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Brill-Noether Theories for Rank 1 Sheaves on $\overline{\mathcal{M}}_g$

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ABSTRACT

Here we discuss some Brill-Noether problems for rank 1 sheaves on stable curves.

RESUMEN

Nosotros discutimos aquí algunos problemas de Brill-Noether para hazes de rango 1 sobre curvas estables.

Key words and phrases: *Stable curve, reducible curve, Brill-Noether theory.*

Math. Subj. Class.: *14H10, 14H51.*

We stress the plural "Brill-Noether theories" in the title. In section 1 we consider reducible curves with large genus with degree 1 spanned sheaves (one could say that they have gonality 1). Since these are our easiest results, we state them in the introduction, before going on to other possible meanings of the word "gonality".

Let X be a connected projective curve such that each point of X lying on at least two irreducible components of X is an ordinary node of X. Let $\mathrm{Sing}(X)''$ be the set of all points of X lying

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on at least two irreducible components of X. By assumption every point of $\mathrm{Sing}(X)''$ is an ordinary node of X. We assume $g := p_a(X) \geq 2$. Let $\mathcal{B}(X)$ denote the set of all irreducible components of X. For the elementary properties of depth 1 coherent sheaves on reduced curves, see [21], parts VII and VIII. A coherent sheaf F on X has depth 1 if and only if no non-zero subsheaf of F is supported by a finite set. Hence on a reduced curve depth 1 sheaves are often called torsion-free sheaves. We say that a depth 1 sheaf F on X has pure pure rank 1 if its restriction to X_{reg} is a line bundle. Let F be sheaf on X with pure rank 1 and with depth 1. Set $\mathrm{Sing}(F) := \{P \in X : F$ is not locally free at $P\}$. Hence $\mathrm{Sing}(F) \subseteq \mathrm{Sing}(X)$. The degree $\deg(F)$ of F may be defined by the Rieman-Roch formula $\chi(F) = \deg(F) + \chi(\mathcal{O}_X)$, even if the curve is not connected. Let $S_k(X)$ denote the set of all spanned coherent sheaves on X with depth 1, pure rank 1 and degree k. Set $D_k(X) := S_k(X) \cap \mathrm{Pic}(X)$.

Theorem 1. *Let X be a connected projective curve such that each point of X lying on at least two irreducible components of X is an ordinary node of X. Assume $g := p_a(X) \geq 2$. Let $\mathfrak{S}(X)$ be the set of all $T \in \mathcal{B}(X)$ such that $T \cong \mathbb{P}^1$ and $X \backslash T$ has $\sharp(T \cap Sing(X))$ connected components. $D_1(X) \neq \emptyset$ if and only if $\mathfrak{S}(X) \neq \emptyset$. There is a bijection between $\mathfrak{S}(X)$ and $D_1(X)$ constructed in the following way. Take $T \in \mathfrak{S}(X)$ and an automorphism $\phi : T \to \mathbb{P}^1$. There is a unique spanned degree 1 line bundle L_T on X such that $h^0(X, L_T) = 2$ and the morphism $h_{L_T} : X \to \mathbb{P}^1$ induces ϕ on T, while it sends the connected components $Y_1, \dots, Y_s$, $s := \sharp(T \cap Sing(X))$, of $\overline{X \backslash T}$ respectively to the points $\phi(T \cap Y_1), \dots, \phi(T \cap Y_s)$. If $D_1(X) \neq \emptyset$, then for every integer $d \geq 2$ there is $L \in D_d(X)$ such that $h^0(X, L) = d + 1$.*

Theorem 2. *Let X be a connected projective curve such that each point of X lying on at least two irreducible components of X is an ordinary node of X. Assume $g := p_a(X) \geq 2$. There is a bijection between the set $\mathbb{D}(X)$ of all disconnecting nodes and $S_1(X) \backslash D_1(X)$ constructed in the following way. For each $P \in \mathbb{D}(X)$ let $f_P : X_P \to X$ be the partial normalization of X in which we only normalize the point P. Then $f_{P*}(\mathcal{O}_{X_P}) \in S_1(X) \backslash D_1(X)$. Any $F \in S_1(X) \backslash D_1(X)$ has a unique singular point, this singular point is a disconnecting node, and $h^0(X, F) = 2$.*

In Theorems 1 and 2 we just required that the sheaves are spanned. The universal propety of projective spaces say that spanned line bundles are essentially equivalent to morphisms to projective spaces. In section 2 we discuss what happens if we add the condition that the morphism contracts no irreducible component, i.e. if the spanned line bundle is ample. We only consider $\mathbb{P}^1$ as the target and study the deformation theory of a morphism $f : X \to \mathbb{P}^1$ which does not contract any irreducible component of X.

The main contender for the title of "best Brill-Noether theory" is the one introduced in [6]: line bundles (or sheaves) satisfying the so-called Basic Inequality. For them see section 3. See [7] for the case of binary curves, i.e. genus g stable curves with two irreducible components, both of them smooth and rational. In the introduction we give the combinatorial data which describe all topological types for stable curves with a fixed genus.

For any nodal curve X let $\mathrm{Sing}(X)'$ (resp. $\mathrm{Sing}(X)''$) be the set of all singular points of X lying on exactly one (resp. two) irreducible components of X. To any nodal projective curve X

we associate the following non-oriented marked graph $\|X\|$. There is a bijection between vertices of $\|X\|$ and the set $\mathcal{B}(X)$ of all irreducible components of X. For any $T \in \mathcal{B}(X)$ let $[T]$ denote the associated vertex of $\|X\|$. For each $T \in \mathcal{B}(X)$ we give as a marking the non-negative integer q_T, where q_T is the geometric genus of T. $\|X\|$ contains $\sharp(\mathrm{Sing}(X)' \cap T)$ loops with $[T]$ as their vertex. For all $T, T' \in \mathcal{B}(X)$, such that $T \neq T'$ the vertices $[T]$ and $[T']$ of $\|X\|$ are joined by $\sharp(T \cap T')$ edges. Call τ the abstract marked graph $\|X\|$. The set of all nodal projective curves Y such that $\|Y\| \cong \tau$ (as marked graphs) is parametrized by an irreducible algebraic variety $\mathcal{M}[\tau]$. If $\mathbb{K} = \mathbb{C}$, then the topological type of the complex analytic space $X(\mathbb{C})$ is uniquely determined by the marked graph τ and two non-isomorphic marked graphs give topologically different complex analytic spaces. If we forget the marking, i.e. if we forget the integer q_T, $T \in \mathcal{B}(X)$, then $\|X\|$ becomes the classical dual graph of the nodal curve X. Fix a topological type τ for nodal connected curves, say $\tau = \|X\|$. For all $T, J \in \mathcal{B}(X)$, $T \neq J$ let $q_T \geq 0$ be the associated marking, a_T the number of looops based at T and $a_{TJ} := \sharp(T \cap J)$. For every $T \in \mathcal{B}(X)$ fix an integer d_T and set $\underline{d} := (d_T)_{T \in \mathcal{B}(X)}$ and call $\underline{d}$ a multidegree for τ or for every curve $A \in \mathcal{M}[\tau]$. We say that $\underline{d}$ is *positive* and write $\underline{d} > 0$ if $d_T > 0$ for all $T \in \mathcal{B}(X)$. A natural question is to study the Brill-Nother theory (for all multidegrees, not just for the total degree of the sheaf or line bundle) of a general element of $\mathcal{M}[\tau]$. See [7] for the case of binary curves

We conclude by giving examples of stable curves X such that ω_X contains no spanned line bundle (Section 4). We don't touch a very important topic: the limit linear series introduced by D. Eisenbud and J. Harris for nodal curves of compact type, i.e. such that $\|X\|$ has no loop and no multiple edge ([11]). See [18] for the positive characteristic case and [13] for the case of nodal curves with two smooth irreducible components. A quick glance at G. Farkas's survey on the geometry of $\overline{\mathcal{M}}_g$ and at the references quoted theirin shows the importance of this theory. However, a quick look at the examples in [11] (resp. [13]) and at Example 2 (resp. Example 3) here shows that the Brill-Noether theory coming from the limit linear series has nothing in common with the one considered here in section 1 or the one studied by L. Caporaso in [7].

Theorems 1 and 1 and Section 1 are contained in [2] (in which also degree 2 sheaves are considered). Section 2 is contained in [3].

1 Curves with Gonality 1

Remark 1. Let X be a reduced and quasi-projective curve, $P \in X$, and F a sheaf on X with pure rank 1 and depth 1. The germ F_P of F at P is a torsion free $\mathcal{O}_{X,P}$-module with rank 1. Hence there exists an inclusion $j : F_P \hookrightarrow M$ with M a free $\mathcal{O}_{X,P}$-module with rank 1. The minimal integer $\dim_{\mathbb{K}}(M/F_P)$ for all such pairs (j, M) is an important invariant of the germ F_P. Call $\ell(F, P)$ this integer. We have $\ell(F, P) \geq 0$ and $\ell(F, P) = 0$ if and only if F_P is a free $\mathcal{O}_{X,P}$-module. This invariant may be computed on the formal completion of $\mathcal{O}_{X,P}$. Let $m_{X,P}$ be the maximal ideal of the local ring $\mathcal{O}_{X,P}$. Notice that $m_{X,P}$ is a free $\mathcal{O}_{X,P}$-module if and only if $P \in X_{reg}$. Hence if $P \in \mathrm{Sing}(X)$ and $F_P \cong m_{X,P}$, then $\ell(F, P) = 1$. Now assume that X is projective.

Fix a finite set $S \subseteq \mathrm{Sing}(X)$ and let $f : C \to X$ be the partial normalization of X in which we normalize only the points of S. The torsion of $f^*(F)$ is supported on the finite set $f^{-1}(S)$. Set $G := f^*(F)/\mathrm{Tors}(f^*(F))$. G is a coherent sheaf on C with depth 1 and pure rank 1. Since X and C are projective, the integers $\deg(F)$ and $\deg(G)$ are well-defined and satisfy the Riemmann-Roch formulas $\chi(F) = \deg(F) + \chi(\mathcal{O}_X)$, $\chi(G) = \deg(G) + \chi(\mathcal{O}_C)$ even if X or C are not connected. We have

$$\deg(G) = \deg(F) - \sum_{P \in S} \ell(F, P) \tag{1}$$

We need this formula only when each point of S is an ordinary node of X. In this case we may decompose f into $\sharp(S)$ partial normalizations of a single node. Hence for nodes it is sufficient to prove it when $\sharp(S) = 1$, say $S = \{P\}$. In this case (1) is obviously true if F_P is free. If $F = \mathcal{I}_P$, then (1) holds, because $\ell(\mathcal{I}_P, P) = 1$ and G is the ideal sheaf of the two points $f^{-1}(P)$. For an arbitrary F_P use the next result.

Remark 2. Take the set-up of the first part of Remark 1. Assume that P is either an ordinary node or an ordinary cusp of X. Assume $P \in \mathrm{Sing}(F)$. By the classification of torsion free modules on simple curves singularities ([15], or, for nodes, [21], pp. 163–166) the germ of F at each P is formally equivalent to the maximal ideal $m_{X,P}$ of the local ring $\mathcal{O}_{X,P}$. Hence Remark 1 gives $\ell(F, P) = 1$.

Remarks 1 and 2 immediately give the following result.

Corollary 1. *Let X be a reduced projective curve and F a coherent sheaf on X with depth 1 and pure rank 1. Fix $S \subseteq Sing(F)$. Assume that each point of S is an ordinary node or an ordinary cusp of X. Let $h : D \to X$ (resp. $f : C \to X$) be the partial normalization of X in which we normalize only the points of S (resp. the points of S and the singular points of X at which F is locally free). Set $L := f^*(F)/Tors(f^*(F))$ and $R := h^*(F)/Tors(h^*(F))$. Then* $\deg(L) = deg(R) = \deg(F) - \sharp(S)$.

Lemma 1. *Let X, Y reduced, projective curves and $f : Y \to X$ a finite surjective morphism. Let A be a coherent sheaf on Y. Then* $\deg(f_*(A)) = \deg(A) + \chi(\mathcal{O}_X) - \chi(\mathcal{O}_Y)$.

Proof. Obviously, $h^0(X, f_*(A)) = h^0(Y, A)$. Since f is finite, $R^1 f_*(A) = 0$. Hence the Leray spectral sequence of f gives $h^1(X, f_*(A)) = h^1(Y, A)$. Thus $\chi(A) = \chi(f_*(A))$. Since $\chi(A) = \deg(A) + \chi(\mathcal{O}_Y)$ and $\chi(f_*(A)) = \deg(f_*(A)) + \chi(\mathcal{O}_X)$, we are done. □

Remark 3. For any X (even not connected) and any $L \in \mathrm{Pic}(X)$ we have

$$\sum_{T \in \mathcal{B}(X)} \deg(L|T) = \deg(L) \tag{2}$$

Notice that (2) is true for non-locally free L if we only assume that L is locally free at each point of X lying on at least two irreducible components.

Proof of Theorem 1. Fix any $L \in D_1(X)$. Since $\deg(L|C) \geq 0$ for all $C \in \mathcal{B}(X)$, there is $T_L \in \mathcal{B}(X)$ such that $\deg(L|T_L) = 1$, while the morphism $h_L : X \to \mathbb{P}^r$, $r := h^0(X, L) - 1$, contracts to points all other components. Let $Y_1, \dots, Y_s$ be the closures in X of the connected components of $X \backslash T_L$. Since $L|T_L$ is spanned, $T_L \cong \mathbb{P}^1$, and $h_L|T_L$ is bijective. This implies that $h_L(Y_i) = h_L(Y_i \cap T)$ for all i. The second part of the statement of Theorem 1 shows how to construct from any $T \in \mathfrak{S}(X)$ a morphism $h_{L_T} : X \to \mathbb{P}^1$ such that the spanned line bundle $h_{L_T}^*(\mathcal{O}_{\mathbb{P}^1}(1))$ has degree 1. Obviously, the last (resp. first) construction is the inverse of the first (resp. last) one. To check the last assertion take L constructed in a similar way by taking the degree d Veronese embedding $T \hookrightarrow \mathbb{P}^d$ of $T \cong \mathbb{P}^1$, instead of the isomorphism ϕ. □

Lemma 2. *Let T be an integral projective curve. There is no spanned rank 1 torsion-free sheaf F on T such that Sing$(F) \neq \emptyset$ and* $\deg(F) = 1$.

Proof. Assume the existence of such a sheaf F. Since $\text{Sing}(F) \neq \emptyset$, T is singular. Hence $p_a(T) \geq 1 = \deg(F)$. Since $F \neq \mathcal{O}_X$ and F is spanned, $h^0(X, F) \geq 2$. The contradiction comes from Clifford's inequality ([12], Theorem A at p. 532). □

Proof of Theorem 2. Assume the existence of $F \in S_1(X) \backslash D_1(X)$ and set $c := \sharp(\text{Sing}(F))$ and $b := \sharp(\text{Sing}(F) \cap \text{Sing}(X)'')$. By assumption $c > 0$. If $b = 0$, then Lemma 2 and the last sentence of Remark 3 gives a contradiction. Hence we may assume $b > 0$. Let $h : D \to X$ be the partial normalization of X in which we normalize only the points of $\text{Sing}(F) \cap \text{Sing}(X)''$. Set $R := h^*(F)/\text{Tors}(h^*(F))$. Corollary 1 gives $\deg(R) = 1 - b \leq 0$. R is a spanned sheaf with pure rank 1 and depth 1. Hence $b = 1$ and $R \cong \mathcal{O}_D$. Hence $c = b$ and F has a unique singular point. Since F has no torsion, the natural map $h^* : H^0(X, F) \to H^0(D, R)$ is injective. Hence $h^0(X, R) \geq 2$. Thus the only singular point, P, of $\text{Sing}(F)$ is a disconnecting node of X. Hence $D = X_P$ and $h = f_P$. Since X_P has two connected components, we get $h^0(X, F) = 2$. Hence to prove all the assertions of Theorem 2 it is sufficient to check that $A := f_{P*}(\mathcal{O}_{X_P}) \in S_1(X) \backslash D_1(X)$. Lemma 1 gives $\deg(A) = 1$. Obviously, A is not locally free at P. We have $h^0(X, A) = h^0(X_P, \mathcal{O}_{X_P}) = 2$. Let A' be the subsheaf of A spanned by $H^0(X, A)$. If $A' = A$, then we are done. Assume $A' \neq A$. Hence $\deg(A') \leq \deg(A) - 1 \leq 0$. Since X is connected, we get $h^0(X, A') \leq 1$. Since $h^0(X, A') = h^0(X, A) = 2$, we get a contradiction. □

Example 1. Fix integers $g > q > 0$. Let X be a genus g stable curve with 2 irreducible components X_1 and X_2 such that $p_a(X_1) = q$ and $p_a(X_2) = g - q$. Since $p_a(X) = 1$, we have $\sharp(X_1 \cap X_2) = 1$. The only point P of $X_1 \cap X_2$ is a disconnecting node of X. It is easy to check that the degree 1 spanned sheaf associated by Theorem 2 to this disconnecting node satisfies the Basic Inequality (5) (see section 3) if and only if $g = 2q$.

2 The Deformation Theory of Maps to $\mathbb{P}^1$

Fix a positive multidegree $\underline{d}$ for τ. Set $\delta(\underline{d}) : \sum_{T\in\mathcal{B}(X)} d_T$ (the *total degree* of $\underline{d}$). For any positive multidegree $\underline{d} = \{d_T\}$ let $G^1(\tau,\underline{d})$ denote the set of all pairs (X,f) such that $X \in \mathcal{M}[\tau]$ and $f : X \to \mathbb{P}^1$ is a morphism with multidegree $\underline{d}$, i.e. such that $\deg(f^*(\mathcal{O}_{\mathbb{P}^1}(1)) = d_T$ for every $T \in \mathcal{B}(X)$. In this section we prove the following result.

Theorem 3. *Fix a topological type τ for nodal and connected projective curves and a positive multidegree $\underline{d}$ for τ. Either $G^1(\tau,\underline{d}) = \emptyset$ or $G^1(\tau,\underline{d})$ is smooth and of pure dimension $2\delta(\underline{d})+2g-2-s$, where X is any element of $\mathcal{M}[\tau]$, $g := p_a(X)$, and $s := \sharp(Sing(X))$.*

Remark 4. Let X be a connected projective curve with only ordinary nodes as singularities. Let Θ_X denote the dual of the cotangent sheaf Ω^1_X. Since Θ_X is the dual of a generically rank 1 coherent sheaf, Θ_X has no torsion. It is easy to check that $\Theta_X = (\Omega^1_X/\mathrm{Tors}(\Omega^1_X))^*$. Fix $P \in \mathrm{Sing}(X)$. It is well-known that the connected component of $\mathrm{Tors}(\Omega^1_X)$ supported by P has length 1 and that $\Omega^1_X/\mathrm{Tors}(\Omega^1_X)$ is not locally free at P (see [10], formula (4.1), or [16], p. 33). Since X is Gorenstein, every depth 1 sheaf on X is reflexive. Thus $\Omega^1_X/\mathrm{Tors}(\Omega^1_X) \cong \Theta^*_X$. Hence $\mathrm{Sing}(\Theta_X) = \mathrm{Sing}(X)$. Here we present an alternative proof. We claim that the germ at P of the sheaf $\Omega^1_X/\mathrm{Tors}(\Omega^1_X)$ is isomorphic to a colength 1 module F of the trivial $\mathcal{O}_{X,P}$-module of $\omega_{X,P}$. It would be sufficient to prove the claim, because no such F is locally free by the classification of torsion free modules over an ordinary node ([21], huitième partie, propositions 2 and 3). The claim is just part (2) of [5], Lemma 6.1.2, in which the following notation is used: λ is the colength which we want compute, μ is the Milnor number of X at P ($\mu = 1$ for an ordinary node), δ is the genus of the singularity ($\delta = 1$ for an ordinary node) κ (the cuspidal number) is equal to the multiplicity minus the number of branches by part (1) of [5], Lemma 6.1.2 (hence $\kappa = 0$ for an ordinary node); the formula says that $\mu \ge \lambda \ge \delta+\kappa$, i.e. $1 \ge \lambda \ge 1+0$.

Lemma 3. *Let X be a connected projective curve with only ordinary nodes as singularities. Then* $\deg(\Theta_X) = 2\cdot\chi(\mathcal{O}_X) + \sharp(Sing(X))$.

Proof. Since X is Gorenstein, every torsion free coherent sheaf F on X is reflexive and $\deg(F^*) = -\deg(F)$. Since $\Theta_X \cong (\Omega^1_X/\mathrm{Tors}(\Omega^1_X))^*$, it is sufficient to prove that

$$\deg((\Omega^1_X/\mathrm{Tors}(\Omega^1_X)) = -2\cdot\chi(\mathcal{O}_X) - \sharp(\mathrm{Sing}(X)).$$

Since $\deg(\omega_X) = -2\cdot\chi(\mathcal{O}_X)$, it is sufficient to prove the existence of an inclusion $j : \Omega^1_X/\mathrm{Tors}(\Omega^1_X) \hookrightarrow \omega_X$ whose cokernel is a torsion sheaf with length $\sharp(\mathrm{Sing}(X))$. This assertion is the last part of Remark 4, i.e. [5], Lemma 6.1.2. $\square$

Let X be a nodal and connected projective curve, M a smooth and projective variety and $f : X \to M$ be a morphism whose restriction to each irreducible component of X is non-constant. The latter condition implies $T_{X/M} = 0$, where $T_{X/M}$ (also denoted with $T_{X/f/Y}$ or T_f) is the subsheaf of Θ_X defined in [21], p. 387. Our assumption on f is called "non-degenerate " in [21], Definition 3.4.5. Let N'_f denote the cokernel of the natural map $\Theta_X \to \Theta_M$; the same sheaf is denoted with N_f in [21], Definition 3.4.5. Since $T_{X/M} = 0$, we have an exact sequence of coherent sheaves on X ([21], p. 162).

$$0 \to \Theta_X \xrightarrow{\widetilde{f}} f^*(\Theta_M) \to N'_f \to 0 \qquad (3)$$

We are interested in the functor of locally trivial deformations of the map f with M fixed, mainly when $M = \mathbb{P}^1$. Since f is non-degenerate, the vector space $H^0(X, N'_f)$ is the tangent space to the functor of locally trivial deformations of the map f with M fixed, while the vector space $H^1(X, N'_f)$ is an obstruction space for the same functor ([21], Lemma 3.4.7 (iii) and Theorem 3.4.8). Since X is nodal, saying "locally trivial " means that we only look at nearby pairs (X', f') with X' a nodal curve with $\|X'\| = \|X\|$, i.e. in which no node is smoothed, i.e. with the topological type of X.

Proposition 1. *Let X be a nodal and connected projective curve. Let $f : X \to \mathbb{P}^1$ be a morphism such that $f|T$ is not constant for every irreducible component T of X. Set $g := p_a(X)$, $s := \sharp(Sing(X))$ and $d := \deg(f)$. The the functor of locally trivial deformations of f is smooth at f and of dimension $2d + 2g - 2 - s$.*

Proof. Both Θ_X and $f^*(\Theta_{\mathbb{P}^1})$ are sheaves with depth 1 and pure rank 1. Hence the injectivity of the map $\widetilde{f}$ in the exact sequence (3) gives that N'_f is supported by finitely many points of X. Hence $h^1(X, N'_f) = 0$ and $h^0(X, N'_f) = \deg(f^*(\Theta_{\mathbb{P}^1})) - \deg(\Theta_X) = 2d + 2g - 2 - s$ (Lemma 3). □

Proof of Theorem 3. The theorem is just a restatement of Proposition 1. □

Let X be a nodal and connected projective curve. Let $\text{gon}_1(X)$ denote the minimal integer d such that there is a degree d morphism $X \to \mathbb{P}^1$. Obviously, $\text{gon}_1(X) \ge \sum_{T \in \mathcal{B}(X)} \text{gon}_1(T)$. Thus $\text{gon}_1(X) \ge \sharp(\mathcal{B}(X))$ and the inequality is strict if at least one of the components of X is not isomorphic to $\mathbb{P}^1$. There are topological types τ such that $h^0(X, f^*(\mathcal{O}_{\mathbb{P}^1}(1)) \ge 3$ for any nodal curve X with topological type τ and f computing $\text{gon}_1(X)$. A stupid example is given by the topological type of a reducible conic. A more interesting example is given by the graph curves X of genus $g \ge 4$ ([4]) for which $\text{gon}_1(X) \ge \sharp(\mathcal{B}(X)) = 3g - 3$ and hence $h^0(X, f^*(\mathcal{O}_{\mathbb{P}^1}(1)) \ge g - 1$ (Riemann-Roch). Let τ be a topological type for nodal and connected projective curves. Let $\text{gon}_{1,-}(\tau)$ (resp. $\text{gon}_{1,+}(\tau)$) denote the minimal (resp. maximal) integer d such that there is $X \in \tau$ such that $\text{gon}_1(X) = d$.

Question 1. Compute $\text{gon}_{1,-}(\tau)$ and $\text{gon}_{1,+}(\tau)$ for every topological type τ.

3 Caporaso's Compactification: The Basic Inequality

Fix an integer $g \ge 2$. In [6] L. Caporaso constructed a compactification over $\overline{\mathcal{M}}_g$ of the set of all line bundles with fixed degree on smooth genus g curves. Recall that for every stable genus g curve X the canonical sheaf ω_X is an ample line bundle. R. Pandharipande proved that Caporaso's compactification is equivalent to the moduli scheme of equivalence classes of all slope-semistable (with respect to the polarization ω_X) coherent sheaves with depth 1, pure rank 1 and degree d ([19], Theorem 10.3.1; see [21] for the construction of this moduli space). This is a very nice result, because it shows that, at least if we take the canonical polarization, the Brill-Noether theory of elements of Caporaso's compactification or of semistable sheaves are the same. We will use the set-up of [6], because it has a very important feature: if X is reducible, then we may refine the degree, prescribing (at least for line bundles) the multidegree, i.e. the degree of the restriction to each irreducible components. Easy examples show that for certain multidegrees a Brill-Noether locus may be empty, while for other multidegrees with the same total degree the corresponding moduli space is non-empty ([6], Proposition 12). This is not an exceptional situation: it happens very frequently.

We recall the definition of semibalanced or balanced line bundle and of Basic Inequality ([17], Definition 1.1). A semistable curve X is called *quasistable* if any two exceptional irreducible components of X (i.e. smooth rational component intersecting the other components at two points) are disjoint. X is quasistable if and only if either X is stable or its stable reduction $u : X \to Y$ have the following property: there is $S \subseteq \mathrm{Sing}(Y)$ such that $u|u^{-1}(X\backslash S) : u^{-1}(X\backslash S) \to X\backslash S$ is an isomorphism and $u^{-1}(P) \cong \mathbb{P}^1$ for all $P \in S$. If X is quasistable, but not stable, then (u, Y, S) are uniquely determined by X, while X is uniquely determined by the pair (Y, S). Let X be a quasistable curve of genus $g \geq 2$. For any proper subcurve A of X set $k_A := \sharp(A \cap \overline{X\backslash A})$ and $w_A := \deg(\omega_X|A) = -2 \cdot \chi(\mathcal{O}_A) + k_A$. Fix $L \in \mathrm{Pic(X)}$. L is said to be *semibalanced* if the following inequality (called the Basic Inequality)

$$\deg(L) \cdot w_Z/(2g-2) - k_Z/2 \leq \deg(L|Z) \leq \deg(L) \cdot w_Z/(2g-2) + k_Z/2 \tag{4}$$

holds for every proper connected subcurve Z of X. L is said to be *balanced* if it is balanced and $\deg(L|E) = 1$ for every exceptional component E of X. Let X be a stable curve. Fix $S \subseteq \mathrm{Sing}(X)$. Let $u_S : X_{[S]} \to X$ be the partial normalization of X in which we normalize exactly the point of S. Thus $X_{[S]}$ is nodal $\sharp(\mathrm{Sing}(X_{[S]}) = \sharp(\mathrm{Sing}(X)) - \sharp(S)$ and $\chi(\mathcal{O}_{[S]}) = \sharp(S) + \chi(\mathcal{O}_X)$. $X_{[S]}$ may be disconnected. Let $v_S : X_S \to X$ denote the stable reduction of the quasistable model of the pair (X, S), i.e. X_S is quasistable with $\sharp(S)$ exceptional components, each of them mapped to a different point of S and $v_S|v_S^{-1}(X\backslash S) : v_S^{-1}(X\backslash S) \to X\backslash S$ is an isomorphism. X_S is connected and $p_a(X_S) = p_a(X)$. $X_{[S]}$ is a subcurve of X_S and $u_S = v_S|X_{[S]}$.

Remark 5. Fix a pure rank 1 torsion free sheaf F on X. Write u_F, v_F, $X_{[F]}$ and X_F instead of X_S, v_F, $X_{[S]}$ and X_S if $S := \mathrm{Sing}(F)$. Set $\widetilde{F} := u_F^*(F)/\mathrm{Tors}(u_F^*(F))$. Since $\widetilde{F}$ has no torsion and it is localy free at each singular point of $X_{[F]}$, $\widetilde{F}$ is a line bundle. Fix $P \in \mathrm{Sing}(F)$. The classification of all pure rank 1 torsion free sheaves on a nodal singularity gives that the germ of F at P is isomorphic to the maximal ideal of the local ring $\mathcal{O}_{X,P}$ ([21], huitième partie, Propositions 2 and 3). Hence $\deg(\widetilde{F}) = \deg(F) - \sharp(S)$. There is a unique line bundle $\overline{F}$ on X_S such that $\overline{F}|X_{[F]} = \widetilde{F}$ and $\deg(\overline{F}|E) = 1$ for every exceptional curve E of X_F. We have $\deg(\overline{F}) = \deg(\widetilde{F}) + \sharp(S) = \deg(F)$. Since F has no torsion, the pull-back map f^* induces an inclusion $f^* : H^0(X, F) \to H^0(X_{[S]}, \widetilde{F})$. Hence $h^0(X_{[S]}, \widetilde{F}) \geq h^0(X, F)$. Now assume that F is spanned by a linear subspace $V \subseteq H^0(X, F)$.

Since the tensor product is a right exact functor, $f^*(V)$ spans $\widetilde{F}$. Recall that X_F is obtained from $X_{[F]}$ adding $\sharp(S)$ disjoint exceptional curves. Since $\deg(\overline{F}|E) = 1$ for every exceptional curve E of X_F, $\sharp(S)$ Mayer-Vietoris exact sequences give $h^0(X_S, \overline{F}) = h^0(X_{[S]}, \widetilde{F})$ and that $\overline{F}$ is spanned if and only if $\widetilde{F}$ is spanned (see Lemmas 4 and 5)).

Lemma 4. *Fix a nodal curve X and $S \subseteq Sing(X)$. Fix $L \in Pic(X_{[S]})$ and let M be the only line bundle on X_S such that $M|X_{[S]} \cong L$ and $\deg(M|E)$ for every exceptional curve E of v_S. We have $h^0(X_{[S]}, L) = h^0(X_S, M)$. If L is spanned, then $h^0(X_S, M) \leq h^0(X_{[S]}, L) + \sharp(S)$. M is spanned if and only if L is spanned and for any exceptional curve E of v_S there is $f \in H^0(X_{[S]}, L)$ vanishing at one of the point of $E \cap X_{[S]}$, but not vanishing at the other point of $E \cap X_{[S]}$.*

Proof. By induction on S we reduce to the case $\sharp(S) = 1$. This inductive procedure is the reason for not requiring the stability of X (if $\sharp(S) \geq 2$), since this assumption would be lost in the inductive step. Set $\{P\} := S$. Let E be the only new exceptional curve of $X_{[S]}$ and P_1, P_2 the points of $E \cup X_{[S]}$. We have a Mayer-Vietoris exact sequence on X_S:

$$0 \to M \to M|X_{[S]} \oplus M|E \to M|\{P_1, P_2\} \to 0 \tag{5}$$

Since the restriction map $H^0(E, M|E) \to H^0(\{P_1, P_2\}, M|\{P_1, P_2\})$ is bijective, (5) gives that the restriction map $\rho : H^0(X_S, M) \to H^0(X_{[S]}, L)$ is bijective. Hence $h^0(X_{[S]}, L) = h^0(X_S, M)$. The bijectivity of ρ gives that L is spanned if and only if M is spanned at each point of $X_{[S]}$. Since $E \cong \mathbb{P}^1$ and $\deg(M|E) = 1$, M is spanned at every point of E if and only if the restriction map $\eta : H^0(X_S, M) \to H^0(E, M|E)$ is surjective. The cohomology exact sequence of (5) gives that η is surjective if and only if the restriction map $\beta : H^0(X_{[S]}, L) \to L|\{P_1, P_2\}$ is surjective. Since L is spanned, the surjectivity of β is equivalent to require the existence of $f \in H^0(X_{[S]}, L)$ vanishing at P_1, but not at P_2. ☐

Lemma 5. *Fix a nodal curve X and a sheaf F on X with depth 1 and pure rank 1. Assume that F is spanned. Then $\widetilde{F}$ and $\overline{F}$ are spanned and $h^0(X_F, \overline{F}) = h^0(X_{[F]}, \widetilde{F})$.*

Proof. We know that $\widetilde{F}$ is spanned. Apply Lemma 4 to $S := \mathrm{Sing}(F)$ and get equality $h^0(X_F, \overline{F}) = h^0(X_{[F]}, \widetilde{F})$. To check the spannedness of $\overline{F}$ we need to check the last condition of Lemma 4. Fix $P \in \mathrm{Sing}(F)$ and let E be the corresponding exceptional curve. Set $\{P_1, P_2\} := E \cap X_{[F]}$. Since the germ of F at P is isomorphic to the maximal ideal of $\mathcal{O}_{X,P}$ and $\mathcal{O}_{X,P}$ is an ordinary node, the fiber $F|\{P\}$ of F at P is a 2-dimensional vector space. Since F is spanned at P and the natural map $u_F^* : H^0(X, F) \to H^0(X_{[F]}, L)$ is injective, we get $h^0(X_{[F]}, \mathcal{I}_{\{P_1,P_2\}} \otimes L) \leq h^0(X_{[F]}, L)$. Hence the last part of Lemma 4 gives the spannedness of $\overline{F}$. ☐

SUMMARY OF THE SECTION: We hope to have convinced the reader that in many cases we may separately study the Basic Inequality and the geometric properties (spannedness, very ampleness and so on) of a potential element of a Brill-Noether theory.

4 Stable Curves with ω_X without Spanned Locally Free Subsheaves

Let X be a stable curve of genus g. The dualizing sheaf ω_X is spanned if and only if X has no disconnecting node, i.e. there is no $P \in \mathrm{Sing}(X)$ such that $X\backslash\{P\}$ is not connected ([8], Theorem D, or [1], part (a) of Theorem 1.2, or [9], part (b) of Theorem 3.3). Since X is nodal, $X\backslash\{P\}$ has two connected comnponents if P is a disconnecting node of X. Here we give two examples of genus g stable curves X such that there is no locally free spanned subsheaf L of ω_X with $h^0(X, L) \geq 2$ (i.e. $L \neq \mathcal{O}_X$). Since ω_X is locally free, in any such example ω_X is not spanned, i.e. X has a disconnecting node. The first example works for any genus $g \geq 2$. For the second example we need to assume g large, say $g \geq 9$. In the second example we have no injective map $\mathcal{O}_X \hookrightarrow \omega_X$.

Example 2. Let X be a chain of g curves of genus 1, i.e. assume $X = \cup_{i=1}^{g} T_i$, $p_a(T_i) = 1$ for all i, $T_i \cap T_j \neq \emptyset$ if and only if $|i - j| \leq 1$ and $\sharp(T_i \cap T_{i+1}) = 1$ for all $i \in \{1, \ldots, g-1\}$. Notice that X has $g - 1$ disconnecting nodes. The proofs of [8], Theorem D, or of of [1], Theorem 1.2, or an easy exercise left to the reader show that the subsheaf F of ω_X spanned by $H^0(X, \omega_X)$ has the property that $\omega_X/F = \oplus_{P\in \mathrm{Sing}(X)''} \mathbb{K}_P$, where $\mathbb{K}_P$ denote the skyscraper sheaf supported by P and such that $h^0(X, \mathbb{K}_P) = 1$. Hence $\deg(F) = 2g - 2 - \sharp(\mathrm{Sing}(X)'') = g - 1$ and F is not locally free at any point of $\mathrm{Sing}(X)''$. Every spanned subsheaf of ω_X is contained in F. If L is a locally free subsheaf of F, then the torsion sheaf F/L must have every point of $\mathrm{Sing}(X)''$ in its support. Thus $\deg(L) \leq \deg(F) - (g - 1) \leq 0$. If L is also spanned, we get $L \cong \mathcal{O}_X$.

Example 3. Let X be a genus g stable curve such that there is an irreducible component $T \cong \mathbb{P}^1$. Set $k := \sharp(X \cap \overline{X\backslash T})$. Notice that $\omega_X|T$ has degree $k - 2$. Assume that at least $k - 1$ of the points of $X \cap \overline{X\backslash T}$ are disconnecting node of X. Let F be the subsheaf of ω_X spanned by $H^0(X, \omega_X)$ and let L be any locally free subsheaf of F. Since ω_X is not spanned at any disconnecting node, the inclusion map $j : L \hookrightarrow \omega_X$ drops rank at each disconnecting node of X. Hence $\deg(L|T) \leq \deg(\omega_X|T) - (k-1) \leq -1$. Thus L is not spanned and any section of L vanishes identically on T. In particular $L \neq \mathcal{O}_X$.

Question 2. Is it possible to describe all genus g stable curves X such that there is no locally free spanned subsheaf L of ω_X with $h^0(X, L) \geq 2$ and/or the ones for which there is no injective map $\mathcal{O}_X \hookrightarrow \omega_X$? Is it true that any example with the latter property has a smooth rational component T such that every section of ω_X vanishes identically on T?

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