2|_\Gamma\right\|_{L^2(\Gamma)} \le C\lambda^{1/2}\left\|u_2\right\|_{H^3(\mathcal{O})}. \tag{2.19}$$

Thus, combining (2.16)-(2.19) and taking ε small enough, independent of λ, we conclude

$$\left\|u_1\right\|_{H^3(\Omega_1)} + \left\|u_2\right\|_{H^3(\mathcal{O})} \le C\lambda^{-2}\left\|\widetilde{u}_1\right\|_{H^1(\Omega_1)} + C\lambda^{-2}\left\|\widetilde{u}_2\right\|_{H^1(\mathcal{O})}. \tag{2.20}$$

By (2.10) and (2.20) we arrive at the estimate

$$\left\|u_1\right\|_{H^3(\Omega_1)} + \left\|u_2\right\|_{H^3(\mathcal{O})} \le C\lambda^{-3}\left\|v_1\right\|_{L^2(\Omega_1)} + C\lambda^{-5/2}\left\|v_2\right\|_{L^2(\mathcal{O})}$$

$$+C\lambda^{-1}\left\|\partial_\nu u_2|_\Gamma\right\|_{L^2(\Gamma)} + C\lambda^{-2}\left\|\Delta u_2|_\Gamma\right\|_{L^2(\Gamma)} + C\lambda^{-3}\left\|\partial_\nu\Delta u_2|_\Gamma\right\|_{L^2(\Gamma)}. \tag{2.21}$$

It is easy to see now that (2.8) follows from combining (2.21) and the following

Lemma 2.6 *We have the estimate*

$$\lambda^{-3}\left\|\partial_\nu\Delta u_2|_\Gamma\right\|_{L^2(\Gamma)} \le C\lambda^{-1}\left\|\partial_\nu u_2|_\Gamma\right\|_{L^2(\Gamma)}$$

$$+C\lambda^{-2}\left\|\Delta u_2|_\Gamma\right\|_{L^2(\Gamma)} + C\lambda^{-5/2}\left\|v_2\right\|_{L^2(\mathcal{O})} + C\lambda^{-1/2}\left\|u_2\right\|_{H^3(\mathcal{O})}. \tag{2.22}$$

Proof. Choose a function $\psi \in C^\infty(\mathbf{R}^n)$ such that $\psi = 1$ on a small neighbourhood of $\mathbf{R}^n \setminus \Omega$, $\psi = 0$ outside another small neighbourhood of $\mathbf{R}^n \setminus \Omega$. Then the function $w = (\Delta + \lambda^2)\psi u_2$ satisfies the equation

$$\begin{cases} (-\Delta + \lambda^2)w = \widetilde{w} = -\psi v_2 - [\Delta^2, \psi]u_2 \quad \text{in} \quad \Omega, \\ w|_\Gamma = \Delta u_2|_\Gamma, \\ \partial_\nu w|_\Gamma = \partial_\nu\Delta u_2|_\Gamma + \lambda^2\partial_\nu u_2|_\Gamma. \end{cases} \tag{2.23}$$

Clearly, to prove (2.22) it suffices to show the estimate

$$\lambda^{-1} \|\partial_\nu w|_\Gamma\|_{L^2(\Gamma)} \le C \|w|_\Gamma\|_{L^2(\Gamma)} + C\lambda^{-3/2} \|\widetilde{w}\|_{L^2(\Omega)} . \tag{2.24}$$

By the trace theorem we have

$$\lambda^{-1} \|\partial_\nu w|_\Gamma\|_{L^2(\Gamma)} + \|w|_\Gamma\|_{L^2(\Gamma)} \le C\lambda^{1/2} \|w\|_{H^2(\Omega)} . \tag{2.25}$$

On the other hand, by Green's formula

$$\mathrm{Re} \left\langle \widetilde{w}, w \right\rangle_{L^2(\Omega)} = \lambda^2 \|w\|^2_{L^2(\Omega)} + \|\nabla w\|^2_{L^2(\Omega)} + \mathrm{Re} \left\langle \partial_\nu w|_\Gamma, w|_\Gamma \right\rangle_{L^2(\Gamma)} ,$$

we get

$$\|w\|_{H^2(\Omega)} \le C\lambda^{-2} \|\widetilde{w}\|_{L^2(\Omega)} + \varepsilon\lambda^{-1/2} \left\|\lambda^{-1}\partial_\nu w|_\Gamma\right\|_{L^2(\Gamma)} + C\varepsilon^{-1}\lambda^{-1/2} \|w|_\Gamma\|_{L^2(\Gamma)} \tag{2.26}$$

for any $\varepsilon > 0$. Combining (2.25) and (2.26), and taking ε small enough, independent of λ, we obtain (2.24). □

3 Resolvent Estimates

We will show that Theorem 2.1 implies the following

Theorem 3.1 *Under (1.2), we have the bound*

$$\left\|(\mathcal{A} - z)^{-1}\right\|_{\mathcal{H}\to\mathcal{H}} \le Const, \quad \forall z \in \mathbf{R}. \tag{3.1}$$

Proof. We will derive (3.1) from (2.2). Clearly, it suffices to prove (3.1) for $|z| \gg 1$. Without loss of generality we may suppose that $z > 0$. Let

$$\begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}(\mathcal{A})$$

satisfy the equation

$$(\mathcal{A} - z) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{H}.$$

Clearly, this equation can be rewritten as follows

$$\begin{pmatrix} -iv - zu \\ i\alpha^2\Delta^2 u - zv \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}.$$

Hence the function

$$u = \begin{cases} u_1 & \text{in} \quad \Omega_1, \\ u_2 & \text{in} \quad \mathcal{O}, \end{cases}$$

satisfies the equation

$$\begin{cases} (-c^2\Delta^2 + z^2)u_1 = ig_1 - izf_1 \quad \text{in} \quad \Omega_1, \\ (-\Delta^2 + z^2)u_2 = ig_2 - izf_2 \quad \text{in} \quad \mathcal{O}, \\ u_1|_{\Gamma_1} = u_2|_{\Gamma_1},\, \partial_\nu u_1|_{\Gamma_1} = \partial_\nu u_2|_{\Gamma_1},\, c\Delta u_1|_{\Gamma_1} = \Delta u_2|_{\Gamma_1},\, c\partial_\nu \Delta u_1|_{\Gamma_1} = \partial_\nu \Delta u_2|_{\Gamma_1}, \\ u_2|_\Gamma = 0,\, \Delta u_2|_\Gamma = a\left(-iz\partial_\nu u_2|_\Gamma + \partial_\nu f_2|_\Gamma\right). \end{cases} \tag{3.2}$$

Clearly, (3.1) is equivalent to the estimate

$$\|\Delta u\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} \le C\,\|\Delta f\|_{L^2(\Omega)} + C\,\|g\|_{L^2(\Omega)}\,. \tag{3.3}$$

To prove (3.3) we write $u = u^g + u^f$, $v = v^g + v^f$, where

$$(\mathcal{A} - z)\begin{pmatrix} u^g \\ v^g \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}, \quad (\mathcal{A} - z)\begin{pmatrix} u^f \\ v^f \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

Clearly, $u^g = (u_1^g, u_2^g)$ (resp. $u^f = (u_1^f, u_2^f)$) satisfies (3.2) with $f \equiv 0$ (resp. $g \equiv 0$). Applying Theorem 2.1 with $\lambda = z^{1/2}$ and $F \equiv 0$ and recalling that $v^g = zu^g$, we get

$$\|\Delta u^g\|_{L^2(\Omega)} + \|v^g\|_{L^2(\Omega)} \le C\,\|g\|_{L^2(\Omega)}\,, \tag{3.4}$$

with a constant $C > 0$ independent of z. It is easy to see that (3.3) would follow from (3.4) and the estimate

$$\left\|\Delta u^f\right\|_{L^2(\Omega)} + \left\|v^f\right\|_{L^2(\Omega)} \le C\,\|Gf\|_H\,, \quad \forall f \in \mathcal{D}(G), \tag{3.5}$$

with a constant $C > 0$ independent of z. In what follows we will derive (3.5) from Theorem 2.1. Choose a function $\phi \in C_0^\infty(\mathbf{R})$, $\phi = 1$ on $[1/2, 2]$, $\phi = 0$ on $(-\infty, 1/3] \cup [3, +\infty)$. Write $f = f^\flat + f^\natural$, where

$$f^\flat = \phi(G/z)f, \quad f^\natural = (1-\phi)(G/z)f.$$

We have $u^f = u^\flat + u^\natural$, $v^f = v^\flat + v^\natural$, where

$$(\mathcal{A} - z)\begin{pmatrix} u^\flat \\ v^\flat \end{pmatrix} = \begin{pmatrix} f^\flat \\ 0 \end{pmatrix}, \quad (\mathcal{A} - z)\begin{pmatrix} u^\natural \\ v^\natural \end{pmatrix} = \begin{pmatrix} f^\natural \\ 0 \end{pmatrix}.$$

Clearly, $u^\flat = (u_1^\flat, u_2^\flat)$ (resp. $u^\natural = (u_1^\natural, u_2^\natural)$) satisfies (3.2) with $g \equiv 0$ and $f = f^\flat$ (resp. $f = f^\natural$). Applying Theorem 2.1 with $\lambda = z^{1/2}$ and $F = \partial_\nu f_2^\flat|_\Gamma$ leads to the estimate

$$\begin{aligned} &\left\|\Delta u^\flat\right\|_{L^2(\Omega)} + \left\|v^\flat\right\|_{L^2(\Omega)} \le \left\|\Delta u^\flat\right\|_{L^2(\Omega)} + z\left\|u^\flat\right\|_{L^2(\Omega)} + \left\|f^\flat\right\|_{L^2(\Omega)} \\ &\le Cz\left\|f^\flat\right\|_{L^2(\Omega)} + C\left\|\partial_\nu f_2^\flat|_\Gamma\right\|_{L^2(\Gamma)} \le Cz\left\|f^\flat\right\|_{L^2(\Omega)} + C\left\|f_2^\flat\right\|_{H^2(\mathcal{O})} \\ &\le Cz\left\|f^\flat\right\|_H + C\left\|Gf_2^\flat\right\|_H \le C\left\|zG^{-1}\phi(G/z)\right\|_{H\to H}\|Gf\|_H \\ &\quad + C\,\|\phi(G/z)\|_{H\to H}\,\|Gf\|_H \le C\,\|Gf\|_H\,, \end{aligned} \tag{3.6}$$

with a constant $C > 0$ independent of z, where the Sobolev space $H^2(\mathcal{O})$ is equipped with the classical norm and we have used the trace theorem together with the fact that the norms on H and $L^2(\Omega)$ are equivalent. We would like to get a similar estimate for the functions $u^\natural$ and $v^\natural$. Set

$$U = (z-G)^{-1}(z+G)^{-1}(-izf^\natural) = -iz(z-G)^{-1}(z+G)^{-1}(1-\phi)(G/z)f.$$

Since the operator G^{-1} is bounded on H, we have

$$\|U\|_H \leq C\|GU\|_H \leq Cz^{-1}\|Gf\|_H \tag{3.7}$$

with a constant $C > 0$ independent of z. It is easy to see that the function $U = (U_1, U_2)$ satisfies the equation

$$\begin{cases} (-c^2\Delta^2 + z^2)U_1 = -izf_1^\natural \quad \text{in} \quad \Omega_1, \\ (-\Delta^2 + z^2)U_2 = -izf_2^\natural \quad \text{in} \quad \mathcal{O}, \\ U_1|_{\Gamma_1} = U_2|_{\Gamma_1},\, \partial_\nu U_1|_{\Gamma_1} = \partial_\nu U_2|_{\Gamma_1},\, c\Delta U_1|_{\Gamma_1} = \Delta U_2|_{\Gamma_1},\, c\partial_\nu\Delta U_1|_{\Gamma_1} = \partial_\nu\Delta U_2|_{\Gamma_1}, \\ U_2|_\Gamma = 0,\, \Delta U_2|_\Gamma = 0. \end{cases}$$

Hence the function $w = u^\natural - U$ satisfies the equation

$$\begin{cases} (-c^2\Delta^2 + z^2)w_1 = 0 \quad \text{in} \quad \Omega_1, \\ (-\Delta^2 + z^2)w_2 = 0 \quad \text{in} \quad \mathcal{O}, \\ w_1|_{\Gamma_1} = w_2|_{\Gamma_1},\, \partial_\nu w_1|_{\Gamma_1} = \partial_\nu w_2|_{\Gamma_1},\, c\Delta w_1|_{\Gamma_1} = \Delta w_2|_{\Gamma_1},\, c\partial_\nu\Delta w_1|_{\Gamma_1} = \partial_\nu\Delta w_2|_{\Gamma_1}, \\ w_2|_\Gamma = 0,\, \Delta w_2|_\Gamma = a\left(-iz\partial_\nu w_2|_\Gamma + \partial_\nu f_2^\natural|_\Gamma - iz\partial_\nu U_2|_\Gamma\right). \end{cases}$$

Therefore, by Theorem 2.1 we obtain

$$\begin{aligned} \left\|\Delta u^\natural - \Delta U\right\|_{L^2(\Omega)} + z\left\|u^\natural - U\right\|_{L^2(\Omega)} &\leq C\left\|\partial_\nu f_2^\natural|_\Gamma\right\|_{L^2(\Gamma)} + Cz\left\|\partial_\nu U_2|_\Gamma\right\|_{L^2(\Gamma)} \\ &\leq C\left\|f_2^\natural\right\|_{H^2(\mathcal{O})} + Cz\left\|U_2\right\|_{H^2(\mathcal{O})} \leq C\left\|Gf^\natural\right\|_H + Cz\left\|GU\right\|_H. \end{aligned} \tag{3.8}$$

By (3.7) and (3.8) we conclude

$$\left\|\Delta u^\natural\right\|_{L^2(\Omega)} + \left\|v^\natural\right\|_{L^2(\Omega)} \leq \left\|\Delta u^\natural\right\|_{L^2(\Omega)} + z\left\|u^\natural\right\|_{L^2(\Omega)} + \left\|f^\natural\right\|_{L^2(\Omega)} \leq C\left\|Gf\right\|_H. \tag{3.9}$$

Clearly, (3.5) follows from (3.6) and (3.9). $\square$

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