CUBO A Mathematical Journal
Vol.12, Nº01, (181–193). March 2010

On $N(k)$-Contact Metric Manifolds

A.A. Shaikh
Department of Mathematics, University of Burdwan, Golapbag, Burdwan-713104, West Bengal, India
email: aask2003@yahoo.co.in

and

C.S. Bagewadi
Department of Mathematics, Kuvempu University, Jana Sahyadri, Shankaraghatta-577 451, Karnataka, India
email: prof_bagewadi@yahoo.com

ABSTRACT

The object of the present paper is to study a type of contact metric manifolds, called $N(k)$-contact metric manifolds admitting a non-null concircular and torse forming vector field. Among others it is shown that such a manifold is either locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$ or a Sasakian manifold. Also it is shown that such a contact metric manifold can be expressed as a warped product $I \times_{e^p} \overset{*}{M}$, where $(\overset{*}{M}, \overset{*}{g})$ is a $2n$-dimensional manifold.

RESUMEN

El objetivo del presente artículo es estudiar un tipo de variedades métricas de contacto, llamadas $N(k)$-variedades métricas de contacto admitiendo un campo de vectores concircular y forma torse. Es demostrado también que tales variedades son o localmente isométricas a productos Riemannianos $E^{n+1}(0) \times S^n(4)$ o una variedade Sasakian. Es demostrado que tales variedades métricas de contacto pueden ser expresadas como un producto deformado $I \times_{e^p} \overset{*}{M}$, donde $(\overset{*}{M}, \overset{*}{g})$ es una variedad $2n$-dimensional.

Key words and phrases: *Contact metric manifold, k-nullity distribution, $N(k)$-contact metric manifold, concircular vector field, torse forming vector field, η-Einstein, Sasakian manifold, warped product.*

Math. Subj. Class.: *53C05, 53C15, 53C25.*

1 Introduction

A contact manifold is a smooth $(2n+1)$-dimensional manifold M^{2n+1} equipped with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Given a contact form η, there exists a unique vector field ξ, called the characteristic vector field of η, satisfying $\eta(\xi)$=1 and $d\eta(X,\xi)$=0 for any vector field X on M^{2n+1}. A Riemannian metric g is said to be associated metric if there exists a tensor field ϕ of type (1, 1) such that

$$\eta(X) = g(X,\xi), d\eta(X,Y) = g(X,\phi Y) \text{ and } \phi^2 X = -X + \eta(X)\xi \tag{1.1}$$

for all vector fields X, Y on M^{2n+1}. Then the structure (ϕ,ξ,η,g) on M^{2n+1} is called a contact metric structure and the manifold M^{2n+1} equipped with such a structure is said to be a contact metric manifold [2]. It can be easily seen that in a contact metric manifold, the following relations hold :

$$\phi\xi = 0, \quad \eta\circ\phi = 0, \quad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{1.2}$$

for any vector field X, Y on M^{2n+1}.
Given a contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ we define a (1, 1) tensor field h by $h = \frac{1}{2}\pounds_\xi\phi$, where $\pounds$ denotes the operator of Lie differentiation.Then h is symmetric and satisfies

$$h\xi = 0, \quad h\phi = -\phi h, \quad Tr.h = Tr.\phi h = 0. \tag{1.3}$$

If ∇ denotes the Riemannian connection of g, then we have the following relation

$$\nabla_X\xi = -\phi X - \phi hX. \tag{1.4}$$

A contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ for which ξ is a Killing vector field is called a K-contact manifold. A contact metric manifold is Sasakian if and only if

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{1.5}$$

where R is the Riemannian curvature tensor of type (1, 3).

In 1988, S. Tanno [7] introduced the notion of k-nullity distribution of a contact metric manifold as a distribution such that the characteristic vector field ξ of the contact metric manifold belongs to the distribution. The contact metric manifold with ξ belonging to the k-nullity distribution is called $N(k)$-contact metric manifold and such a manifold is also studied by various authors. Generalizing this notion in 1995, Blair, Koufogiorgos and Papantoniou [4] introduced the notion of a contact metric manifold with ξ belonging to the $(k,\ \mu)$-nullity distribution, where k and μ are real constants. In particular, if $\mu = 0$, then the notion of $(k,\ \mu)$-nullity distribution reduces to the notion of k-nullity distribution.

The present paper deals with a study of $N(k)$-contact metric manifolds. The paper is organised as follows. Section 2 is concerned with the discussion of $N(k)$-contact metric manifolds. In section 3, we obtain a necessary and sufficient condition for a $N(k)$-contact metric manifold to be an $\eta-$ Einstein manifold. Section 4 is devoted to the study of $N(k)$-contact metric manifolds admitting a non-null concircular vector field and it is proved that such a manifold is either locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$ or a Sasakian manifold. The last section deals with a study of $N(k)$-contact metric manifolds admitting a non-null torse forming vector field and it is shown that such a torse forming vector field reduces to a unit proper concircular vector field. Hence a $N(k)$-contact metric manifold admits a proper concircular vector field, namely, the characteristic vector field ξ, and it is proved that a $N(k)$-contact metric manifold is a subprojective manifold in the sense of Kagan [1]. Finally it is shown that a $N(k)$-contact metric manifold can be expressed as a warped product $I \times_{e^p} \overset{*}{M}$, where $(\overset{*}{M},\ \overset{*}{g})$ is a $2n$-dimensional manifold.

2 $N(k)$-Contact Metric Manifolds

Let us consider a contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$. The k-nullity distribution [7] of a Riemainnian manifold (M, g) for a real number k is a distribution

$$N(k) : p \to N_p(k) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\}$$

for any $X, Y \in T_pM$. Hence if the characteristic vector field ξ of a contact metric manifold belongs to the k-nullity distribution, then we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y]. \tag{2.1}$$

Thus a contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ satisfying the relation (2.1) is called a $N(k)$-contact metric manifold. From (1.5) and (2.1) it follows that a $N(k)$-contact metric manifold is a Sasakian manifold if and only if $k = 1$. Also in a $N(k)$-contact metric manifold, k is always a constant such that $k \leq 1$ [7].

The $(k,\ \mu)$-nullity distribution of a contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is a distribution [4]

$$\begin{aligned} N(k,\mu) : p \to N_p(k,\mu) \quad = \quad & \Big[Z \in T_pM : R(X,Y)Z = k\{g(Y,Z)X - g(X,Z)Y\} \\ & +\mu\{g(Y,Z)hX - g(X,Z)hY\}\Big] \end{aligned}$$

for any X, Y$\in T_pM$, where k, μ are real constants. Hence if the characteristic vector field ξ belongs to the $(k,\ \mu)$-nullity distribution, then we have

$$R(X,Y)\xi = k\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\}. \tag{2.2}$$

A contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ satisfying the relation (2.2) is called a $N(k,\mu)$-contact metric manifold or simply a $(k,\ \mu)$-contact metric manifold. In particular, if $\mu = 0$, then the relation (2.2) reduces to (2.1) and hence a $N(k)$-contact metric manifold is a $N(k,0)$-contact metric manifold.

Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a $N(k)$-contact metric manifold. Then the following relations hold ([5], [7]):

$$Q\phi - \phi Q = 4(n-1)h\phi, \tag{2.3}$$

$$h^2 = (k-1)\phi^2, \qquad k \leq 1, \tag{2.4}$$

$$Q\xi = 2nk\xi, \tag{2.5}$$

$$R(\xi,X)Y = k[g(X,Y)\xi - \eta(Y)X], \tag{2.6}$$

where Q is the Ricci operator, i.e., $g(QX,Y) = S(X,Y)$, S being the Ricci tensor of type (0, 2). In view of (1.1)-(1.2), it follows from (2.3)– (2.6) that in a $N(k)$-contact metric manifold, the following relations hold:

$$Tr.h^2 = 2n(1-k), \tag{2.7}$$

$$S(X,\phi Y) + S(\phi X,Y) = 2(2n-2)g(\phi X,hY), \tag{2.8}$$

$$S(\phi X,\phi Y) = S(X,Y) - 2nk\eta(X)\eta(Y) - 2(2n-2)g(hX,Y), \tag{2.9}$$

$$Q\phi + \phi Q = 2\phi Q + 2(2n-2)h\phi, \tag{2.10}$$

$$\eta(R(X,Y)Z) = k[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.11}$$

$$S(\phi X,\xi) = 0 \tag{2.12}$$

for any vector field X, Y on M^{2n+1}. Also in a $N(k)$-contact metric manifold the scalar curvature r is given by ([4], [5])

$$r = 2n(2n-2+k). \tag{2.13}$$

We now state a result as a lemma which will be used later on.

Lemma 2.1. *[3] Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a contact metric manifold with $R(X,Y)\xi$=0 for all vector fields X, Y. Then the manifold is locally isometric to the Riemannian product $E^{n+1}(0)\times S^n(4)$.*

3 η-Einstein $N(k)$-Contact Metric Manifolds

Definition 3.1. *A $N(k)$-contact metric manifold M^{2n+1} is said to be η-Einstein if its Ricci tensor S of type (0, 2) is of the form*

$$S = ag + b\eta \otimes \eta, \tag{3.1}$$

where a, b are smooth functions on M^{2n+1}.

From (3.1) it follows that

$$(i) \quad r = (2n+1)a + b, \quad (ii) \quad 2nk = a + b, \tag{3.2}$$

which yields by virtue of (2.13) that $a = 2n - 2$ and $b = 2n(k-1) + 2$. Obviously a and b are constants as k is a constant. Hence by virtue of (3.1) we can state the following:

Proposition 3.1 *In an η-Einstein $N(k)$-contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)(n > 1)$, the Ricci tensor is of the form*

$$S = (2n-2)g + \{2n(k-1) + 2\}\eta \otimes \eta. \tag{3.3}$$

Let $M^{2n+1}(\phi, \xi, \eta, g)(n > 1)$ be a $N(k)$-contact metric manifold. Now we have

$$(R(X,Y) \cdot S)(U,V) = -S(R(X,Y)U,V) - S(U,R(X,Y)V),$$

which implies that

$$(R(X,\xi) \cdot S)(U,V) = -S(R(X,\xi)U,V) - S(U,R(X,\xi)V). \tag{3.4}$$

First we suppose that a $N(k)$-contact metric manifold is an η-Einstein manifold. Then we have

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{3.5}$$

where a and b are given by $a = 2n - 2$ and $b = 2n(k-1) + 2$. Using (3.5), (2.5) and (2.6) in (3.4) we obtain

$$\begin{aligned}(R(X,\xi) \cdot S)(U,V) &= k[(2nk - a)g(X,U)\eta(V) + g(X,V)\eta(U) \\ &\quad -2b\eta(X)\eta(U)\eta(V)].\end{aligned} \tag{3.6}$$

From (3.2)(ii) it follows that

$$b = 2nk - a. \tag{3.7}$$

In view of (3.7), (3.6) reduces to

$$\begin{aligned}(R(X,\xi) \cdot S)(U,V) &= kb[g(X,U)\eta(V) + g(X,V)\eta(U) \\ &\quad -2\eta(X)\eta(U)\eta(V)].\end{aligned} \tag{3.8}$$

Putting $V = \xi$ in (3.8) we obtain

$$(R(X, \xi) \cdot S)(U, \xi) = k\{2n(k-1) + 2\}[g(X, U) - \eta(X)\eta(U)]. \tag{3.9}$$

Hence we can state the following:

Theorem 3.1. *If a $N(k)$-contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ $(n > 1)$ is η-Einstein, then the relation (3.9) holds.*

Next, we suppose that in a $N(k)$-contact metric manifold $M^{2n+1}(n > 1)$ the relation (3.9) holds. Then using (2.5) and (2.6) in (3.4) we get

$$(R(X, \xi) \cdot S)(U, \xi) = k[2nkg(X, U) - S(X, U)]. \tag{3.10}$$

By virtue of (3.9) and (3.10) we obtain

$$k[S(X, U) - (2n-2)g(X, U) - \{2n(k-1) + 2\}\eta(X)\eta(U)] = 0.$$

This implies either $k = 0$,

$$\text{or,} \quad S(X, U) = (2n-2)g(X, U) + \{2n(k-1) + 2\}\eta(X)\eta(U). \tag{3.11}$$

If $k = 0$, then from (2.1) we have

$$R(X, Y)\xi = 0 \quad \text{for all } X, Y.$$

Hence by Lemma 2.1, it follows that the manifold is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$. Again (3.11) implies that the manifold is η-Einstein. Hence we can state the following:

Theorem 3.2. *If in a $N(k)$-contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)(n > 1)$ the relation (3.9) holds, then either the manifold is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$ or the manifold is η-Einstein.*

Combining Theorem 3.1 and Theorem 3.2 we can state the following:

Theorem 3.3. *A $N(k)$-contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)(n > 1)(k \neq 0)$ is an η-Einstein manifold if and only if the relation (3.9) holds.*

4 $N(k)$-Contact Metric Manifolds Admitting a Non-null Concircular Vector Field

Definition 4.1. *A vector field V on a Riemannian manifold is said to be concircular vector field [6] if it satisfies an equation of the form*

$$\nabla_X V = \rho X \qquad \textit{for all } X, \tag{4.1}$$

where ρ is a scalar.

We suppose that a $N(k)$-contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)(n>1)$ admits a non-null concircular vector field. Then we have (4.1). Differentiating (4.1) covariantly we get

$$\nabla_Y\nabla_X V=\rho\nabla_Y X+d\rho(Y)X. \tag{4.2}$$

From (4.2) it follows that (since the torsion tensor $T(X,Y)=\nabla_X Y-\nabla_Y X-[X,Y]=0$)

$$\nabla_Y\nabla_X V-\nabla_X\nabla_Y V-\nabla_{[X,Y]}V=d\rho(X)Y-d\rho(Y)X. \tag{4.3}$$

Hence by Ricci identity we obtain from (4.3)

$$R(X,Y)V=d\rho(X)Y-d\rho(Y)X, \tag{4.4}$$

which implies that

$$\tilde{R}(X,Y,V,Z)=d\rho(X)g(Y,Z)-d\rho(Y)g(X,Z), \tag{4.5}$$

where $\tilde{R}(X,Y,V,Z)=g(R(X,Y)V,Z)$.
Replacing Z by ξ in (4.5) we get

$$\eta(R(X,Y)V)=d\rho(X)\eta(Y)-d\rho(Y)\eta(X). \tag{4.6}$$

Again from (2.11) we have

$$\eta(R(X,Y)V)=k[g(Y,V)\eta(X)-g(X,V)\eta(Y)]. \tag{4.7}$$

By virtue of (4.6) and (4.7) we have

$$d\rho(X)\eta(Y)-d\rho(Y)\eta(X)=k[g(Y,V)\eta(X)-g(X,V)\eta(Y)]. \tag{4.8}$$

Putting $X=\phi X$ and $Y=\xi$ in (4.8), and then using (1.2) we get

$$d\rho(\phi X)=-kg(\phi X,V). \tag{4.9}$$

Substituting X by ϕX in (4.9), we obtain by virtue of (1.1) that

$$d\rho(X)-d\rho(\xi)\eta(X)=k[g(X,V)-\eta(X)\eta(V)]. \tag{4.10}$$

Now we have $g(X,V)\neq 0$ for all X. For, if $g(X,V)=0$ for all X, then $g(V,V)=0$ which means that V is a null vector field, contradicts to our assumption. Hence multiplying both sides of (4.10) by $g(X,V)$ we have

$$d\rho(X)g(X,V)-d\rho(\xi)g(X,V)\eta(X)=kg(X,V)[g(X,V)-\eta(X)\eta(V)]. \tag{4.11}$$

Also from (4.5) we get for $Z=V$ (since $\tilde{R}(X,Y,V,V)=0$)

$$d\rho(X)g(Y,V)=d\rho(Y)g(X,V). \tag{4.12}$$

Putting $Y=\xi$ in (4.12) and then using (1.1) we obtain

$$d\rho(X)\eta(V)=d\rho(\xi)g(X,V). \tag{4.13}$$

Since $\eta(X) \neq 0$ for all X, multiplying both sides of (4.13) by $\eta(X)$, we have

$$d\rho(X)\eta(X)\eta(V) = d\rho(\xi)\eta(X)g(X,V). \tag{4.14}$$

By virtue of (4.11) and (4.14) we get

$$[d\rho(X) - kg(X,V)][g(X,V) - \eta(X)\eta(V)] = 0. \tag{4.15}$$

Hence it follows from (4.15) that

$$\text{either} \quad d\rho(X) = kg(X,V) \qquad \text{for all } X \tag{4.16}$$

$$\text{or,} \quad g(X,V) - \eta(X)\eta(V) = 0 \qquad \text{for all } X. \tag{4.17}$$

First we consider the case of (4.16). By virtue of (4.16) we obtain from (4.5) that

$$\tilde{R}(X,Y,V,Z) = k[-g(Y,V)g(X,Z) + g(X,V)g(Y,Z)]. \tag{4.18}$$

Let $\{e_i : \ i = 1, 2, ..., 2n+1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = Z = e_i$ in (4.18) and taking summation over i, $1 \leq i \leq 2n+1$, we get

$$S(Y,V) = -2nkg(Y,V). \tag{4.19}$$

Now

$$(\nabla_Z S)(Y,V) = \nabla_Z S(Y,V) - S(\nabla_Z Y,V) - S(Y,\nabla_Z V). \tag{4.20}$$

Using (4.1) and (4.19) in (4.20) we obtain

$$(\nabla_Z S)(Y,V) = \rho[-2nkg(Y,Z) + S(Y,Z)]. \tag{4.21}$$

Setting $Y = Z = e_i$ in (4.21) and then taking summation over $1 \leq i \leq 2n+1$, we get

$$\frac{1}{2}dr(V) = \rho[-2nk(2n+1) + r], \tag{4.22}$$

where r denotes the scalar curvature of the manifold. Since in a $N(k)$-contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)(n > 1)$ k is a constant, by virtue of (2.13) it follows that r is constant and hence (4.22) yields (since $r \neq 2nk(2n+1)$) $\rho = 0$, which implies by virtue of (4.4) that $R(X,Y)V = 0$ for all X and Y. This yields $S(Y,V) = 0$, which implies by virtue of (4.19) that $k = 0$. If $k = 0$ then from (2.1) we have $R(X,Y)\xi = 0$ for all X and Y and hence by Lemma 2.1, it follows that the manifold is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$.

Next we consider the case (4.17). Differentiating (4.17) covariantly along Z, we get

$$(\nabla_Z \eta)(X)\eta(V) + (\nabla_Z \eta)(V)\eta(X) = (\nabla_Z g)(X,V) = 0. \tag{4.23}$$

Now we have

$$\begin{aligned}(\nabla_X\eta)(Y) &= \nabla_X\eta(Y) - \eta(\nabla_X Y)\\ &= \nabla_X g(Y,\xi) - g(\nabla_X Y,\xi).\\ &= (\nabla_X g)(Y,\xi) + g(Y,\nabla_X\xi).\end{aligned}$$

$$\text{That is,}\quad (\nabla_X\eta)(Y) = g(Y,\nabla_X\xi). \tag{4.24}$$

By virtue of (4.24) we get from (4.23) that

$$\eta(V)g(X,\nabla_Z\xi) + \eta(X)g(V,\nabla_Z\xi) = 0. \tag{4.25}$$

In view of (1.4), (4.25) yields

$$[g(X,\phi Z) + g(X,\phi hZ)]\eta(V) + [g(V,\phi Z) + g(V,\phi hZ)]\eta(X) = 0. \tag{4.26}$$

Putting $X = \xi$ in (4.26) we get

$$g(X,\phi Z) + g(V,\phi hZ) = 0. \tag{4.27}$$

Substituting Z by ϕZ in (4.27), we obtain by virtue of (1.1), $h\phi = -\phi h$ and $h\xi = 0$ that

$$-g(V,Z) + \eta(V)\eta(Z) + g(V,hZ) = 0. \tag{4.28}$$

Using (4.17) in (4.28) we get

$$g(V,hZ) = 0 \quad \text{for all} \ \ Z.$$

Since h is symmetric, the above relation implies that $g(hV,Z) = 0$ for all Z, which gives us $hV = 0$. But since V is non-null, by our assumption, we must have $h = 0$ and hence from (2.4) it follows that $k = 1$. Therefore the manifold is Sasakian. Hence summing up all the cases we can state the following:

Theorem 4.1. *If a $N(k)$-contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)(n > 1)$ admits a non-null concircular vector field, then either the manifold is locally isometric to the Riemannian product $E^{n+1}(0)\times S^n(4)$ or the manifold is Sasakian.*

5 $N(k)$-Contact Metric Manifolds Admitting a Non-null Torse Forming Vector Field

Definition 5.1. *A vector field V on a Riemannian manifold is said to be torse forming vector field ([6], [8]) if the 1-form $\omega(X) = g(X,\ V)$ satisfies the equation of the form*

$$(\nabla_X\omega)Y = \rho g(X,\ Y) + \pi(X)\omega(Y), \tag{5.1}$$

where ρ is a non-vanishing scalar and π is a non-zero 1-form given by $\pi(X) = g(X,\ P)$.

If the 1-form π is closed, then the vector field V is called a *proper concircular vector field.* In particular if the the 1-form π is zero, then the vector field V reduces to a concircular vector field.

Let us consider a $N(k)$-contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)(n > 1)$ admitting a unit torse forming vector field U corresponding to the non-null torse forming vector field V. Hence if $T(X) = g(X,\ U)$, then we have

$$T(X) = \frac{\omega(X)}{\sqrt{\omega(X)}}. \tag{5.2}$$

By virtue of (5.2), it follows from (5.1) that

$$(\nabla_X T)(Y) = \beta g(X,\ Y) + \pi(X)T(Y), \tag{5.3}$$

where $\beta = \dfrac{\alpha}{\sqrt{\omega(V)}}$ is a non-zero scalar. Since U is a unit vector field, substituting Y by U in (5.3) yields

$$\pi(X) = -\beta T(X)$$

and hence (5.3) reduces to the following

$$(\nabla_X T)(Y) = \beta[g(X,\ Y) + T(X)T(Y)]. \tag{5.4}$$

The relation (5.4) implies that the 1-form T is closed. Differentiating (5.4) covariantly we obtain by virtue of Ricci identity that

$$\begin{aligned} -T(R(X,Y)Z) \quad = \quad & (X\beta)[g(Y,\ Z) + T(Y)T(Z)] - (Y\beta)[g(X,\ Z) + T(X)T(Z)] \\ & +\beta^2[g(Y,\ Z)T(X) + g(X,\ Z)T(Y)]. \end{aligned} \tag{5.5}$$

Setting $Z = \xi$ in (5.5) and then using (2.1) we get

$$\begin{aligned} & (X\beta)[\eta(Y) + T(Y)\eta(U)] - (Y\beta)[\eta(X) + T(X)\eta(U)] \\ & +(k + \beta^2)[g(Y,\ Z)T(X) + g(X,\ Z)T(Y)] = 0. \end{aligned} \tag{5.6}$$

Putting $X = U$ in (5.6) we obtain

$$[k + \beta^2 + (U\beta)][\eta(Y) - \eta(U)T(Y)] = 0,$$

which implies that

$$\text{either} \quad [k + \beta^2 + (U\beta)] = 0 \tag{5.7}$$

$$\text{or,} \ \eta(Y) - \eta(U)T(Y) = 0. \tag{5.8}$$

We first consider the case of (5.7). From (5.5) it follows that

$$S(Y,U) = [2n\beta^2 + (U\beta)]T(Y) - (2n-1)(Y\beta), \tag{5.9}$$

which yields for $Y = \xi$ that

$$(\xi\beta) = (U\beta)\eta(U). \tag{5.10}$$

Again, setting $Y = \xi$ in (5.6) we obtain by virtue of (5.10) that

$$[1 - (\eta(U))^2][(X\beta) - (k + \beta^2)T(X)] = 0. \tag{5.11}$$

In this case $\eta(Y) - \eta(U)T(Y) \neq 0$ for all Y and hence $1 - (\eta(U))^2 \neq 0$. Consequently, (5.11) gives us

$$(X\beta) = (k + \beta^2)T(X). \tag{5.12}$$

Again, from $\pi(X) = -\beta T(X)$ it follows that

$$Y\pi(X) = -[(Y\beta)T(X) + \beta(YT(X))]. \tag{5.13}$$

In view of (5.13) we obtain

$$d\pi(X,\ Y) = -\beta dT(X,\ Y).$$

Since T is closed, π is also closed and hence the vector field V is a proper concircular vector field in this case.

Next, we consider the case of (5.8). The relation (5.8) implies that

$$(\eta(U))^2 = 1$$

and hence $\eta(U) = \pm 1$. Consequently (5.8) reduces to

$$\eta(Y) = \pm T(Y). \tag{5.14}$$

Differentiating (5.14) covariantly along X, we obtain by virtue of (5.14) that

$$(\nabla_X \eta)(Y) = \pm\beta[g(X,\ Y) - \eta(X)\eta(Y)], \tag{5.15}$$

which yields by virtue of (1.4) that

$$g(X + hX,\ \phi Y) = \pm\beta[g(X,\ Y) - \eta(X)\eta(Y)]. \tag{5.16}$$

Replacing Y by ϕY in (5.16) and then using (1.2) we get

$$-g(X,\ Y) - g(hX,\ Y) + \eta(X)\eta(Y) = \pm\beta g(X,\ \phi Y). \tag{5.17}$$

Again setting $X = hX$ in (5.17) we obtain by virtue of (1.1) and (2.4) that

$$-g(hX,\ Y) + (k-1)[g(X,\ Y) - \eta(X)\eta(Y)] = \pm\beta g(hX,\ \phi Y). \tag{5.18}$$

Putting $X = Y = e_i$ in (5.18) and then taking summation over $1 \leq i \leq 2n+1$ we get by virtue of (1.3) that

$$k = 1 \tag{5.19}$$

and hence the manifold is Sasakian.

Let us now suppose that the manifold is non-Sasakian. Then $k < 1$ [4]. Hence from (5.17) and (5.18) it follows that

$$(k-\beta^2)[g(X,\ Y)-\eta(X)\eta(Y)] = \mp 2\beta g(X,\phi Y) \tag{5.20}$$

which yields by contraction $k = \pm\beta^2$. Since $\beta \neq 0$, it follows that $(X\beta) = 0$ for any X and hence β is constant. Consequently we obtain $\pi(X) = -\beta T(X)$ where β is constant, it follows that the 1-form π is also closed and hence the vector field V is a proper concircular vector field. Considering all the cases we can state the following:

Theorem 5.1. *In a $N(k)$-contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)(n>1)$, a non-null torse forming vector field is a proper concircular vector field.*

From (1.4) and (5.4) it follows that in a $N(k)$-contact metric manifold the characteristic vector field ξ is a unit torse forming vector field and hence by virtue of Theorem 5.1, we can state the following:

Theorem 5.2. *A $N(k)$-contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)(n>1)$ admits a proper concircular vector field.*

Again, it is known that if a Riemannian manifold admits a proper concircular vector field, then the manifold is a subprojective manifold in the sense of Kagan ([1]). Since a $N(k)$-contact metric manifold admits a concircular vector field, namely, the vector field ξ, in view of the known result we can state the following:

Theorem 5.3. *A $N(k)$- contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)(n>1)$ is a subprojective manifold in the sense of Kagan.*

By virtue of Theorem 5.2 and Theorem 4.1 we can state the following:

Theorem 5.4. *A $N(k)$-contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)(n>1)$ is either locally isometric to the Riemannian product $E^{n+1}(0)\times S^n(4)$ or a Sasakian manifold.*

K. Yano [8] proved that if a Riemannian manifold M^{2n+1} admits a concircular vector field, it is necessary and sufficient that there exists a coordinate system with respect to which the fundamental quadratic differential form may be written as

$$ds^2 = (dx^1)^2 + e^p \overset{*}{g}_{\lambda\mu}\, dx^\lambda dx^\mu, \tag{5.21}$$

where $\overset{*}{g}_{\lambda\mu}=\overset{*}{g}_{\lambda\mu}\ (x^\nu)$ are the function of x^ν only $(\lambda,\ \mu,\ \nu = 2,\ 3,\ \ldots\ldots,\ 2n)$ and $p = p(x^1) \neq$ constant, is a function of x^1 only. Since a $N(k)$-contact metric manifold admits a proper concircular vector field, namely, the characteristic vector field ξ, by virtue of the above it follows that there exists a coordinate system with respect to which the fundamental quadratic differential form can be written as (5.21). Consequently the manifold can be expressed as a warped product $I\times_{e^p}\overset{*}{M}$, where $(\overset{*}{M},\ \overset{*}{g})$ is a $2n$-dimensional manifold. Hence we can state the following:

Theorem 5.5. *A $N(k)$-contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)(n > 1)$ can be expressed as a warped product $I \times_{e^p} \overset{*}{M}$, where $(\overset{*}{M}, \overset{*}{g})$ is a $2n$-dimensional manifold.*

Received: July, 2008. Revised: January, 2009.

References

[1] Adati, T., *On Subprojective spaces III*, Tohoku Math. J., **3**(1951), 343–358.

[2] Blair, D.E., *Contact manifolds in Riemannian geometry*, Lecture Notes in Math., **509**, Springer-Verlag, 1976.

[3] Blair, D.E., *Two remarks on contact metric structure* , Tohoku Math. J., **29**(1977), 319–324.

[4] Blair, D.E., Koufogiorgos, T. and Papantoniou, B.J., *Contact metric manifolds satisfying a nullity condition*, Israel J. of Math., **19**(1995), 189–214.

[5] Shaikh, A.A. and Baishya, K.K., *On (k, μ)-contact metric manifolds*, J. Diff. Geom. and Dyn. Sys., **11**(1906), 253–261.

[6] Schouten, J.A., *Ricci Calculas (Second Edition)*, Springer-Verlag, 1954, 322.

[7] Tanno, S., *Ricci curvatures of contact Riemannian manifolds*, Tohoku Math. J., **40**(1988), 441–448.

[8] Yano, K., *On the torse forming direction in Riemannian spaces*, Proc. Imp. Acad. Tokyo, **20**(1940), 340–345.

[9] Yano, K., *Concircular geometry I-IV*, Proc. Imp. Acad. Tokyo, **16**(1940), 195- 200, 354–350.