CUBO A Mathematical Journal
Vol.11, N$^{\underline{o}}$05, (23–38). December 2009

Resonances and SSF Singularities for Magnetic Schrödinger Operators

Jean-François Bony, Vincent Bruneau
Université Bordeaux I, Institut de Mathématiques de Bordeaux,
UMR CNRS 5251, 351, Cours de la Libération, 33405 Talence, France
emails: bony@math.u-bordeaux1.fr,
vbruneau@math.u-bordeaux1.fr

Philippe Briet
Centre de Physique Théorique,
CNRS-Luminy, Case 907, 13288 Marseille, France
email: briet@cpt.univ-mrs.fr

and

Georgi Raikov
Departamento de Matemáticas, Facultad de Matemáticas,
Pontificia Universidad Católica de Chile,
Vicuña Mackenna 4860, Santiago de Chile
email: graikov@mat.puc.cl

ABSTRACT

The aim of this note is to review recent articles on the spectral properties of magnetic Schrödinger operators. We consider H_0, a 3D Schrödinger operator with constant magnetic field, and $\tilde{H}_0$, a perturbation of H_0 by an electric potential which depends only on the variable along the magnetic field. Let H (resp. $\tilde{H}$) be a short range perturbation of H_0 (resp. of $\tilde{H}_0$). In the case of (H, H_0), we study the local singularities of the Krein spectral shift function (SSF) and the distribution of the resonances of H near the Landau levels which play the role of spectral thresholds. In the case of $(\tilde{H}, \tilde{H}_0)$, we study similar problems near the eigenvalues of $\tilde{H}_0$ of infinite multiplicity.

RESUMEN

El objetivo de esta nota es reseñar artículos recientes sobre las propiedades espectrales de operadores de Schrödinger magnéticos. Consideramos H_0, el operador tridimensional de Schrödinger con campo magnético constante, y $\tilde{H}_0$, perturbación de H_0 por un potencial eléctrico que depende sólo de la variable a lo largo del campo magnético. Sea H, respectivamente $\tilde{H}$, perturbación de corto alcance de H_0, respectivamente $\tilde{H}_0$. En el caso del par (H, H_0), estudiamos las singularidades locales de la función de corrimiento espectral (SSF) de Krein y la distribución de las resonancias de H cerca de los niveles de Landau que tienen el papel de umbrales espectrales. En el caso del par $(\tilde{H}, \tilde{H}_0)$, investigamos problemas similares cerca de los valores propios de multiplicidad infinita de $\tilde{H}_0$.

Key words and phrases: *Magnetic Schrödinger operators, resonances, spectral shift function.*

Math. Subj. Class.: *35P25, 35J10, 47F05, 81Q10.*

1 Introduction

Let

$$H_0 := (-i\nabla - A)^2 - b,$$

be the (shifted) 3D Schrödinger operator with constant magnetic field $\mathbf{B} = (0, 0, b)$, $b > 0$, self-adjoint in $L^2(\mathbb{R}^3)$, and essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$. Here $A = \left(-\frac{bx_2}{2}, \frac{bx_1}{2}, 0\right)$ is a magnetic potential generating the magnetic field $\mathbf{B} = \operatorname{curl} A$. It is well-known that the spectrum of H_0 is absolutely continuous and coincides with the interval $[0, \infty)$, i.e.

$$\sigma(H_0) = \sigma_{\rm ac}(H_0) = [0, \infty).$$

Moreover, *the Landau levels* $2bq$, $q \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\}$, play the role of thresholds in $\sigma(H_0)$ (see [10, 15, 3] and Section 3 below).

For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ we write $x = (X_\perp, x_3)$ where $X_\perp := (x_1, x_2) \in \mathbb{R}^2$ are the variables on the plane perpendicular to the magnetic field $\mathbf{B}$.

Further, let $V : \mathbb{R}^3 \to \mathbb{R}$ be the electric potential. Throughout the article, we assume that $V \in L^\infty(\mathbb{R}^3) \cap C(\mathbb{R}^3)$, and V does not vanish identically. Moreover, we will suppose that V satisfies one of the following decay assumptions:

$$|V(x)| = \mathcal{O}(\langle X_\perp\rangle^{-m_\perp}\langle x_3\rangle^{-m_3}), \quad m_\perp > 2, \quad m_3 > 1, \quad x = (X_\perp, x_3) \in \mathbb{R}^3, \tag{1.1}$$

$$|V(x)| = \mathcal{O}(\langle x\rangle^{-m_0}), \quad m_0 > 3, \quad x \in \mathbb{R}^3, \tag{1.2}$$

$$|V(x)| = \mathcal{O}(\langle X_\perp\rangle^{-m_\perp}\exp(-N|x_3|)), \quad m_\perp > 2, \quad \forall N > 0, \quad x = (X_\perp, x_3) \in \mathbb{R}^3. \tag{1.3}$$

Note that each of the conditions (1.2) and (1.3) implies (1.1).

On the domain of H_0 define the perturbed operator

$$H := H_0 + V.$$

We introduce the Krein spectral shift function (SSF) $\xi(\cdot; H, H_0)$ for the operator pair (H, H_0) (see Section 2 for its definition), and study its behavior near the Landau levels. In Theorem 4.1 we show that at least in the case of V of constant sign, the SSF $\xi(\cdot; H, H_0)$ has a singularity at each Landau levels, i.e. it either blows up to $+\infty$ if $V \geq 0$, or blows down to $-\infty$ if $V \leq 0$, as the energy approaches the Landau level in an appropriate manner. The singularities of $\xi(\cdot; H, H_0)$ are described in the terms of the spectral characteristics of compact operators of Berezin-Toeplitz type, studied in Section 3.

One of the possible explanation of the singularities of the SSF $\xi(\cdot; H, H_0)$ is the accumulation of resonances of H to the Landau levels. In Section 5 we define the resonances of H as the poles of an appropriate meromorphic extension of its resolvent, and in Theorem 5.1 we obtain upper and lower estimates of the number of the resonances in a vicinity of a given Landau level. The lower bound confirms our conjecture that the singularities of the SSF at the Landau levels and the accumulation of the resonances of H to these levels, are intimately related.

Further, in Section 6 we introduce a modified operator pair $(\tilde{H}, \tilde{H}_0)$. Here $\tilde{H}_0 := H_0 + v_0$ where $v_0 = v_0(x_3)$ decays fast enough at infinity, while $\tilde{H} := \tilde{H}_0 + V$ where V satisfies (1.1). The remarkable property of the operator $\tilde{H}_0$ is that, under appropriate assumption on v_0, it has infinitely many eigenvalues of infinite multiplicity, most of which are embedded in the continuous spectrum. We study the asymptotic behavior of the SSF $\xi(\cdot; \tilde{H}, \tilde{H}_0)$ near these eigenvalues of $\tilde{H}_0$ of infinite multiplicity. Finally, in Section 7, under the assumption that the perturbation V is axisymmetric, we investigate the asymptotic behavior as $\varkappa \to 0$ of the unitary group associated with $\tilde{H}_0 + \varkappa V$, and relate it to certain dynamic resonances for the operator $\tilde{H}$.

2 The Spectral Shift Function

Assume that V satisfies (1.1). Then the resolvent difference $(H - i)^{-1} - (H_0 - i)^{-1}$ is a trace-class operator, and there exists a unique $\xi = \xi(\cdot; H, H_0) \in L^1(\mathbb{R}; (1 + E^2)^{-1}dE)$ such that *the Lifshits-Krein trace formula*

$$\operatorname{Tr}\left(f(H) - f(H_0)\right) = \int_{\mathbb{R}} \xi(E; H, H_0) f'(E) dE$$

holds for each $f \in C_0^\infty(\mathbb{R})$, and $\xi(E; H, H_0) = 0$ for each $E \in (-\infty, \inf \sigma(H))$ (see the original works [17, 14] or [24, Chapter 8]). The function $\xi(\cdot; H, H_0)$ is called *the spectral shift function* (SSF) for the operator pair (H, H_0).

For almost every $E \in \sigma_{\rm ac}(H) = [0, \infty)$ the SSF $\xi(E; H, H_0)$ coincides with *the scattering phase* for the operator pair (H, H_0) according to *the Birman-Krein formula*

$$\det S(E; H, H_0) = e^{-2\pi i \xi(E; H, H_0)}$$

where $S(E; H, H_0)$ is the scattering matrix for the operator pair (H, H_0) (see the original work [4] or [24, Chapter 8]). On the other hand, for almost every $E < 0$ the value $-\xi(E; H, H_0)$ coincides with the number of the eigenvalues of H less than E, counted with their multiplicity.

A priori, the SSF $\xi(\cdot; H, H_0)$ is defined as an element of $L^1(\mathbb{R}; (1+E^2)^{-1}dE)$. The following proposition provides more precise information on $\xi(\cdot; H, H_0)$:

Proposition 2.1. [7, Proposition 2.5] (i) *The SSF $\xi(\cdot; H, H_0)$ is bounded on every compact subset of $\mathbb{R} \setminus 2b\mathbb{Z}_+$.*
(ii) *The SSF $\xi(\cdot; H, H_0)$ is continuous on $\mathbb{R} \setminus (2b\mathbb{Z}_+ \cup \sigma_{\rm pp}(H))$ where $\sigma_{\rm pp}(H)$ is the set of the eigenvalues of H.*

3 Auxiliary Toeplitz Operators

We have

$$H_0 = H_{0,\perp} \otimes I_\parallel + I_\perp \otimes H_{0,\parallel}$$

where $I_\perp$ and $I_\parallel$ are the identities in $L^2(\mathbb{R}^2_{X_\perp})$ and $L^2(\mathbb{R}_{x_3})$ respectively,

$$H_{0,\perp} := \left(-i\frac{\partial}{\partial x_1} + \frac{bx_2}{2}\right)^2 + \left(-i\frac{\partial}{\partial x_2} - \frac{bx_1}{2}\right)^2 - b$$

is the (shifted) *Landau Hamiltonian*, self-adjoint in $L^2(\mathbb{R}^2_{X_\perp})$, and

$$H_{0,\parallel} := -\frac{d^2}{dx_3^2}$$

is the 1D free Hamiltonian, self-adjoint in $L^2(\mathbb{R}_{x_3})$. Note that

$$H_{0,\perp} = a^* a$$

where

$$a^* := -2ie^{b|z|^2/4}\frac{\partial}{\partial z}e^{-b|z|^2/4}, \ z = x_1 + ix_2,$$

is *the creation operator*, and

$$a := -2ie^{-b|z|^2/4}\frac{\partial}{\partial \bar{z}}e^{b|z|^2/4}, \ \bar{z} = x_1 - ix_2,$$

is *the annihilation operator*. Moreover,

$$[a, a^*] = 2b.$$

Therefore, $\sigma(H_{0,\perp}) = \cup_{q=0}^{\infty}\{2bq\}$. Furthermore,

$$\mathrm{Ker}\, H_{0,\perp} = \mathrm{Ker}\, a = \left\{ f \in L^2(\mathbb{R}^2) | f = ge^{-b|z|^2/4}, \ \frac{\partial g}{\partial \bar{z}} = 0 \right\}$$

is the classical *Fock-Segal-Bargmann* space (see e.g. [11]), and

$$\mathrm{Ker}\,(H_{0,\perp} - 2bq) = (a^*)^q \mathrm{Ker}\, H_{0,\perp}, \quad q \geq 1.$$

Evidently,

$$\dim \operatorname{Ker}(H_{0,\perp} - 2bq) = \infty$$

for each $q \in \mathbb{Z}_+$.
Let $U : \mathbb{R}^2 \to \mathbb{R}$. Fix $q \in \mathbb{Z}_+$. Introduce *the Toeplitz operator*

$$p_q U p_q : p_q L^2(\mathbb{R}^2) \to p_q L^2(\mathbb{R}^2)$$

where p_q is the orthogonal projection onto $\operatorname{Ker}(H_{0,\perp} - 2bq)$. Obviously, if $U \in L^\infty(\mathbb{R}^2)$, then the operator $p_q U p_q$ is bounded. Denote by S_r the Schatten-von Neumann class of order $r \in [1, \infty)$. Then $U \in L^r(\mathbb{R}^2)$ implies $p_q U p_q \in S_r$, $r \in [1, \infty)$ (see [20, Lemma 5.1] or [9, Lemma 3.1]).
Moreover, if U is sufficiently regular, then $p_0 U p_0$ is unitarily equivalent to a ΨDO with anti-Wick symbol

$$\omega(y, \eta) := U(b^{-1/2}\eta, b^{-1/2}y), \ (y, \eta) \in T^*\mathbb{R},$$

(see [20]), while $p_q U p_q$ with $q \geq 1$ is unitarily equivalent to

$$p_0 \left(\sum_{s=0}^{q} \frac{q!}{(2b)^s (s!)^2 (q-s)!} \Delta^s U \right) p_0$$

(see [7, Lemma 9.2]).
For further references, in the following three theorems we describe the eigenvalue asymptotics for the Toeplitz operator $p_q U p_q$, $q \in \mathbb{Z}_+$, under the assumptions that U admits a power-like decay, exponential decay, or is compactly supported, respectively.

Theorem 3.1. [20, Theorem 2.6] *Let* $0 \leq U \in C^1(\mathbb{R}^2)$*, and*

$$U(X_\perp) = u_0(X_\perp/|X_\perp|)|X_\perp|^{-\alpha}(1 + o(1)),$$

$$|\nabla U(X_\perp)| = \mathcal{O}(|X_\perp|^{-\alpha-1}),$$

as $|X_\perp| \to \infty$*, with* $\alpha > 0$*, and* $0 < u_0 \in C(\mathbb{S}^1)$*. Fix* $q \in \mathbb{Z}_+$*. Then*

$$\operatorname{Tr} \mathbf{1}_{(s,\infty)}(p_q U p_q) = \psi_\alpha(s)(1 + o(1)), \quad s \downarrow 0, \tag{3.1}$$

where

$$\psi_\alpha(s) := s^{-2/\alpha} \frac{b}{4\pi} \int_{\mathbb{S}^1} u_0(t)^{2/\alpha} dt. \tag{3.2}$$

Theorem 3.2. [22, Theorem 2.1, Proposition 4.1] *Let* $0 \leq U \in L^\infty(\mathbb{R}^2)$ *and*

$$\ln U(X_\perp) = -\mu |X_\perp|^{2\beta}(1 + o(1)), \quad |X_\perp| \to \infty,$$

with $\beta \in (0, \infty)$*,* $\mu \in (0, \infty)$*. Fix* $q \in \mathbb{Z}_+$*. Then*

$$\operatorname{Tr} \mathbf{1}_{(s,\infty)}(p_q U p_q) = \varphi_\beta(s)(1 + o(1)), \quad s \downarrow 0, \tag{3.3}$$

where

$$\varphi_\beta(s) := \begin{cases} \frac{b}{2\mu^{1/\beta}}|\ln s|^{1/\beta} \text{ if } 0 < \beta < 1, \\ \frac{1}{\ln(1+2\mu/b)}|\ln s| \text{ if } \beta = 1, \\ \frac{\beta}{\beta-1}(\ln|\ln s|)^{-1}|\ln s| \text{ if } 1 < \beta < \infty, \end{cases} \quad s \in (0, e^{-1}). \tag{3.4}$$

Theorem 3.3. [22, Theorem 2.2, Proposition 4.1] *Let* $0 \leq U \in L^\infty(\mathbb{R}^2)$*. Assume that* $\operatorname{supp} U$ *is compact, and* $U \geq C > 0$ *on an open subset of* $\mathbb{R}^2$*. Fix* $q \in \mathbb{Z}_+$*. Then*

$$\operatorname{Tr} \mathbf{1}_{(s,\infty)}(p_q U p_q) = \varphi_\infty(s)(1 + o(1)), \quad s \downarrow 0, \tag{3.5}$$

where

$$\varphi_\infty(s) := (\ln|\ln s|)^{-1}|\ln s|, \quad s \in (0, e^{-1}). \tag{3.6}$$

Remark: Relations (3.1) and (3.3) with $\beta < 1$ are semiclassical in the sense that they are equivalent to

$$\operatorname{Tr} \mathbf{1}_{(s,\infty)}(p_q U p_q) = \frac{b}{2\pi}\left|\left\{X_\perp \in \mathbb{R}^2 | U(X_\perp) > s\right\}\right|(1 + o(1)), \quad s \downarrow 0.$$

The asymptotic order in relation (3.3) with $\beta = 1$ which corresponds to Gaussian decay of V, is semiclassical, but the coefficient is not. Finally, even the asymptotic order in (3.3) with $\beta > 1$, and (3.5) is not semiclassical. Moreover the main terms of these asymptotics do not depend on the Landau levels.

4 Singularities of the SSF at the Landau levels

Let V satisfy (1.1). For $X_\perp \in \mathbb{R}^2$, $\lambda \geq 0$, set

$$W(X_\perp) := \int_{\mathbb{R}} |V(X_\perp, x_3)| dx_3, \tag{4.1}$$

$$\mathcal{W}_\lambda = \mathcal{W}_\lambda(X_\perp) := \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix},$$

where

$$w_{11} := \int_{\mathbb{R}} |V(X_\perp, x_3)| \cos^2(\sqrt{\lambda} x_3) dx_3,$$

$$w_{12} = w_{21} := \int_{\mathbb{R}} |V(X_\perp, x_3)| \cos(\sqrt{\lambda} x_3) \sin(\sqrt{\lambda} x_3) dx_3,$$

$$w_{22} := \int_{\mathbb{R}} |V(X_\perp, x_3)| \sin^2(\sqrt{\lambda} x_3) dx_3.$$

We have

$$\operatorname{rank} p_q W p_q = \infty, \quad \operatorname{rank} p_q \mathcal{W}_\lambda p_q = \infty, \ \lambda \geq 0.$$

Theorem 4.1. [9, Theorems 3.1, 3.2] *Let V satisfy* (1.2)*, and $V \geq 0$ or $V \leq 0$. Fix $q \in \mathbb{Z}_+$. Then we have*

$$\xi(2bq - \lambda; H, H_0) = \mathcal{O}(1), \quad \lambda \downarrow 0,$$

if $V \geq 0$, and for each $\varepsilon \in (0,1)$ we have

$$-\mathrm{Tr}\, \mathbf{1}_{((1-\varepsilon)2\sqrt{\lambda},\infty)}\,(p_q W p_q) + \mathcal{O}(1) \leq$$
$$\xi(2bq - \lambda; H, H_0) \leq$$
$$-\mathrm{Tr}\, \mathbf{1}_{((1+\varepsilon)2\sqrt{\lambda},\infty)}\,(p_q W p_q) + \mathcal{O}(1),\ \lambda \downarrow 0,$$

if $V \leq 0$.
Moreover, for each $\varepsilon \in (0,1)$ we have

$$\pm\frac{1}{\pi}\mathrm{Tr}\ \arctan\left(((1 \pm \varepsilon)2\sqrt{\lambda})^{-1} p_q \mathcal{W}_\lambda p_q\right) + \mathcal{O}(1) \leq$$
$$\xi(2bq + \lambda; H, H_0) \leq$$
$$\pm\frac{1}{\pi}\mathrm{Tr}\ \arctan\left(((1 \mp \varepsilon)2\sqrt{\lambda})^{-1} p_q \mathcal{W}_\lambda p_q\right) + \mathcal{O}(1),\ \lambda \downarrow 0,$$

if $\pm V \geq 0$.

Remark: The proof of Theorem 4.1 is based on a representation of the SSF obtained by A. Pushnitski in [19].
If we assume that W admits a power-like or exponential decay at infinity, or has a compact support, we can combine the results of Theorems 3.1 − 3.3 for $U = W$, with Theorem 4.1 and obtain explicitly the main asymptotic term of $\xi(2bq + \lambda; H, H_0)$ as $\lambda \downarrow 0$ or $\lambda \uparrow 0$:

Corollary 4.1. [9, Corollaries 3.1 − 3.2], [21, Corollary 2.1] *Let* (1.2) *hold with $m_0 > 3$.*
(i) *Assume that the hypotheses of Theorem 3.1 hold with $U = W$ and $\alpha > 2$. Then we have*

$$\xi(2bq - \lambda; H, H_0) = -\frac{b}{2\pi}\left|\left\{X_\perp \in \mathbb{R}^2 | W(X_\perp) > 2\sqrt{\lambda}\right\}\right|\ (1 + o(1)) =$$
$$-\psi_\alpha(2\sqrt{\lambda})\ (1 + o(1)), \quad \lambda \downarrow 0, \tag{4.2}$$

if $V \leq 0$, and

$$\xi(2bq + \lambda; H, H_0) = \pm\frac{b}{2\pi^2}\int_{\mathbb{R}^2} \arctan\ ((2\sqrt{\lambda})^{-1} W(X_\perp)) dX_\perp\ (1 + o(1)) =$$
$$\pm\ \frac{1}{2\cos(\pi/\alpha)}\ \psi_\alpha(2\sqrt{\lambda})\,(1 + o(1)), \quad \lambda \downarrow 0,$$

if $\pm V \geq 0$, the function ψ_α being defined in (3.2)*.*
(ii) *Assume that the hypotheses of Theorem 3.2 hold with $U = W$. Then we have*

$$\xi(2bq - \lambda; H, H_0) = -\varphi_\beta(2\sqrt{\lambda})\ (1 + o(1)), \quad \lambda \downarrow 0, \quad \beta \in (0,\infty),$$

if $V \leq 0$, *the functions* φ_β *being defined in* (3.4). *If, in addition,* V *satisfies* (1.1) *for some* $m_\perp > 2$ *and* $m_3 > 2$, *we have*

$$\xi(2bq + \lambda; H, H_0) = \pm \frac{1}{2} \varphi_\beta(2\sqrt{\lambda})\, (1 + o(1)), \quad \lambda \downarrow 0, \quad \beta \in (0, \infty),$$

if $\pm V \geq 0$.
(iii) *Assume that the hypotheses of Theorem 3.3 hold with* $U = W$. *Then we have*

$$\xi(2bq - \lambda; H, H_0) = -\varphi_\infty(2\sqrt{\lambda})\, (1 + o(1)), \quad \lambda \downarrow 0,$$

if $V \leq 0$, *the function* φ_∞ *being defined in* (3.6). *If, in addition,* V *satisfies* (1.1) *for some* $m_\perp > 2$ *and* $m_3 > 2$, *we have*

$$\xi(2bq + \lambda; H, H_0) = \pm \frac{1}{2} \varphi_\infty(2\sqrt{\lambda})\, (1 + o(1)), \quad \lambda \downarrow 0,$$

if $\pm V \geq 0$.

As a corollary we obtain the following result which could be regarded as *generalized Levinson formulae*:

Corollary 4.2. *Let* V *satisfy* (1.2), *and* $V \leq 0$. *Fix* $q \in \mathbb{Z}_+$. *Then*

$$\lim_{\lambda \downarrow 0} \frac{\xi(2bq + \lambda; H, H_0)}{\xi(2bq - \lambda; H, H_0)} = \frac{1}{2\cos\frac{\pi}{\alpha}}$$

if W *admits a power-like decay with decay rate* $\alpha > 2$ *(i.e. if* $U = W$ *satisfies the hypotheses of Theorem 3.1), or*

$$\lim_{\lambda \downarrow 0} \frac{\xi(2bq + \lambda; H, H_0)}{\xi(2bq - \lambda; H, H_0)} = \frac{1}{2}$$

if W *decays exponentially or has a compact support (i.e. if* $U = W$ *satisfies the hypotheses of Theorems 3.2 – 3.3).*

Remark: The classical Levinson formula relates the number of the negative eigenvalues of $-\Delta + V$ and $\lim_{E \downarrow 0} \xi(E; -\Delta + V, -\Delta)$ (see the original work [16] or the survey article [23]).

5 Resonances Near the Landau Levels

One of the possible explanations of the singularities of the SSF described in Theorem 4.1, is the accumulation of resonances of H at the Landau levels. In this section we define the resonances of H as an appropriate extension of the resolvent $(H - z)^{-1}$ defined a priori for $z \in \mathbb{C}_+ := \{\zeta \in \mathbb{C} \,|\, \mathrm{Im}\, \zeta > 0\}$, following the exposition of [5]. In Theorem 5.1 below we establish upper and lower estimates on the number of resonances of H in a ring centered at a given Landau level; the lower

bounds imply accumulation of the resonances to the Landau level.
For $z \in \mathbb{C}_+$ we have

$$(H_0 - z)^{-1} = \sum_{q=0}^{\infty} p_q \otimes \left(H_{0,\|} + 2bq - z\right)^{-1}.$$

For each $q \in \mathbb{Z}_+$ and $N > 0$ the operator-valued function

$$z \mapsto \left(H_{0,\|} + 2bq - z\right)^{-1} \in \mathcal{L}(e^{-N\langle x_3\rangle} L^2(\mathbb{R}), e^{N\langle x_3\rangle} L^2(\mathbb{R})),$$

admits a holomorphic extension from $\mathbb{C} \setminus [2bq, \infty)$ to the 2-sheeted covering

$$\mathcal{P}_q : \{\zeta \in \mathbb{C} \setminus \{0\},\ \mathrm{Im}\,\zeta > -N\} \ni k \mapsto k^2 + 2bq \in \mathbb{C} \setminus \{2bq\}.$$

This extension however depends on $q \in \mathbb{Z}_+$.
Let $\pi_1(\mathbb{C} \setminus 2b\mathbb{Z}_+)$ be the fundamental group of $\mathbb{C} \setminus 2b\mathbb{Z}_+$, and G be the subgroup of $\pi_1(\mathbb{C} \setminus 2b\mathbb{Z}_+)$ generated by

$$\left\{a_1^2, a_2 a_1 a_2^{-1} a_1^{-1} \mid a_1, a_2 \in \pi_1(\mathbb{C} \setminus 2b\mathbb{Z}_+)\right\}.$$

We define $\mathcal{P}_G : \mathcal{M} \mapsto \mathbb{C} \setminus 2b\mathbb{Z}_+$ as the connected infinite-sheeted covering such that $\mathcal{P}_G^*(\pi_1(\mathcal{M})) = G$.
Fix a base point in $\mathcal{M}$. Let $\mathcal{F}$ be the connected component of $\mathcal{P}_G^{-1}(\mathbb{C} \setminus [0, \infty))$ containing this base point. By definition, the functions $\mathcal{M} \ni z \mapsto \sqrt{z - 2bq}$ have a positive imaginary part on $\mathcal{F}$. Set $\mathcal{F}_+ := \mathcal{F} \cap \mathcal{P}_G^{-1}(\mathbb{C}_+)$.
For $\lambda_0 \in \mathbb{C}$ and $\varepsilon > 0$ put

$$D(\lambda_0, \varepsilon) := \{\lambda \in \mathbb{C} |\, |\lambda - \lambda_0| < \varepsilon\},$$

$$D(\lambda_0, \varepsilon)^* := \{\lambda \in \mathbb{C} |\, 0 < |\lambda - \lambda_0| < \varepsilon\}.$$

Let $D_q^* \subset \mathcal{M}$ be the connected component of $\mathcal{P}_G^{-1}(D(2bq, 2b)^*)$ that intersects $\mathcal{F}_+$. There exists an analytic bijection z_q:

$$D(0, \sqrt{2b})^* \ni k \mapsto z_q(k) \in D_q^*,$$

such that $\mathcal{P}_G(z_q(k)) = 2bq + k^2$ and $z_q^{-1}(D_q^* \cap \mathcal{F}_+)$ is the first quadrant of $D(0, \sqrt{2b})^*$.
For $N > 0$ set

$$\mathcal{M}_N := \left\{z \in \mathcal{M} \,|\, \mathrm{Im}\,\sqrt{z - 2bq} > -N, \forall q \in \mathbb{Z}_+\right\}.$$

Evidently, $\cup_{N>0} \mathcal{M}_N = \mathcal{M}$.

Proposition 5.1. [5, Proposition 1] (i) *For each $N > 0$ the operator-valued function*

$$z \mapsto (H_0 - z)^{-1} \in \mathcal{L}\left(e^{-N\langle x_3\rangle} L^2(\mathbb{R}^3); e^{N\langle x_3\rangle} L^2(\mathbb{R}^3)\right)$$

has a holomorphic extension from the open upper half-plane to $\mathcal{M}_N$.
(ii) *Suppose that V satisfies* (1.3) *with $m_\perp > 0$. Then for each $N > 0$ the operator-valued function*

$$z \mapsto (H - z)^{-1} \in \mathcal{L}\left(e^{-N\langle x_3\rangle} L^2(\mathbb{R}^3); e^{N\langle x_3\rangle} L^2(\mathbb{R}^3)\right),$$

has a meromorphic extension from the open upper half plane to $\mathcal{M}_N$. Moreover, the poles and the range of the residues of this extension do not depend on N.

We define the resonances of H as the poles of the meromorphic extension of the resolvent $(H-z)^{-1}$. The multiplicity of a resonance z_0 is defined as

$$\text{rank}\ \frac{1}{2i\pi}\int_\gamma (H-z)^{-1}dz,$$

where γ is an appropriate circle centered at z_0. In what follows, $\text{Res}_q(H_0+\varkappa V)$ denotes the set of the resonances of $H_0+\varkappa V$ in D_q^*.

Theorem 5.1. [5, Theorem 2] *Assume that V satisfies* (1.3)*, and $V \geq 0$ or $V \leq 0$. Fix $q \in \mathbb{Z}_+$. Then for any $\delta > 0$ there exist $\varkappa_0, r_0 > 0$ such that:*
(i) *For any $0 < r < r_0$ and $0 \leq \varkappa \leq \varkappa_0$, we have*

$$\#\{z = z_q(k) \in \text{Res}_q(H_0+\varkappa V) \,|\, r < |k| < 2r\} = \mathcal{O}(\text{Tr}\ \mathbf{1}_{(r,8r)}(\varkappa p_q W p_q)).$$

(ii) *For any $0 \leq \varkappa \leq \varkappa_0$, $H_0+\varkappa V$ has no resonances in*

$$\{z = z_q(k) \,|\, 0 < |k| < r_0,\ \mp \text{Im}\ k \leq \frac{1}{\delta}|\text{Re}\ k|\}$$

if $\pm V \geq 0$.
(iii) *If W satisfies $\ln W(X_\perp) \leq -C\langle X_\perp\rangle^2$, then for any $0 \leq \varkappa \leq \varkappa_0$, $H_0+\varkappa V$ has an infinite number of resonances in $\{z = z_q(k) \,|\, 0 < |k| < r_0,\ \mp\text{Im}\ k > \frac{1}{\delta}|\text{Re}\ k|\}$ if $\pm V \geq 0$. More precisely, there exists a decreasing sequence $(r_\ell)_{\ell\in\mathbb{N}}$ of positive numbers, $r_\ell \downarrow 0$ such that,*

$$\#\{z = z_q(k) \in \text{Res}_q(H_0+\varkappa V) \,|\, r_{\ell+1} < |k| < r_\ell,\ \mp\,\text{Im}\ k > \frac{1}{\delta}|\text{Re}\ k|\} \geq \text{Tr}\ \mathbf{1}_{(2r_{\ell+1},2r_\ell)}(\varkappa p_q W p_q).$$

The setting is summarized by the following figure:

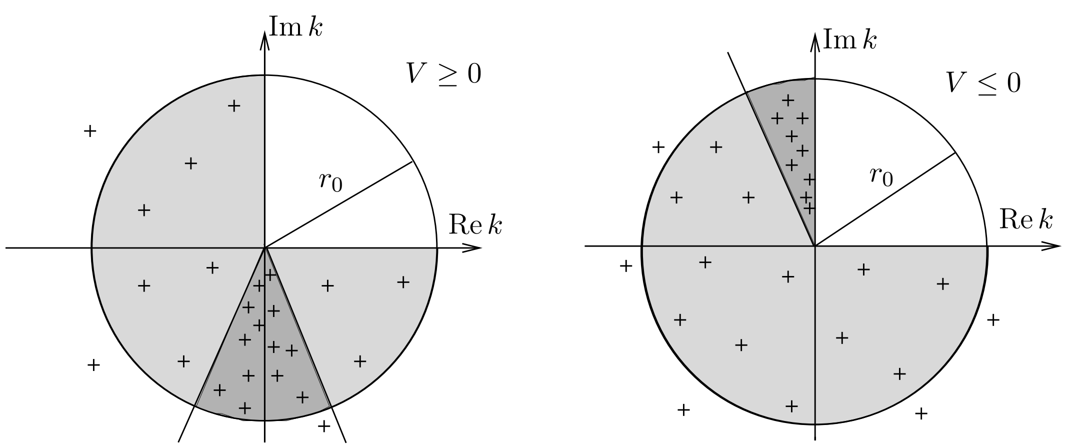

Figure 1: Resonances near a Landau level for V of definite sign. They are essentially concentrated near the semi-axis $k = \mp i[0,\infty)$ for $\pm V \geq 0$. The physical sheet is in white.

6 SSF for a Pair of Modified Magnetic Schrödinger Operators

In this section we replace the unperturbed operator H_0 by $\tilde{H}_0 = H_0 + v_0$ where v_0 is an electric potential which depends only on the variable x_3, and decays fast enough as $|x_3| \to \infty$. Under appropriate assumptions on v_0, the operator $\tilde{H}_0$ has infinitely many eigenvalues of infinite multiplicity, most of which are embedded in the continuous spectrum. We introduce the perturbed operator $\tilde{H} := \tilde{H}_0 + V$ where V satisfies (1.1), and investigate the asymptotic behavior of the SSF $\xi(\cdot; \tilde{H}, \tilde{H}_0)$ near the eigenvalues of $\tilde{H}_0$ of infinite multiplicity.

Let $v_0 \in C^1(\mathbb{R}; \mathbb{R})$ satisfy the assumption

$$|v_0(x)| = \mathcal{O}(\langle x \rangle^{-\delta_0}), \quad x \in \mathbb{R}, \quad \delta_0 > 1. \tag{6.1}$$

Set

$$H_{\parallel} := H_{0,\parallel} + v_0 = -\frac{d^2}{dx_3^2} + v_0(x_3).$$

We have

$$\sigma_{\rm ess}(H_{\parallel}) = \sigma_{\rm ac}(H_{\parallel}) = [0, \infty),$$

$$\sigma_{\rm pp}(H_{\parallel}) \cap (0, \infty) = \emptyset, \quad \sigma_{\rm sc}(H_{\parallel}) = \emptyset.$$

For simplicity, assume that $\inf \sigma(H_{\parallel}) > -2b$. Then

$$\tilde{H}_0 := H_{0,\perp} \otimes I_{\parallel} + I_{\perp} \otimes H_{\parallel} = H_0 + I_{\perp} \otimes v_0$$

is the Hamiltonian of a non-relativistic spinless quantum particle subject to a classical electromagnetic field $(\mathbf{E}, \mathbf{B})$ with $\mathbf{B} = (0, 0, b)$ and $\mathbf{E} = -(0, 0, v_0')$.
Let $\Lambda \in \sigma_{\rm disc}(H_{\parallel})$. Then $2bq + \Lambda$ with $q \in \mathbb{Z}_+$ is an eigenvalue of $\tilde{H}_0$ of infinite multiplicity; if $q \geq 1$, this eigenvalue is embedded in $\sigma_{\rm ac}(\tilde{H}_0)$.
Assume now that $V : \mathbb{R}^3 \to \mathbb{R}$ satisfies (1.1) with $m_{\perp} > 2$ and $m_3 > 1$, and set

$$\tilde{H} := \tilde{H}_0 + V.$$

Then $(\tilde{H} - i)^{-1} - (\tilde{H}_0 - i)^{-1}$ is trace-class, and the SSF $\xi(\cdot; \tilde{H}, \tilde{H}_0)$ is well defined.
Set

$$\mathcal{Z} := \{E = 2bq + \mu \mid q \in \mathbb{Z}_+, \ \mu \in \sigma_{\rm disc}(H_{\parallel}) \text{ or } \mu = 0\}.$$

Similarly to Proposition 2.1, we can show that the SSF $\xi(\cdot; \tilde{H}, \tilde{H}_0)$ is bounded on every compact subset of $\mathbb{R} \setminus \mathcal{Z}$, and is continuous on $\mathbb{R} \setminus (\mathcal{Z} \cup \sigma_{\rm pp}(\tilde{H}))$.
Let $\Lambda \in \sigma_{\rm disc}(H_{\parallel})$, and ψ be the eigenfunction satisfying

$$H_{\parallel}\psi = \Lambda\psi, \ \|\psi\|_{L^2(\mathbb{R})} = 1, \ \psi = \bar{\psi}.$$

By analogy with (4.1) set

$$\tilde{W}(X_{\perp}) := \int_{\mathbb{R}} |V(X_{\perp}, x_3)| \psi(x_3)^2 dx_3.$$

Theorem 6.1. [2, Theorem 6.1] *Let v_0 satisfy* (6.1), *and V satisfy* (1.1). *Assume* $\inf \sigma(H_\|) > -2b$. *Let $\Lambda \in \sigma_{\rm disc}(H_\|)$. Fix $q \in \mathbb{Z}_+$. Then for each $\varepsilon \in (0,1)$ we have*

$$\operatorname{Tr} \mathbf{1}_{((1+\varepsilon)\lambda,\infty)}(p_q\tilde{W}p_q) + \mathcal{O}(1) \leq \xi(2bq+\Lambda+\lambda; \tilde{H}, \tilde{H}_0) \leq \operatorname{Tr} \mathbf{1}_{((1-\varepsilon)\lambda,\infty)}(p_q\tilde{W}p_q) + \mathcal{O}(1),$$

$$\xi(2bq+\Lambda-\lambda; \tilde{H}, \tilde{H}_0) = \mathcal{O}(1),$$

as $\lambda \downarrow 0$, if $V \geq 0$, and

$$-\operatorname{Tr} \mathbf{1}_{((1-\varepsilon)\lambda,\infty)}(p_q\tilde{W}p_q) + \mathcal{O}(1) \leq \xi(2bq+\Lambda-\lambda; \tilde{H}, \tilde{H}_0) \leq -\operatorname{Tr} \mathbf{1}_{((1+\varepsilon)\lambda,\infty)}(p_q\tilde{W}p_q) + \mathcal{O}(1),$$

$$\xi(2bq+\Lambda+\lambda; \tilde{H}, \tilde{H}_0) = \mathcal{O}(1),$$

as $\lambda \downarrow 0$, if $V \leq 0$.

Similarly to Corollary 4.2, we can combine the result of Theorem 6.1 with those of Theorems 3.1 – 3.3, and obtain explicit asymptotic formulae describing the singularity of $\xi(\cdot; \tilde{H}, \tilde{H}_0)$ at $2bq+\Lambda$ under explicit assumptions about the decay of $\tilde{W}$ at infinity. We omit here this obvious corollary, and refer the reader to [2, Corollary 6.1].

7 Dynamical Resonances for Axisymmetric Perturbations V

In this section we assume that the perturbation V is axisymmetric, and investigate the asymptotics as $\varkappa \to 0$ of the unitary group $e^{-it(\tilde{H}_0+\varkappa V)}$, $t \geq 0$. For $m \in \mathbb{Z}$ introduce the operator

$$H_{0,\perp}^{(m)} := -\frac{1}{\varrho}\frac{d}{d\varrho}\varrho\frac{d}{d\varrho} + \left(\frac{m}{\varrho} - \frac{b\varrho}{2}\right)^2 - b,$$

self-adjoint in $L^2(\mathbb{R}_+; \varrho d\varrho)$. We have

$$\sigma(H_{0,\perp}^{(m)}) = \cup_{q=m_-}^{\infty}\{2bq\}, \quad m_- := \max\{0, -m\},$$

$$\dim \operatorname{Ker}(H_{0,\perp}^{(m)} - 2bq) = 1, \quad \forall q \geq m_-.$$

For $m \in \mathbb{Z}$, $q \in \mathbb{Z}$, $q \geq m_-$, let $\varphi_{q,m}$ satisfy

$$H_{0,\perp}^{(m)}\varphi_{q,m} = 2bq\varphi_{q,m}, \quad \|\varphi_{q,m}\|_{L^2(\mathbb{R}_+;\varrho d\varrho)} = 1.$$

Let the multiplier by v_0 be $H_{0,\|}$-compact. Set

$$H_0^{(m)} := H_{0,\perp}^{(m)} \otimes I_\| + \mathcal{I}_\perp \otimes H_{0,\|},$$

$$\tilde{H}_0^{(m)} := H_{0,\perp}^{(m)} \otimes I_\| + \mathcal{I}_\perp \otimes H_\| = H_0^{(m)} + \mathcal{I}_\perp \otimes v_0,$$

where $\mathcal{I}_\perp$ is the identity operator in $L^2(\mathbb{R}_+; \varrho d\varrho)$;

$$\sigma(H_0^{(m)}) = \sigma_{\rm ess}(\tilde{H}_0^{(m)}) = [2m_-b, \infty).$$

Let (ϱ, ϕ, x_3) be the cylindrical coordinates in $\mathbb{R}^3$. The operator $H_0^{(m)}$ (resp. $\tilde{H}_0^{(m)}$), $m \in \mathbb{Z}$, is unitarily equivalent to the restriction of H_0 (resp., of $\tilde{H}_0$) onto $\mathrm{Ker}\,(L-m)$ with $L := -i\frac{\partial}{\partial\phi}$. Hence, the operator H_0 (resp., $\tilde{H}_0$) is unitarily equivalent to the orthogonal sum $\oplus_{m\in\mathbb{Z}} H_0^{(m)}$ (resp., $\oplus_{m\in\mathbb{Z}} \tilde{H}_0^{(m)}$).
Let the multiplier by $V : \mathbb{R}^3 \to \mathbb{R}$ be H_0-bounded with zero relative bound. Suppose that V is axisymmetric i.e. $\frac{\partial V}{\partial \phi} = 0$.
Let $\varkappa \in \mathbb{R}$. On $D(\tilde{H}_0)$ introduce the operator

$$\tilde{H}_\varkappa := \tilde{H}_0 + \varkappa V$$

self-adjoint in $L^2(\mathbb{R}^3)$, and on $D(\tilde{H}_0^{(m)})$ introduce the operator

$$\tilde{H}_\varkappa^{(m)} := \tilde{H}_0^{(m)} + \varkappa V, \quad m \in \mathbb{Z},$$

self-adjoint in $L^2(\mathbb{R}_+ \times \mathbb{R}; \varrho d\varrho dx_3)$.
Let $\Lambda \in \sigma_{\rm disc}(H_\|)$. For $z \in \mathbb{C}_+$, $m \in \mathbb{Z}$, $q \in \mathbb{Z}_+$, $q \geq m_-$, set

$$F_{q,m}(z) := \langle (\tilde{H}_0^{(m)} - z)^{-1}(I - \mathcal{P}_{q,m})V\Phi_{q,m}, V\Phi_{q,m}\rangle$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $L^2(\mathbb{R}_+ \times \mathbb{R}; \varrho d\varrho dx_3)$, $\Phi_{q,m}(\varrho, x_3) = \varphi_{q,m}(\varrho)\psi(x_3)$ so that $\tilde{H}_0^{(m)}\Phi_{q,m} = (2bq + \Lambda)\Phi_{q,m}$, and $\mathcal{P}_{q,m} := |\Phi_{q,m}\rangle\langle\Phi_{q,m}|$.
We will say that *the Fermi Golden Rule* $\mathcal{F}_{q,m,\Lambda}$ is valid if the limit

$$F_{q,m}(2bq + \Lambda) = \lim_{\delta \downarrow 0} F_{q,m}(2bq + \Lambda + i\delta)$$

exists and is finite, and

$$\mathrm{Im}\, F_{q,m}(2bq + \Lambda) > 0.$$

For $j \in \mathbb{Z}_+$ set $v_j(x_3) := x_3^j \frac{d^j v_0}{dx_3^j}$, $V_j = x_3^j \frac{\partial^j V}{\partial x_3^j}$.
We will say that the condition $\mathcal{C}_\nu$, $\nu \in \mathbb{Z}_+$, holds true if the multipliers by v_j, $j = 0, 1$, are $H_{0,\|}$-compact, the multipliers by v_j, $j \leq \nu$, are $H_{0,\|}$-bounded, the multiplier by V is H_0-bounded with zero relative bound, and the multipliers by V_j, $j = 1, \ldots, \nu$, are H_0-bounded.

Theorem 7.1. [2, Theorem 4.1] *Let V be axisymmetric. Fix $n \in \mathbb{Z}_+$, and assume that the condition $\mathcal{C}_\nu$ holds with $\nu \geq n+5$. Suppose that* $\inf \sigma(H_\|) > -2b$. *Let $\Lambda \in \sigma_{\rm disc}(H_\|)$. Fix $m \in \mathbb{Z}$, $q \in \mathbb{Z}_+$, $q > m_-$, and suppose that the Fermi Golden Rule $\mathcal{F}_{q,m,\Lambda}$ is valid.*
Then there exists a function $g \in C_0^\infty(\mathbb{R}; \mathbb{R})$ such that $g = 1$ near $2bq + \Lambda$, and

$$\langle e^{-i\tilde{H}_\varkappa^{(m)}t} g(\tilde{H}_\varkappa^{(m)})\Phi_{q,m}, \Phi_{q,m}\rangle = a(\varkappa)e^{-i\Lambda_{q,m}(\varkappa)t} + b(\varkappa, t), \quad t \geq 0, \tag{7.1}$$

where

$$\Lambda_{q,m}(\varkappa) = 2bq + \Lambda + \varkappa\langle V\Phi_{q,m}, \Phi_{q,m}\rangle - \varkappa^2 F_{q,m}(2bq + \Lambda) + o_{q,m,V}(\varkappa^2), \quad \varkappa \to 0. \tag{7.2}$$

Moreover, a and b satisfy the estimates

$$|a(\varkappa) - 1| = \mathcal{O}(\varkappa^2),$$

$$|b(\varkappa, t)| = \mathcal{O}(\varkappa^2 |\ln|\varkappa||(1+t)^{-n}),$$

$$|b(\varkappa, t)| = \mathcal{O}(\varkappa^2 (1+t)^{-n+1}),$$

as $\varkappa \to 0$ *uniformly with respect to* $t \geq 0$.

Remarks: (i) The numbers $\Lambda_{q,m}$ appearing in (7.1) can be interpreted as resonances for the operator $\tilde{H}_m$, and hence of $\tilde{H}$. Another definition of the resonances of $\tilde{H}$ which is in the spirit of [1] and [12] and includes x_3-analyticity assumptions concerning v_0 and V, can be found in [2, Section 3] (see also [13] and [6] where no axial symmetry of V is assumed).
(ii) Note that the numbers $\langle V\Phi_{q,m}, \Phi_{q,m}\rangle$ appearing in (7.2) are eigenvalues of the Toeplitz operator $p_q \tilde{W} p_q$ appearing in Theorem 6.1.
(iii) The proof of Theorem 7.1 is based on appropriate Mourre estimates (see [18]), and the approach developed in [8].

Acknowledgements. G. Raikov thanks the organizers of the Second Symposium on Scattering and Spectral Theory, Pernambuco, Brazil, August 2008, for having invited him to present there the results of this paper. J.-F. Bony and V. Bruneau were partially supported by the French *ANR* Grant no. JC0546063. V. Bruneau and G. Raikov were partially supported by the Chilean Science Foundation *Fondecyt* under Grants 1050716 and 7060245 and by the cooperation program CNRS-Conicyt RQWUS41561. G. Raikov was partially supported by *Núcleo Científico ICM* P07-027-F "*Mathematical Theory of Quantum and Classical Magnetic Systems*".

Received: December, 2008. Revised: March, 2009.

References

[1] Aguilar, J. and Combes, J.M., *A class of analytic perturbations for one-body Schrödinger Hamiltonians*, Comm. Math. Phys., **22** (1971), 269–279.

[2] Astaburuaga, M.A., Briet, Ph., Bruneau, V., Fernández, C. and G. Raikov, *Dynamical resonances and SSF singularities for a magnetic Schrödinger operator*, Serdica Math. J., **34** (2008), 179–218.

[3] Avron, J., Herbst, I. and Simon, B., *Schrödinger operators with magnetic fields. I. General interactions*, Duke Math. J., **45** (1978), 847–883.

[4] Birman, M.Š. and Kreĭn, M.G., *On the theory of wave operators and scattering operators*, Dokl. Akad. Nauk SSSR, **144** (1962), 475–478 (Russian); English translation in Soviet Math. Doklady, **3** (1962).

[5] Bony, J.-F., Bruneau, V. and Raikov, G.D., *Resonances and spectral shift function near Landau levels*, Ann. Inst. Fourier, **57** (2007), 629–671.

[6] Bruneau, V., Khochman, A. and Raikov, G., *Perturbation of a magnetic Schrödinger operator near an embedded infinite-multiplicity eigenvalue*, to appear in: Spectral and Scattering Theory for Quantum Magnetic Systems, Contemporary Mathematics, **500** (2009), 47–62.

[7] Bruneau, V., Pushnitski, A. and Raikov, G.D., *Spectral shift function in strong magnetic fields*, Algebra i Analiz, **16** (2004), 207–238; see also St. Petersburg Math. J., **16** (2005), 181–209.

[8] Cattaneo, L., Graf, G.M. and Hunziker, W., *A general resonance theory based on Mourre's inequality*, Ann. Henri Poincaré, **7** (2006), 583–601.

[9] Fernández, C. and Raikov, G.D., *On the singularities of the magnetic spectral shift function at the Landau levels*, Ann. Henri Poincaré, **5** (2004), 381–403.

[10] Fock, V, *Bemerkung zur Quantelung des harmonischen Oszillators im Magnetfeld*, Z. Physik, **47** (1928), 446–448.

[11] Hall, B.C., *Holomorphic methods in analysis and mathematical physics*, In: First Summer School in Analysis and Mathematical Physics, Cuernavaca Morelos, 1998, 1–59, Contemp. Math., **260**, AMS, Providence, RI, 2000.

[12] Hunziker, W., *Distortion analyticity and molecular resonance curves*, Ann. Inst. H. Poincaré Phys. Théor., **45** (1986), 339–358.

[13] Khochman, A., *Resonances and spectral shift function for a magnetic Schrödinger operator*, J. Math. Phys. **50** (2009), no. 4, 043507, 16 pp.

[14] Krein, M.G., *On the trace formula in perturbation theory*, Mat. Sb., **33** (1953), 597–626 (Russian).

[15] Landau, L., *Diamagnetismus der Metalle*, Z. Physik, **64** (1930), 629–637.

[16] Levinson, N., *On the uniqueness of the potential in a Schrödinger equation for a given asymptotic phase*, Danske Vid. Selsk. Mat.-Fys. Medd., **25** (1949), no. 9, 1–29.

[17] Lifshits, I.M., *On a problem in perturbation theory*, Uspekhi Mat. Nauk, **7** (1952), 171–180 (Russian).

[18] Mourre, E., *Absence of singular continuous spectrum for certain self-adjoint operators*, Comm. Math. Phys., **78** (1981), 391–408.

[19] Pushnitskiĭ, A., *A representation for the spectral shift function in the case of perturbations of fixed sign*, Algebra i Analiz, **9** (1997), 197–213 (Russian); English translation in St. Petersburg Math. J., **9** (1998), 1181–1194.

[20] Raikov, G.D., *Eigenvalue asymptotics for the Schrödinger operator with homogeneous magnetic potential and decreasing electric potential. I. Behaviour near the essential spectrum tips*, Comm. PDE, **15** (1990), 407–434; Errata: Comm. PDE, **18** (1993), 1977–1979.

[21] Raikov, G.D., *Spectral shift function for Schrödinger operators in constant magnetic fields*, Cubo **7** (2005), 171–199.

[22] Raikov, G.D. and Warzel, S., *Quasi-classical versus non-classical spectral asymptotics for magnetic Schrödinger operators with decreasing electric potentials*, Rev. Math. Phys., **14** (2002), 1051–1072.

[23] Robert, D., *Semiclassical asymptotics for the spectral shift function*, In: Differential Operators and Spectral theory, AMS Translations Ser. 2, **189**, 187–203, AMS, Providence, RI, 1999.

[24] Yafaev, D.R., *Mathematical scattering theory. General theory.* Translations of Mathematical Monographs, **105** AMS, Providence, RI, 1992.