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Operator Homology and Cohomology in Clifford Algebras

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ABSTRACT

In recent work, the authors used canonical lowering and raising operators to define Appell systems on Clifford algebras of arbitrary signature. Appell systems can be interpreted as polynomial solutions of generalized heat equations, and in probability theory they have been used to obtain non-central limit theorems. The natural grade-decomposition of a Clifford algebra of arbitrary signature lends it a natural Appell system decomposition. In the current work, canonical raising and lowering operators defined on a Clifford algebra of arbitrary signature are used to define chains and cochains of vector spaces underlying the Clifford algebra, to compute the associated homology and cohomology groups, and to derive long exact sequences of underlying vector spaces. The vector spaces appearing in the chains and cochains correspond to the Appell system decomposition of the Clifford algebra. Using Mathematica, kernels of lowering operators ∇ and raising operators $\mathcal{R}$ are explicitly computed, giving solutions to equations $\nabla x = 0$ and $\mathcal{R} x = 0$. Connections with quantum probability and graphical interpretations of the lowering and raising operators are discussed.

RESUMEN

En recientes trabajos, los autores usaron operadores canónicos de bajada y de elevación para definir sistemas de Appell sobre algebras de Clifford de signo arbitrario. Los sistemas de Appell pueden ser interpretados como soluciones polinomiales de ecuaciones del calor generalizadas, y en teoría de probabilidades estos han sido usados para obtener teoremas de límite no central. La natural malla-descomposición para una algebra de Clifford de signo arbitrario presta una descomposición natural del sistema de Appel. En este trabajo, operadores canónicos de elevación y de bajada definidos sobre una algebra de Clifford de signo arbitrario son usados para definir cadenas y cocadenas de espacios vectoriales de llegada de algebras de Clifford; para calcular los grupos de homología y cohomología asociados; y para obtener el tamaño de las sucesiones exactas de los espacios vectoriales de llegada. Los espacios vectoriales que aparecen en las cadenas y cocadenas corresponden a la descomposición de sistemas de Appell de la algebra de Clifford. Usando MATHEMATICA, son calculados explícitamente los núcleos de operadores de bajada ∇ y de operadores de elevación $\mathcal{R}$ dando soluciones para las ecuaciones $\nabla x = 0$ y $\mathcal{R} x = 0$. Son discutidas conecciones con probabilidad cuantica y interpretaciones graficas para los operadores de bajada y de elevación.

Key words and phrases: *Operator calculus, Clifford algebras, Appell systems, quantum probability, homology, cohomology, Fock space, fermion*

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1 Introduction

In a recent work [12], the authors used canonical raising and lowering operators on Clifford algebras of arbitrary signature to define Appell systems. For any operator $\mathcal{A}$, set

$$\mathcal{Z}_n = \{\psi : \mathcal{A}^{n+1}\psi = 0\}$$

for $n \geq 0$. An $\mathcal{A}$-*Appell system* is a sequence of nonzero functions $\{\psi_0, \psi_1, \ldots, \psi_n, \ldots\}$ satisfying the following:

1. $\psi_n \in \mathcal{Z}_n$, $\forall n \geq 0$

2. $\mathcal{A}\psi_n = \psi_{n-1}$, for $n \geq 1$.

The system of embeddings $\mathcal{Z}_0 \subset \mathcal{Z}_1 \subset \mathcal{Z}_2 \subset \cdots$ is a canonical $\mathcal{A}$-*Appell system decomposition.*

These systems can be interpreted as polynomial solutions of generalized heat equations, and in probability theory they are also used to obtain non-central limit theorems. Analogues of Appell systems have previously been defined on Lie groups [6], the Schrödinger algebra [5], and quantum groups [4].

In the current work, raising and lowering operators are used to define chains and cochains on vector spaces underlying a Clifford algebra of arbitrary signature. The associated homology and cohomology groups are computed, and long exact sequences of vector spaces are derived.

Kernels and images of lowering and raising operators are computed explicitly with Mathematica, yielding solutions to equations $\nabla x = 0$ and $\mathcal{R} x = 0$ where ∇ and $\mathcal{R}$ are the canonical lowering and raising operators, respectively. Interpretations of lowering and raising operators as adjacency matrices of directed graphs are discussed and examples are constructed.

Connections with quantum probability are also discussed. In particular, necessary and sufficient conditions are obtained under which the matrix representation of the operator $i(\nabla + \mathcal{R})$ is a *quantum observable*.

The reader is referred to [9, Chapter 22] for details on the exterior algebra and contraction operators appearing herein.

Definition 1.1. *For fixed $n \geq 0$, let V be an n-dimensional vector space having orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$. The 2^n-dimensional Clifford algebra of signature (p, q), where $p + q = n$, is defined as the associative algebra generated by the collection $\{\mathbf{e}_i\}$ along with the scalar $\mathbf{e}_\emptyset = 1 \in \mathbb{R}$, subject to the following multiplication rules:*

$$\mathbf{e}_i \, \mathbf{e}_j + \mathbf{e}_j \, \mathbf{e}_i = 0 \text{ for } i \neq j, \text{ and} \tag{1.1}$$

$$\mathbf{e}_i{}^2 = \begin{cases} 1, & \text{if } 1 \leq i \leq p \\ -1, & \text{if } p+1 \leq i \leq p+q = n. \end{cases} \tag{1.2}$$

The Clifford algebra of signature (p, q) is denoted $\mathcal{C}\ell_{p,q}$.

Let $[n] = \{1, , 2, \ldots, n\}$ and denote arbitrary, canonically ordered subsets of $[n]$ by underlined Roman characters. The basis elements of $\mathcal{C}\ell_{p,q}$ can then be indexed by these finite subsets by writing

$$\mathbf{e}_{\underline{i}} = \prod_{k \in \underline{i}} \mathbf{e}_k. \tag{1.3}$$

Arbitrary elements of $\mathcal{C}\ell_{p,q}$ have the form

$$u = \sum_{\underline{i} \in 2^{[n]}} u_{\underline{i}} \, \mathbf{e}_{\underline{i}}, \tag{1.4}$$

where $u_{\underline{i}} \in \mathbb{R}$ for each $\underline{i} \in 2^{[n]}$.

The basis elements $\{\mathbf{e}_{\underline{i}}\}$ are called *blades*. The *grade* of a blade is defined as the cardinality of its multi-index; i.e., $\mathrm{gr}\left(\mathbf{e}_{\underline{i}}\right) = |\underline{i}|$. Given an element $u \in \mathcal{C}\ell_{p,q}$, $p + q = n$, the *grade-k part* of u is defined by

$$\langle u \rangle_k = \sum_{|\underline{i}| = k} u_{\underline{i}} \, \mathbf{e}_{\underline{i}}. \tag{1.5}$$

Clifford algebras thereby have a natural *grade decomposition*. For any $u \in \mathcal{C}\ell_{p,q}$,

$$u = \sum_{k=0}^{n} \langle u \rangle_k \, .$$

Two involutions on $\mathcal{C}\ell_{p,q}$ will also be useful. The *grade involution* is defined by linear extension of $\hat{\mathbf{e}_{\underline{i}}} := (-1)^{|\underline{i}|}\mathbf{e}_{\underline{i}}$. *Reversion* is defined by linear extension of $\tilde{\mathbf{e}_{\underline{i}}} = (-1)^{\frac{1}{2}(|\underline{i}|+1)}\mathbf{e}_{\underline{i}}$.

The exterior algebra of V is denoted $\bigwedge V$ and has a graded structure made explicit by writing $\bigwedge V = \bigoplus_{k=0}^{n} \bigwedge^{k} V$. There is a canonical vector space isomorphism $\mathcal{C}\ell_{p,q} \to \bigwedge V$.

Letting Q denote the following quadratic form on V:

$$Q(\mathbf{x}) = {x_1}^2 + \cdots + {x_p}^2 - {x_{p+1}}^2 - \cdots - {x_n}^2, \tag{1.6}$$

the algebra $\mathcal{C}\ell_{p,q}$ is also denoted by $\mathcal{C}\ell(Q)$. Associate with Q the symmetric bilinear form

$$\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{2}\left[Q(\mathbf{x}+\mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y})\right], \tag{1.7}$$

and extend to simple k-vectors in $\bigwedge^k V$ by

$$\langle \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \cdots \wedge \mathbf{x}_k, \mathbf{y}_1 \wedge \mathbf{y}_2 \wedge \cdots \wedge \mathbf{y}_k \rangle = \det \langle \mathbf{x}_i, \mathbf{y}_j \rangle . \tag{1.8}$$

This inner product extends linearly to all of $\bigwedge^k V$ and by orthogonality to $\bigwedge V$.

The inner product and exterior product extend to $\mathcal{C}\ell(Q)$ via the canonical vector space isomorphism. Left contraction is defined by (cf. [9, Chapter 14])

$$\mathbf{x} \lrcorner \mathbf{y} = \langle \mathbf{x}, \mathbf{y} \rangle \ \ \forall \mathbf{x}, \mathbf{y} \in V; \tag{1.9}$$

$$\mathbf{x} \lrcorner (u \wedge v) = (\mathbf{x} \lrcorner u) \wedge v + \hat{u} \wedge (\mathbf{x} \lrcorner v), \ \ \forall u, v \in \bigwedge V, \mathbf{x} \in V; \tag{1.10}$$

$$(u \wedge v) \lrcorner w = u \lrcorner (v \lrcorner w), \ \ \forall u, v, w \in \bigwedge V. \tag{1.11}$$

In particular, left and right contraction are dual to the exterior product and satisfy the following:

$$\langle u \lrcorner v, w \rangle = \langle v, \tilde{u} \wedge w \rangle , \tag{1.12}$$

$$\langle u \llcorner v, w \rangle = \langle u, w \wedge \tilde{v} \rangle . \tag{1.13}$$

The *Clifford product* of $\mathbf{x} \in V$ and $u \in \bigwedge V$ is defined in terms of exterior product and left contraction by $\mathbf{x}\, u = \mathbf{x} \wedge u + \mathbf{x} \lrcorner u$. This is extended by linearity and associativity to all of $\bigwedge V$, resulting in an associative algebra isomorphic to $\mathcal{C}\ell_{p,q}$.

Of paramount importance, the exterior product and left contraction satisfy the following in $\mathcal{C}\ell_{p,q}$:

$$\mathbf{e}_j \wedge \mathbf{e}_{\underline{i}} = \left\langle \mathbf{e}_j\, \mathbf{e}_{\underline{i}} \right\rangle_{|\underline{i}|+1} = \begin{cases} \pm \mathbf{e}_{\underline{i} \cup \{j\}} & \text{if } j \notin \underline{i}, \\ 0 & \text{otherwise;} \end{cases} \tag{1.14}$$

and

$$\mathbf{e}_j \lrcorner \mathbf{e}_{\underline{i}} = \left\langle \mathbf{e}_j\, \mathbf{e}_{\underline{i}} \right\rangle_{|\underline{i}|-1} = \begin{cases} \pm \mathbf{e}_{\underline{i} \setminus \{j\}} & \text{if } j \in \underline{i}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.15}$$

In addition to the inner product described above, the following Clifford inner product will also be convenient:

$$\langle u, v \rangle_2 := \langle \tilde{u} v \rangle_0 = \sum_{\underline{i} \in 2^{[n]}} u_{\underline{i}} v_{\underline{i}}. \tag{1.16}$$

Note that $\langle\cdot,\cdot\rangle_2$ is the Euclidean inner product on $\mathbb{R}^{2^n}$.

In this paper, the basis blades of $\mathcal{C}\ell_{p,q}$ are ordered according to

$$\mathbf{e}_{\underline{i}} \prec \mathbf{e}_{\underline{j}} \Leftrightarrow \sum_{i\in\underline{i}} 2^{i-1} < \sum_{j\in\underline{j}} 2^{j-1},\ \underline{i},\underline{j} \neq \emptyset,\ \underline{i} \neq \underline{j}, \tag{1.17}$$

$$\mathbf{e}_{\emptyset} \prec \mathbf{e}_{\underline{i}},\ \forall \underline{i} \neq \emptyset. \tag{1.18}$$

For example, under $\prec$ the following collection is canonically ordered:

$$\{\mathbf{e}_{\emptyset}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_{12}, \mathbf{e}_3, \mathbf{e}_{13}, \mathbf{e}_{23}, \mathbf{e}_{123}, \mathbf{e}_4, \mathbf{e}_{14}, \mathbf{e}_{24}, \mathbf{e}_{124}, \mathbf{e}_{34}, \mathbf{e}_{134}, \mathbf{e}_{234}, \mathbf{e}_{1234}\}.$$

Remark 1.1. *The ordering specified by $\prec$ is one of the four "admissible" monomial orders in the Grassmann algebra defined in [3] and appearing also in [1]. In those works, the ordering is referred to as InvLex.*

As a consequence of the graded structure, if $\mathcal{A}$ is any operator on $\mathcal{C}\ell_{p,q}$ mapping terms of grade not exceeding ℓ to terms of grade not exceeding $\ell-1$ for all $1 \leq \ell \leq n$, then the Clifford algebra has a natural $\mathcal{A}$-Appell system decomposition of the form $\mathcal{Z}_0 \subset \mathcal{Z}_1 \subset \cdots \subset \mathcal{Z}_n$ where $\psi_\ell \in \mathcal{Z}_\ell \Leftrightarrow \psi_\ell = \sum_{k=0}^{\ell} \langle\psi_\ell\rangle_k$.

This natural decomposition is used in conjunction with the *canonical lowering operator* to define Appell systems on Clifford algebras. The grade decomposition appears naturally in the chains and cochains defined later.

Specific examples of Clifford algebras include the following: $\mathcal{C}\ell_{0,2}$ is canonically isomorphic to the algebra of quaternions; $\mathcal{C}\ell_{3,0}$ is isomorphic to the algebra of physical space (APS) spanned by the Pauli spin matrices [2]; $\mathcal{C}\ell_{n,0}$ is canonically isomorphic to the n-particle fermionic Fock space; and $\mathcal{C}\ell_{n,n}$ is isomorphic to the n-particle algebra of fermion creation and annihilation operators (cf. [7]). The reader is referred to works such as [9], [11], and [10] for more background on Clifford algebras.

2 Operator calculus on Clifford algebras

Throughout the remainder of the paper, we define $n := p+q$ whenever a Clifford algebra of signature (p,q) is specified.

Definition 2.1. *Let $\mathcal{C}\ell_{p,q}$ be a Clifford algebra of arbitrary signature. For each $1 \leq j \leq n$, define the j^{th} raising operator R_j by linear extension of*

$$R_j\,\mathbf{e}_{\underline{i}} = \left\langle \mathbf{e}_j\,\mathbf{e}_{\underline{i}} \right\rangle_{|\underline{i}|+1} = \mathbf{e}_j \wedge \mathbf{e}_{\underline{i}}. \tag{2.1}$$

Define the j^{th} lowering operator D_j by linear extension of

$$D_j\,\mathbf{e}_{\underline{i}} = \left\langle \mathbf{e}_j\,\mathbf{e}_{\underline{i}} \right\rangle_{|\underline{i}|-1} = \mathbf{e}_j \lrcorner \mathbf{e}_{\underline{i}}. \tag{2.2}$$

Now the canonical lowering operator ∇ is defined by $\nabla = \bigoplus_{j=1}^{n} D_j$, and the canonical raising operator $\mathcal{R}$ is defined by $\mathcal{R} = \bigoplus_{j=1}^{n} R_j$.

It is apparent (cf. [12]) that the collection $\{R_i\}$, $1 \leq i \leq n$ generates an algebra isomorphic to the algebra of fermion creation operators, and the collection $\{D_i\}$, $1 \leq i \leq n$ generates an algebra isomorphic to the fermion annihilation operators.

Remark 2.1. *Note that the j^{th} raising and lowering operators are defined herein using left multiplication by $\mathbf{e}_j$. This is considered more natural than the corresponding operators previously defined by the authors in which right multiplication was used [12].*

Example 2.1. *In the Clifford algebra $\mathcal{C}\ell_{2,2}$ the raising and lowering operators act according to:*

$$D_1\, \mathbf{e}_{\{1,2,3\}} = \mathbf{e}_{\{2,3\}} \tag{2.3}$$

$$R_2\, \mathbf{e}_{\{1,3\}} = -\mathbf{e}_{\{1,2,3\}} \tag{2.4}$$

$$\nabla\, \mathbf{e}_{\{1,2,3\}} = \mathbf{e}_{\{2,3\}} - \mathbf{e}_{\{1,3\}} - \mathbf{e}_{\{1,2\}} \tag{2.5}$$

$$\mathcal{R}\, \mathbf{e}_{\{1,2\}} = \mathbf{e}_{\{1,2,3\}} + \mathbf{e}_{\{1,2,4\}}. \tag{2.6}$$

The following lemma details some essential properties of the raising and lowering operators. While some of these properties were established in the earlier work [12], they are reframed here in the context of exterior products and left contraction. Moreover, additional properties such as duality with respect to the inner products are new. The authors credit the anonymous referee for suggesting this lemma.

Lemma 2.1. *Fix nonnegative integers p, q and let $n = p + q$. In $\mathcal{C}\ell_{p,q}$ the operators $\{D_j\}$, $\{R_j\}$, ∇ and $\mathcal{R}$ satisfy the following:*

(a) $R_j \circ D_k + D_k \circ R_j = 0,\ \ j \neq k,\ 1 \leq j, k \leq n,$

(b) $R_j \circ D_j + D_j \circ R_j = {\mathbf{e}_j}^2,\ \ 1 \leq j \leq n,$

(c) $R_j \circ R_k = -R_k \circ R_j,\ \ j \neq k,\ 1 \leq j, k \leq n,$

(d) ${R_j}^2 := R_j \circ R_j = 0,\ \ 1 \leq j \leq n,$

(e) $D_j \circ D_k = -D_k \circ D_j,\ \ \ j \neq k,\ 1 \leq j, k \leq n,$

(f) ${D_j}^2 := D_j \circ D_j = 0,\ \ 1 \leq j \leq n,$

(g) $\nabla^2 := \nabla \circ \nabla = 0,$

(h) $\mathcal{R}^2 := \mathcal{R} \circ \mathcal{R} = 0.$

(i) The operators R_j and D_j are dual to each other with respect to the inner product $\langle \cdot, \cdot \rangle$; i.e., $\langle D_j\, u, w \rangle = \langle u, R_j\, w \rangle$ for any $u, w \in \mathcal{C}\ell_{p,q}$, $1 \leq j \leq n$.

(j) The operators $\mathcal{R}$ and ∇ are dual to each other with respect to the inner product $\langle \cdot, \cdot \rangle$; i.e., $\langle \nabla\, u, w \rangle = \langle u, \mathcal{R}\, w \rangle$ for all $u, w \in \mathcal{C}\ell_{p,q}$.

(k) The operators R_j and D_j are dual to each other with respect to the inner product $\langle \cdot, \cdot \rangle_2$; i.e., $\langle D_j\, u, w \rangle_2 = \langle u, R_j\, w \rangle$ for any $u, w \in \mathcal{C}\ell_{p,q}$, if and only if $j \leq p$.

(l) The operators ∇ and $\mathcal{R}$ are dual to each other with respect to $\langle \cdot, \cdot \rangle_2$ only for Euclidean signatures $(n, 0)$ and are not dual for any other signature. That is, $\langle \nabla\, u, w \rangle_2 = \langle u, \mathcal{R}\, w \rangle_2$ for all $u, w \in \mathcal{C}\ell_{n,0}$.

Proof. Let R_j, D_j, ∇, and $\mathcal{R}$ be as stated.

(a) First, note that $(R_j\circ D_k)(\mathbf{e}_{\underline{i}}) = R_j(\mathbf{e}_k \lrcorner \mathbf{e}_{\underline{i}}) = \mathbf{e}_j\wedge(\mathbf{e}_k\lrcorner\mathbf{e}_{\underline{i}})$. Moreover, $(D_k\circ R_j)(\mathbf{e}_{\underline{i}}) = D_k(\mathbf{e}_j\wedge\mathbf{e}_{\underline{i}}) = \mathbf{e}_k\lrcorner(\mathbf{e}_j\wedge\mathbf{e}_{\underline{i}}) = (\mathbf{e}_k\lrcorner\mathbf{e}_j)\wedge\mathbf{e}_{\underline{i}} - \mathbf{e}_j\wedge(\mathbf{e}_k\lrcorner\mathbf{e}_{\underline{i}}) = -\mathbf{e}_j\wedge(\mathbf{e}_k\lrcorner\mathbf{e}_{\underline{i}})$ since $j\neq k \Rightarrow \mathbf{e}_j\lrcorner\mathbf{e}_k = 0$.

(b) Observe that $(R_j\circ D_j)(\mathbf{e}_{\underline{i}}) = R_j(\mathbf{e}_j\lrcorner\mathbf{e}_{\underline{i}}) = \mathbf{e}_j\wedge(\mathbf{e}_j\lrcorner\mathbf{e}_{\underline{i}})$. Then $(D_j\circ R_j)(\mathbf{e}_{\underline{i}}) = D_j(\mathbf{e}_j\wedge\mathbf{e}_{\underline{i}}) = \mathbf{e}_j\lrcorner(\mathbf{e}_j\wedge\mathbf{e}_{\underline{i}}) = (\mathbf{e}_j\lrcorner\mathbf{e}_j)\wedge\mathbf{e}_{\underline{i}} - \mathbf{e}_j\wedge(\mathbf{e}_j\lrcorner\mathbf{e}_{\underline{i}}) = -\mathbf{e}_j\wedge(\mathbf{e}_j\lrcorner\mathbf{e}_{\underline{i}}) + {\mathbf{e}_j}^2\mathbf{e}_{\underline{i}} = -(R_j\circ D_j)(\mathbf{e}_{\underline{i}}) + {\mathbf{e}_j}^2\mathbf{e}_{\underline{i}}$.

(c) When $j\neq k$,

$$\begin{aligned}(R_j\circ R_k)(\mathbf{e}_{\underline{i}}) &= R_j(\mathbf{e}_k\wedge\mathbf{e}_{\underline{i}}) = \mathbf{e}_j\wedge(\mathbf{e}_k\wedge\mathbf{e}_{\underline{i}}) = (\mathbf{e}_j\wedge\mathbf{e}_k)\wedge\mathbf{e}_{\underline{i}}\\ &= -(\mathbf{e}_k\wedge\mathbf{e}_j)\wedge\mathbf{e}_{\underline{i}} = -\mathbf{e}_k\wedge(\mathbf{e}_j\wedge\mathbf{e}_{\underline{i}}) = -(R_k\circ R_j)(\mathbf{e}_{\underline{i}}).\end{aligned}$$

(d) $(R_j\circ R_j)(\mathbf{e}_{\underline{i}}) = \mathbf{e}_j\wedge(\mathbf{e}_j\wedge\mathbf{e}_{\underline{i}}) = (\mathbf{e}_j\wedge\mathbf{e}_j)\wedge\mathbf{e}_{\underline{i}} = 0$.

(e) When $j\neq k$,

$$\begin{aligned}(D_j\circ D_k)(\mathbf{e}_{\underline{i}}) &= D_j(\mathbf{e}_k\lrcorner\mathbf{e}_{\underline{i}}) = \mathbf{e}_j\lrcorner(\mathbf{e}_k\lrcorner\mathbf{e}_{\underline{i}}) = (\mathbf{e}_j\wedge\mathbf{e}_k)\lrcorner\mathbf{e}_{\underline{i}}\\ &= -(\mathbf{e}_k\wedge\mathbf{e}_j)\lrcorner\mathbf{e}_{\underline{i}} = -\mathbf{e}_k\lrcorner(\mathbf{e}_j\lrcorner\mathbf{e}_{\underline{i}}) = -(D_k\circ D_j)(\mathbf{e}_{\underline{i}}).\end{aligned}$$

(f) $(D_j\circ D_j)(\mathbf{e}_{\underline{i}}) = \mathbf{e}_j\lrcorner(\mathbf{e}_j\lrcorner\mathbf{e}_{\underline{i}}) = (\mathbf{e}_j\wedge\mathbf{e}_j)\lrcorner\mathbf{e}_{\underline{i}} = 0$.

(g) This follows immediately from $\nabla := \displaystyle\bigoplus_{j=1}^{n} D_j$ and properties (e) and (f) above.

(h) This follows immediately from $\mathcal{R} := \displaystyle\bigoplus_{j=1}^{n} R_j$ and properties (c) and (d) above.

(i) For $1\le j\le n$, (1.12) implies

$$\langle D_j\,u, w\rangle = \langle \mathbf{e}_j\lrcorner u, w\rangle = \langle u, \tilde{\mathbf{e}_j}\wedge w\rangle = \langle u, \mathbf{e}_j\wedge w\rangle = \langle u, R_j\,w\rangle. \tag{2.7}$$

(j) This follows from (i) and the definitions of ∇ and $\mathcal{R}$.

(k) Let $1\le j\le n$, and suppose $u, w\in \mathcal{C}\ell_{p,q}$. For multi-index $\underline{i}\in 2^{[n]}$ and integer $\ell\in[n]$, let $\varsigma(\underline{i},\ell) := |\{k\in\underline{i} : k<\ell\}|$. When $j\le p$, the action of D_j on u is

$$D_j\,u = \mathbf{e}_j\lrcorner\sum_{\underline{i}\in 2^{[n]}} u_{\underline{i}}\mathbf{e}_{\underline{i}} = \sum_{\underline{i}\in 2^{[n]}} u_{\underline{i}}\,\mathbf{e}_j\lrcorner\mathbf{e}_{\underline{i}} = \sum_{\{\underline{k}:j\in\underline{k}\}} (-1)^{\varsigma(\underline{k},j)} u_{\underline{k}}\mathbf{e}_{\underline{k}\setminus\{j\}}. \tag{2.8}$$

Hence, the inner product $\langle D_j\,u, w\rangle_2$ has the following expansion:

$$\langle D_j\,u, w\rangle_2 = \sum_{\{\underline{k}:j\in\underline{k}\}} (-1)^{\varsigma(\underline{k},j)} u_{\underline{k}} w_{\underline{k}\setminus\{j\}}. \tag{2.9}$$

Similarly, with the observation that $\varsigma((\underline{k}\setminus\{j\}),j) = \varsigma(\underline{k},j)$,

$$\begin{aligned}R_j\,w = \mathbf{e}_j\wedge\sum_{\underline{i}\in 2^{[n]}} w_{\underline{i}}\mathbf{e}_{\underline{i}} &= \sum_{\underline{i}\in 2^{[n]}} u_{\underline{i}}\,\mathbf{e}_j\wedge\mathbf{e}_{\underline{i}} = \sum_{\{\underline{k}:j\notin\underline{k}\}} (-1)^{\varsigma(\underline{k},j)} w_{\underline{k}}\mathbf{e}_{\underline{k}\cup\{j\}}\\ &= \sum_{\{\underline{k}:j\in\underline{k}\}} (-1)^{\varsigma((\underline{k}\setminus\{j\}),j)} w_{\underline{k}\setminus\{j\}}\mathbf{e}_{\underline{k}} = \sum_{\{\underline{k}:j\in\underline{k}\}} (-1)^{\varsigma(\underline{k},j)} w_{\underline{k}\setminus\{j\}}\mathbf{e}_{\underline{k}}.\end{aligned} \tag{2.10}$$

Whence,

$$\langle u, R_j\, w\rangle_2 = \sum_{\{\underline{k}: j\in\underline{k}\}} (-1)^{\varsigma(\underline{k},j)} u_{\underline{k}} w_{\underline{k}\setminus\{j\}} = \langle D_j\, u, w\rangle_2 \,. \tag{2.11}$$

Note that when $q > 0$ in the signature (p,q), $p < j \le n$ implies

$$\langle D_j\, u, w\rangle_2 = \sum_{\{\underline{k}: j\in\underline{k}\}} (-1)^{\varsigma(\underline{k},j)+1} u_{\underline{k}} w_{\underline{k}\setminus\{j\}}, \tag{2.12}$$

while

$$\langle u, R_j\, w\rangle_2 = \sum_{\{\underline{k}: j\in\underline{k}\}} (-1)^{\varsigma(\underline{k},j)} u_{\underline{k}} w_{\underline{k}\setminus\{j\}}. \tag{2.13}$$

Therefore, D_j and R_j are dual to each other with respect to $\langle\cdot,\cdot\rangle_2$ in signature (p,q) if and only if $j \le p$.

(l) Follows from (k) and the definitions of ∇ and $\mathcal{R}$.

□

By parts (g) and (h) of Lemma 2.1, ∇ and $\mathcal{R}$ are nilpotent linear operators of index 2. Several consequences follow from the standard theory of nilpotent linear operators [8, Ch. 7]. Each has minimal polynomial $m(t) = t^2$ and characteristic polynomial $\phi(t) = t^{2^n}$. For each there exists an ordered basis of $\mathcal{C}\ell_{p,q} \simeq \bigoplus_{k=0}^{n} \bigwedge^{k} V$ such that the operator's matrix representation with respect to this basis is triangular. This is seen explicitly in Sections 2.2 and 2.4.

Grade decompositions of $\mathcal{C}\ell_{p,q} \simeq \bigoplus_{k=0}^{n} \bigwedge^{k} V$ are naturally induced by the operators ∇ and $\mathcal{R}$. The decompositions expressed here are signature-independent, and they provide a theoretical context for results involving exact sequences in Sections 2.2 and 2.4.

Define $W_0 = \mathcal{R}$, and for each $k = 1,\ldots,n$, define $W_k := \bigwedge^k V$, which implies $\nabla(W_k) \subset W_{k-1}$. Observe that $u \in W_k$ and $v \in \nabla(W_k)$ implies $\langle u, v\rangle = 0$; i.e., $u \perp v$. For $k = 1,2,\ldots,n-1$, denote by $\nabla(W_{k+1})^\perp$ the subspace of W_k complementary to $\nabla(W_{k+1})$; i.e., $W_k = \nabla(W_{k+1}) \oplus \nabla(W_{k+1})^\perp$.

Defining $V_n := W_n$ and $V_k := \nabla(W_{k+1})^\perp \oplus \nabla(W_k)$ for $1 \le k \le n-1$, one finds $\nabla(V_k) \subset V_k$; i.e., V_k is ∇-invariant. Now ∇ has the decomposition $\nabla = \nabla_1 \oplus \nabla_2 \oplus \cdots \oplus \nabla_n$, where ∇_k is the restriction of ∇ to the ∇-invariant subspace V_k. Note that $V_i \perp V_j$ for $1 \le i \ne j \le n$, and V has the following decomposition into ∇-invariant subspaces:

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_n. \tag{2.14}$$

Similarly, for each $k = 1,\ldots,n-1$, $\mathcal{R}(W_k) \subset W_{k+1}$. Writing $W_k = \mathcal{R}(W_{k-1}) \oplus \mathcal{R}(W_{k-1})^\perp$, define $U_1 := W_1$ and $U_k := \mathcal{R}(W_{k-1})^\perp \oplus \mathcal{R}(W_k)$ for $2 \le k \le n$. It follows that $\mathcal{R}(U_k) \subset U_k$; i.e., U_k is $\mathcal{R}$-invariant and $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \cdots \oplus \mathcal{R}_n$, where $\mathcal{R}_k$ is the restriction of $\mathcal{R}$ to the $\mathcal{R}$-invariant subspace U_k. Finally, note that $U_i \perp U_j$ for $1 \le i \ne j \le n$, and V has the following decomposition into $\mathcal{R}$-invariant subspaces:

$$V = U_1 \oplus U_2 \oplus \cdots \oplus U_n. \tag{2.15}$$

Note that for $1 \le i \le n-1$, $V_i \subset W_i \cap W_{i-1}$. Similarly, for $2 \le i \le n$, $U_i \subset W_i \cap W_{i+1}$.

Remark 2.2. *For each $i = 1, 2, \ldots, n$, the lowering operator D_i is nilpotent of index 2 by part f of Lemma 2.1. It follows that each lowering operator induces a natural grade decomposition of $\mathcal{C}\ell_{p,q}$ as described above for the operator ∇. Similarly, each raising operator R_i induces a natural grade decomposition of $\mathcal{C}\ell_{p,q}$ as described for the operator $\mathcal{R}$.*

2.1 $\mathcal{C}\ell_{p,q}$ and the lowering operators $\{D_i\}_{1\le i\le n}$

For $1 \le i \le n$, the lowering operator $D_i : \mathcal{C}\ell_{p,q} \to \mathcal{C}\ell_{p,q}$ satisfies $D_i \circ D_i = 0$ and thus leads to the chain complex

$$\cdots \xrightarrow{D_i} \mathcal{C}\ell_{p,q} \xrightarrow{D_i} \mathcal{C}\ell_{p,q} \xrightarrow{D_i} \cdots . \tag{2.16}$$

The cycles associated with D_i are the same at each stage and are defined by

$$\begin{aligned} Z_i = \{u \in \mathcal{C}\ell_{p,q} : D_i\, u = 0\} &= \{u \in \mathcal{C}\ell_{p,q} : \left\langle u, \mathbf{e}_{\underline{j}} \right\rangle = 0 \text{ whenever } i \in \underline{j}\} \\ &= \{u \in \mathcal{C}\ell_{p,q} : \mathbf{e}_i \lrcorner u = 0\}. \end{aligned} \tag{2.17}$$

Similarly, the boundaries associated with D_i are the same at each stage and are defined by

$$\begin{aligned} B_i &= \{u \in \mathcal{C}\ell_{p,q} : u = D_i\, w, \text{ for some } w \in \mathcal{C}\ell_{p,q}\} \\ &= \{u \in \mathcal{C}\ell_{p,q} : \left\langle u, \mathbf{e}_{\underline{j}} \right\rangle = 0 \text{ whenever } i \in \underline{j}\} \\ &= \{u \in \mathcal{C}\ell_{p,q} : \mathbf{e}_i \lrcorner u = 0\}. \end{aligned} \tag{2.18}$$

In other words, the following condition is satisfied at each stage of the chain complex:

$$\mathrm{Ker} D_i = \mathrm{Im} D_i, \tag{2.19}$$

leading to the trivial homology group $\mathrm{Ker}D_i \big/ \mathrm{Im}D_i \cong \langle e \rangle$ at each stage.

It follows that the image of D_i is a subalgebra of dimension 2^{n-1} generated by the collection $\{\mathbf{e}_j\}_{j\neq i}$. In particular,

$$\mathcal{C}\ell_{p,q} \big/ \mathrm{Ker} D_i \cong \begin{cases} \mathcal{C}\ell_{p-1,q} & \text{if } 1 \le i \le p \\ \mathcal{C}\ell_{p,q-1} & \text{if } p+1 \le i \le p+q. \end{cases} \tag{2.20}$$

The collection $\{D_i\}_{1\le i\le n}$ then induces the following sequence of epimorphisms:

$$\mathcal{C}\ell_{p,q} \xrightarrow{D_1} \mathcal{C}\ell_{p-1,q} \xrightarrow{D_2} \cdots \xrightarrow{D_p} \mathcal{C}\ell_{0,q} \xrightarrow{D_{p+1}} \cdots \xrightarrow{D_n} \mathcal{C}\ell_{0,0}. \tag{2.21}$$

2.2 Homology and the canonical lowering operator ∇

Let p, q be fixed nonnegative integers, and let ∇ be the canonical lowering operator defined on the Clifford algebra $\mathcal{C}\ell_{p,q}$ by

$$\nabla = \bigoplus_{i=1}^{n} D_i. \tag{2.22}$$

Let $\{\mathbf{e}_{\underline{i}}\}_{\underline{i}\in 2^{[n]}}$ denote the collection of blades spanning the algebra, and recall that the basis blades of $\mathcal{C}\ell_{p,q}$ are canonically ordered by $\prec$, as defined by (1.17) and (1.18).

Given this canonical ordering, the matrix representation of $\nabla : \mathcal{C}\ell_{p,q} \to \mathcal{C}\ell_{p,q}$ is defined as the $2^n \times 2^n$ matrix

$$\mathcal{L}^{(p,q)}_{\underline{i},\underline{j}} = \begin{cases} 1 & \text{if } |\underline{j}| = |\underline{i}| - 1 \text{ and } \mathbf{e}_\ell\,\mathbf{e}_{\underline{i}} = \mathbf{e}_{\underline{j}} \text{ for some } 1 \leq \ell \leq n, \\ -1 & \text{if } |\underline{j}| = |\underline{i}| - 1 \text{ and } \mathbf{e}_\ell\,\mathbf{e}_{\underline{i}} = -\mathbf{e}_{\underline{j}} \text{ for some } 1 \leq \ell \leq n, \\ 0 & \text{otherwise.} \end{cases} \tag{2.23}$$

Here the rows and columns of $\mathcal{L}^{(p,q)}$ have been labeled in one-to-one fashion by multi-indices $\underline{i}, \underline{j} \in 2^{[n]}$.

Using the canonical vector space isomorphism $\mathcal{C}\ell_{p,q} \cong \mathbb{R}^{2^n}$ via

$$u = \sum_{\underline{i}\in 2^{[n]}} u_{\underline{i}}\,\mathbf{e}_{\underline{i}} \simeq \vec{u} = \left(u_\emptyset, \ldots, u_{\{1,2,\ldots,n\}}\right), \tag{2.24}$$

where the basis blades of $\mathcal{C}\ell_{p,q}$ are ordered by $\prec$, one has

$$\nabla\, u \simeq \vec{u}\mathcal{L}^{(p,q)}. \tag{2.25}$$

Define the $2^n \times 2^n$ diagonal matrix Ξ with rows and columns ordered by $\prec$ as follows

$$\Xi_{\underline{i},\underline{i}} := (-1)^{|\underline{i}|}. \tag{2.26}$$

Note that Ξ acts as grade involution; i.e., $\hat{u} \simeq \vec{u}\,\Xi$.

Lemma 2.2. *Let p, q be nonnegative integers. Let $\mathbf{0}$ denote the zero matrix, and let Ξ denote the $2^n \times 2^n$ matrix as defined in (2.26). Then, with respect to the canonically ordered basis of $\mathcal{C}\ell_{p,q}$, the matrix representations of ∇ satisfy the following recurrence relations:*

$$\mathcal{L}^{(0,0)} = \mathbf{0}; \tag{2.27}$$

$$\mathcal{L}^{(p,0)} = \begin{pmatrix} \mathcal{L}^{(p-1,0)} & \mathbf{0} \\ \Xi & \mathcal{L}^{(p-1,0)} \end{pmatrix}, \ p > 0; \tag{2.28}$$

$$\mathcal{L}^{(p,q)} = \begin{pmatrix} \mathcal{L}^{(p,q-1)} & \mathbf{0} \\ -\Xi & \mathcal{L}^{(p,q-1)} \end{pmatrix}, \ q > 0. \tag{2.29}$$

Proof. In the case $q = 0$, it is clear from the definition of $\mathcal{L}^{(1,0)}$ that

$$\mathcal{L}^{(1,0)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \mathcal{L}^{(0,0)} & \mathbf{0} \\ \Xi & \mathcal{L}^{(0,0)} \end{pmatrix}. \tag{2.30}$$

Proceed by induction on p. By canonical ordering of basis multi-vectors, rows and columns 1 through 2^p of $\mathcal{L}^{(p+1,0)}$ represent the action of $\nabla : \mathcal{C}\ell_{p,0} \to \mathcal{C}\ell_{p,0}$ since multi-vectors containing the generator $\mathbf{e}_{p+1}$ are all found at the end of the ordering.

There are 2^p basis multi-vectors containing $\mathbf{e}_{p+1}$, so that the matrix is partitioned into four blocks of equal size. Thus, writing the block matrix

$$\mathcal{L}^{(p+1,0)} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{2.31}$$

one finds $A = \mathcal{L}^{(p,0)}$. Moreover, because (i) $\mathcal{L}^{(p+1,0)}$ represents a lowering operator, (ii) $\mathbf{e}_{p+1}$ is not found in any of the first 2^p multi-vectors in the basis, and (iii) columns 2^p+1 through 2^{p+1} correspond to images containing $\mathbf{e}_{p+1}$, it follows that $B = \mathbf{0}$.

Rows 2^p+1 through 2^{p+1} correspond to pre-images containing $\mathbf{e}_{p+1}$. Columns 1 through 2^p correspond to lowering these to elements of $\mathcal{C}\ell_{p,0}$. This is accomplished by squaring $\mathbf{e}_{p+1}$, which lies at the end of each multi-vector in canonical order. Hence, $|\underline{i}|-1$ transpositions are required before squaring $\mathbf{e}_{p+1}$, and ${\mathbf{e}_{p+1}}^2 = 1$. Hence, writing $\underline{i}^- = \underline{i} \setminus \{p+1\}$ gives ${\mathcal{L}^{(p+1,0)}}_{\underline{i},\underline{i}^-} = (-1)^{|\underline{i}|-1} = \Xi_{\underline{i}^-,\underline{i}^-}$ for $\underline{i} \ni p+1$, and $C = \Xi$.

Entries in D correspond to the action of the lowering operator on multi-vectors containing $\mathbf{e}_{p+1}$ that leave $\mathbf{e}_{p+1}$ in place. In accordance, the action is identical to that of $\mathcal{L}^{(p,0)}$.

Turning now to the case $q > 0$, $p \geq 0$ is fixed and proof is by induction on q. The basis step follows from the first part of the proof. In particular, write

$$\mathcal{L}^{(p,1)} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}. \tag{2.32}$$

As before, rows and columns 1 through 2^p of $\mathcal{L}^{(p,1)}$ represent the action of $\nabla : \mathcal{C}\ell_{p,0} \to \mathcal{C}\ell_{p,0}$ since multi-vectors containing the generator $\mathbf{e}_{p+1}$ are all found at the end of the ordering. Hence, one finds $A' = \mathcal{L}^{(p,0)}$.

Rows 2^p+1 through 2^{p+1} correspond to pre-images containing $\mathbf{e}_{p+1}$. Columns 1 through 2^p correspond to lowering these to elements of $\mathcal{C}\ell_{p,0}$. This is accomplished by squaring $\mathbf{e}_{p+1}$, which lies at the end of each multi-vector in canonical order. Hence, $|\underline{i}|-1$ transpositions are required before squaring $\mathbf{e}_{p+1}$, and ${\mathbf{e}_{p+1}}^2 = -1$. Again writing $\underline{i}^- = \underline{i} \setminus \{p+1\}$ gives ${\mathcal{L}^{(p,1)}}_{\underline{i},\underline{i}^-} = (-1)^{|\underline{i}|} = -\Xi_{\underline{i}^-,\underline{i}^-}$ for $\underline{i} \ni p+1$, so $C' = -\Xi$.

By the reasoning applied previously, $B' = \mathbf{0}$ and $D' = \mathcal{L}^{(p,0)}$. Hence,

$$\mathcal{L}^{(p,1)} = \begin{pmatrix} \mathcal{L}^{(p,0)} & \mathbf{0} \\ -\Xi & \mathcal{L}^{(p,0)} \end{pmatrix},$$

and the basis step is complete.

Now assuming

$$\mathcal{L}^{(p,q)} = \begin{pmatrix} \mathcal{L}^{(p,q-1)} & \mathbf{0} \\ -\Xi & \mathcal{L}^{(p,q-1)} \end{pmatrix}$$

for some $q > 0$, consider

$$\mathcal{L}^{(p,q+1)} = \begin{pmatrix} A'' & B'' \\ C'' & D'' \end{pmatrix}. \tag{2.33}$$

By the reasoning applied previously, $B'' = \mathbf{0}$. Rows and columns 1 through 2^p of $\mathcal{L}^{(p,q)}$ represent the action of $\nabla : \mathcal{C}\ell_{p,q-1} \to \mathcal{C}\ell_{p,q-1}$, so $A'' = \mathcal{L}^{(p,q-1)}$. Similarly, $D'' = \mathcal{L}^{(p,q-1)}$.

Rows 2^p+1 through 2^{p+1} correspond to pre-images containing $\mathbf{e}_{p+1}$. Columns 1 through 2^p correspond to lowering these to elements of $\mathcal{C}\ell_{p,q-1}$. This is accomplished by squaring $\mathbf{e}_{p+q}$, which lies at the end of each multi-vector in canonical order. Hence, ${\mathcal{L}^{(p,q)}}_{\underline{i},\underline{i}} = (-1)^{|\underline{i}|} = -\Xi_{\underline{i}^-,\underline{i}^-}$ for $\underline{i} \ni p+1$, so $C'' = \Xi$.

□

Lemma 2.3. *Let signature (p,q) be fixed. The operator Ξ satisfies the following:*

$$\Xi^2 = I, \text{ and} \tag{2.34}$$
$$\Xi\mathcal{L}^{(p,q)} = -\mathcal{L}^{(p,q)}\Xi. \tag{2.35}$$

Proof. As mentioned earlier, Ξ is the matrix representation of grade involution in the canonically ordered basis. It follows immediately that Ξ is self-inverse. Each basis blade in the canonical expansion of ∇u is of grade one less than the corresponding blade in the pre-image. Hence, $\nabla\,\hat{u} = -\widehat{\nabla\, u}$. Applying the vector space isomorphism $u \simeq \vec{u}$ and the representation $\mathcal{L}^{(p,q)} \simeq \nabla$ completes the proof. □

Lemma 2.4.

$$\mathrm{Im}\nabla = \mathrm{Ker}\nabla. \tag{2.36}$$

Proof. Because $\nabla^2 = 0$, it is clear that $\mathrm{Im}\nabla \subseteq \mathrm{Ker}\nabla$. The lemma is proved by showing the reverse inclusion.

Assuming $q = 0$ and $p > 0$, Lemma 2.2 says

$$\mathcal{L}^{(p,0)} = \begin{pmatrix} \mathcal{L}^{(p-1,0)} & \mathbf{0} \\ \Xi & \mathcal{L}^{(p-1,0)} \end{pmatrix}. \tag{2.37}$$

Assuming $\vec{x} = (\vec{x_1}, \vec{x_2}) \in \mathrm{Ker}\mathcal{L}^{(p,0)}$ gives

$$\vec{x}\mathcal{L}^{(p,0)} = (\vec{x_1}\mathcal{L}^{(p-1,0)} + \vec{x_2}\Xi, \vec{x_2}\mathcal{L}^{(p-1,0)}) = (\vec{0}, \vec{0}). \tag{2.38}$$

This implies $\vec{x_2}\Xi = -\vec{x_1}\mathcal{L}^{(p-1,0)}$ and $\vec{x_2} \in \mathrm{Ker}\mathcal{L}^{(p-1,0)}$. Thus,

$$(\vec{x_1}, \vec{x_2}) = (\vec{x_1}, -\vec{x_1}\Psi),$$

where $\Psi := \mathcal{L}^{(p-1,0)}\Xi$.

To see that $(\vec{x_1}, -\vec{x_1}\Psi) \in \mathrm{Im}\mathcal{L}^{(p,0)}$, solve the equation

$$(\vec{s}, \vec{t})\mathcal{L}^{(p,0)} = (\vec{x_1}, -\vec{x_1}\Psi) \tag{2.39}$$

for $\vec{s}$ and $\vec{t}$. One solution is to let $\vec{t} = \vec{x_1}\Xi$ and choose any $\vec{s} \in \mathrm{Ker}\mathcal{L}^{(p-1,0)}$. Then, using Lemma 2.3,

$$\begin{aligned}(\vec{s}, \vec{t})\mathcal{L}^{(p,0)} &= (\vec{s}, \vec{x_1}\Xi)\mathcal{L}^{(p,0)} = (\vec{s}\mathcal{L}^{(p-1,0)} + \vec{x_1}, \vec{x_1}\Xi\mathcal{L}^{(p-1,0)}) \\ &= (\vec{x_1}, \vec{x_1}\Xi\mathcal{L}^{(p-1,0)}) = (\vec{x_1}, -\vec{x_1}\mathcal{L}^{(p-1,0)}\Xi) = (\vec{x_1}, -\vec{x_1}\Psi).\end{aligned} \tag{2.40}$$

$$\nabla_{0,2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}, \quad \nabla_{1,1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad \nabla_{2,0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}$$

Figure 1: The canonical lowering operator in the case $n = 2$.

In the case $q > 0$,

$$\mathcal{L}^{(p,q)} = \begin{pmatrix} \mathcal{L}^{(p,q-1)} & \mathbf{0} \\ -\Xi & \mathcal{L}^{(p,q-1)} \end{pmatrix}. \tag{2.41}$$

Whence, $\vec{x} = (\vec{x_1}, \vec{x_2}) \in \mathrm{Ker}\mathcal{L}^{(p,q)}$ implies

$$\vec{x}\mathcal{L}^{(p,q)} = (\vec{x_1}\mathcal{L}^{(p,q-1)} - \vec{x_2}\Xi, \vec{x_2}\mathcal{L}^{(p,q-1)}) = (\vec{0}, \vec{0}), \tag{2.42}$$

so that $\vec{x_2}\Xi = \vec{x_1}\mathcal{L}^{(p,q-1)}$ and $\vec{x_2} \in \mathrm{Ker}\mathcal{L}^{(p,q-1)}$.

Thus, when $q = 0$, $(x_1, x_2) \in \mathrm{Ker}\mathcal{L}^{(p,q)}$ is of the form

$$(\vec{x_1}, \vec{x_2}) = (\vec{x_1}, \vec{x_1}\Psi),$$

where $\Psi := \mathcal{L}^{(p,q-1)}\Xi$.

Solving the equation

$$(\vec{s}, \vec{t})\mathcal{L}^{(p,q)} = (\vec{x_1}, \vec{x_1}\Psi) \tag{2.43}$$

for $\vec{s}$ and $\vec{t}$ will show that $(\vec{x_1}, \vec{x_2}) \in \mathrm{Im}\mathcal{L}^{(p,q)}$. Letting $\vec{t} = -\vec{x_1}\Xi$ and $\vec{s} \in \mathrm{Ker}\mathcal{L}^{(p,q-1)}$, one finds

$$\begin{aligned}(\vec{s}, \vec{t})\mathcal{L}^{(p,q)} &= (\vec{s}, -\vec{x_1}\Xi)\mathcal{L}^{(p,q)} = (\vec{s}\mathcal{L}^{(p,q-1)} + \vec{x_1}, -\vec{x_1}\Xi\mathcal{L}^{(p,q-1)}) \\ &= (\vec{x_1}, -\vec{x_1}\Xi\mathcal{L}^{(p,q-1)}) = (\vec{x_1}, \vec{x_1}\mathcal{L}^{(p,q-1)}\Xi) = (\vec{x_1}, \vec{x_1}\Psi). \end{aligned} \tag{2.44}$$

Hence, $\mathrm{Im}\mathcal{L}^{(p,q)} = \mathrm{Ker}\mathcal{L}^{(p,q)}$ for any signature (p, q). Therefore $\mathrm{Im}\nabla = \mathrm{Ker}\nabla$ for the canonical lowering operator on a Clifford algebra of arbitrary signature. □

Example 2.2. *Figure 1 depicts the structure of the canonical lowering operators over the four dimensional Clifford algebras.*

Turning now to the chain complex

$$\cdots \xrightarrow{\nabla} \mathcal{C}\ell_{p,q} \xrightarrow{\nabla} \mathcal{C}\ell_{p,q} \xrightarrow{\nabla} \cdots, \tag{2.45}$$

the homology group $\mathrm{Ker}\nabla \diagup \mathrm{Im}\nabla$ is trivial at each stage.

Let $\mathcal{C}\ell_{p,q}^{\lfloor k \rfloor}$ denote the linear span of the collection $\{\mathbf{e}_{\underline{i}}\}_{|\underline{i}| \leq k}$. This is the projection of $\mathcal{C}\ell_{p,q}$ onto a linear subspace of dimension $2^n - \displaystyle\sum_{\ell=0}^{n-k} \binom{n}{\ell}$.

It should be clear that $\mathcal{C}\ell_{p,q}^{\lfloor n \rfloor}$ is the vector space underlying $\mathcal{C}\ell_{p,q}$ and that $\mathcal{C}\ell_{p,q}^{\lfloor 0 \rfloor} \cong \mathbb{R}$.

Because Ker∇ = Im∇ at each stage, the canonical lowering operator induces the following exact sequence:

$$\mathcal{C}\ell_{p,q} \xrightarrow{\nabla} \mathcal{C}\ell_{p,q}^{\lfloor n-1\rfloor} \xrightarrow{\nabla} \mathcal{C}\ell_{p,q}^{\lfloor n-2\rfloor} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathbb{R} \xrightarrow{\nabla} 0. \tag{2.46}$$

At each step k of the sequence, one finds the vector space homomorphism $\nabla : \mathcal{C}\ell_{p,q}^{\lfloor n-k\rfloor} \to \mathcal{C}\ell_{p,q}^{\lfloor n-k-1\rfloor}$. By rank-nullity,

$$\mathcal{C}\ell_{p,q}^{\lfloor n-k\rfloor} \diagup \mathrm{Ker}\nabla \cong \mathrm{Im}\nabla. \tag{2.47}$$

As k runs from 0 to $n-1$, the following vector space isomorphisms are apparent:

$$\mathcal{C}\ell_{p,q}^{\lfloor n-k\rfloor} \cong \mathbb{R}^m, \tag{2.48}$$

where

$$m = \begin{cases} 2^n & \text{if } k=0, \\ 2^n - \displaystyle\sum_{\ell=0}^{k-1} \binom{n}{\ell} & \text{if } 1 \le k \le n. \end{cases} \tag{2.49}$$

Theorem 2.1. *At the k^{th} step of the exact sequence* (2.46)*, where $0 \le k \le n-1$, the vector space homomorphism*

$$\nabla : \mathcal{C}\ell_{p,q}^{\lfloor n-k\rfloor} \to \mathcal{C}\ell_{p,q}^{\lfloor n-k-1\rfloor}$$

satisfies the following condition:

$$\mathcal{C}\ell_{p,q}^{\lfloor n-k\rfloor} \diagup \mathrm{Ker}\nabla \cong \mathbb{R}^m, \tag{2.50}$$

where

$$m = \begin{cases} 2^{n-1} & \text{when } k=0 \\ 2^{n-1} - \displaystyle\sum_{\ell=0}^{k-1} \binom{n-1}{\ell} & \text{when } 1 \le k \le n-1. \end{cases} \tag{2.51}$$

Proof. Proof is by induction on k.

Consider the vector space homomorphism $\nabla : \mathcal{C}\ell_{p,q}^{\lfloor n\rfloor} \to \mathcal{C}\ell_{p,q}^{\lfloor n-1\rfloor}$. By Lemma 2.4, Im$\nabla$ = Ker∇.

Rank-nullity implies

$$\mathrm{Dim}\,(\mathrm{Ker}\nabla) + \mathrm{Dim}\,(\mathrm{Im}\nabla) = \mathrm{Dim}\left(\mathcal{C}\ell_{p,q}^{\lfloor n\rfloor}\right) = 2^n. \tag{2.52}$$

Thus,

$$\mathrm{Dim}\,(\mathrm{Ker}\nabla) = \mathrm{Dim}\,(\mathrm{Im}\nabla) = 2^{n-1}. \tag{2.53}$$

Hence, when $k=0$,

$$\mathcal{C}\ell_{p,q}^{\lfloor n-k\rfloor} \diagup \mathrm{Ker}\nabla \cong \mathbb{R}^{2^{n-1}}. \tag{2.54}$$

Assume $\mathcal{C}\ell_{p,q}^{\lfloor n-k\rfloor} \diagup \mathrm{Ker}\,\nabla \cong \mathbb{R}^m$, where $m = 2^{n-1} - \displaystyle\sum_{\ell=0}^{k-1}\binom{n-1}{\ell}$. Consider the homomorphism $\nabla : \mathcal{C}\ell_{p,q}^{\lfloor n-(k+1)\rfloor} \to \mathcal{C}\ell_{p,q}^{\lfloor n-(k+2)\rfloor}$.

$$
\begin{aligned}
\operatorname{Dim}\left(\mathcal{C}\ell_{p,q}^{\lfloor n-(k+1)\rfloor}\big/\operatorname{Ker}\nabla\right) &= \operatorname{Dim}\left(\mathcal{C}\ell_{p,q}^{\lfloor n-(k+1)\rfloor}\right) - \operatorname{Dim}\left(\operatorname{Ker}\nabla\right)\\
&= \left(2^n - \sum_{\ell=0}^{k}\binom{n}{\ell}\right) - \left(2^{n-1} - \sum_{\ell=0}^{k-1}\binom{n-1}{\ell}\right)\\
&= 2^{n-1} - \sum_{\ell=0}^{k}\binom{n}{\ell} + \sum_{\ell=1}^{k}\binom{n-1}{\ell-1}\\
&= 2^{n-1} - \binom{n}{0} - \sum_{\ell=1}^{k}\left(\binom{n}{\ell} - \binom{n-1}{\ell-1}\right)\\
&= 2^{n-1} - \binom{n}{0} - \sum_{\ell=1}^{k}\binom{n-1}{\ell} = 2^{n-1} - \sum_{\ell=0}^{k}\binom{n-1}{\ell}. \qquad (2.55)
\end{aligned}
$$

□

Example 2.3. *For the Clifford algebra $\mathcal{C}\ell_{4,0}$, one has the following exact sequence:*

$$
\underset{16}{\mathcal{C}\ell_{4,0}^{\lfloor 4\rfloor}} \xrightarrow{\nabla} \underset{15}{\mathcal{C}\ell_{4,0}^{\lfloor 3\rfloor}} \xrightarrow{\nabla} \underset{11}{\mathcal{C}\ell_{4,0}^{\lfloor 2\rfloor}} \xrightarrow{\nabla} \underset{5}{\mathcal{C}\ell_{4,0}^{\lfloor 1\rfloor}} \xrightarrow{\nabla} \underset{1}{\mathbb{R}} \xrightarrow{\nabla} \underset{0}{0}. \qquad (2.56)
$$

Here the number beneath each vector space indicates the dimension of the space. Observe the appearance of binomial coefficients in the sequence of differences of dimension. The homology sequence is

$$\mathcal{C}\ell_{4,0}^{\lfloor 4\rfloor}\big/\operatorname{Ker}\nabla \cong \mathbb{R}^8 \qquad (2.57)$$
$$\mathcal{C}\ell_{4,0}^{\lfloor 3\rfloor}\big/\operatorname{Ker}\nabla \cong \mathbb{R}^7 \qquad (2.58)$$
$$\mathcal{C}\ell_{4,0}^{\lfloor 2\rfloor}\big/\operatorname{Ker}\nabla \cong \mathbb{R}^4 \qquad (2.59)$$
$$\mathcal{C}\ell_{4,0}^{\lfloor 1\rfloor}\big/\operatorname{Ker}\nabla \cong \mathbb{R} \qquad (2.60)$$
$$\mathcal{C}\ell_{4,0}^{\lfloor 0\rfloor}\big/\operatorname{Ker}\nabla \cong \{0\}. \qquad (2.61)$$

2.3 $\mathcal{C}\ell_{p,q}$ and the raising operators $\{R_i\}_{1\le i\le n}$

For $1 \le i \le n$, the raising operator $R_i : \mathcal{C}\ell_{p,q} \to \mathcal{C}\ell_{p,q}$ satisfies $R_i \circ R_i = 0$ and thus leads to the chain complex

$$\cdots \xrightarrow{R_i} \mathcal{C}\ell_{p,q} \xrightarrow{R_i} \mathcal{C}\ell_{p,q} \xrightarrow{R_i} \cdots. \qquad (2.62)$$

The cycles associated with R_i are the same at each stage and are defined by

$$
\begin{aligned}
Z_i = \{u \in \mathcal{C}\ell_{p,q} : R_i\, u = 0\} &= \{u \in \mathcal{C}\ell_{p,q} : \left\langle u, \mathbf{e}_{\underline{j}}\right\rangle \neq 0 \text{ whenever } i \in \underline{j}\}\\
&= \{u \in \mathcal{C}\ell_{p,q} : \mathbf{e}_i \wedge u \neq 0\}. \qquad (2.63)
\end{aligned}
$$

Similarly, the boundaries associated with R_i are the same at each stage and are defined by

$$\begin{aligned} B_i &= \{u \in \mathcal{C}\ell_{p,q} : u = R_i\, w,\ \text{for some } w \in \mathcal{C}\ell_{p,q}\} \\ &= \{u \in \mathcal{C}\ell_{p,q} : \left\langle u, \mathbf{e}_{\underline{j}} \right\rangle \neq 0 \,\text{whenever } i \in \underline{j}\} \\ &= \{u \in \mathcal{C}\ell_{p,q} : \mathbf{e}_i \wedge u \neq 0\}. \end{aligned} \tag{2.64}$$

Hence, the following condition is satisfied at each stage of the chain complex:

$$\operatorname{Ker} R_i = \operatorname{Im} R_i, \tag{2.65}$$

leading to the trivial homology group $\operatorname{Ker} R_i \big/ \operatorname{Im} R_i \cong \langle e \rangle$ at each stage.

Considering the algebra isomorphism $\mathcal{C}\ell_{p,q}\big/\mathrm{Ker}R_i \cong \mathrm{Im}R_i$, it follows that the image of R_i is isomorphic to the 2^{n-1}-dimensional subalgebra generated by the collection $\{\mathbf{e}_j\}_{j\neq i}$. In other words,

$$\mathcal{C}\ell_{p,q}\big/\operatorname{Ker} R_i \cong \begin{cases} \mathcal{C}\ell_{p-1,q} & \text{if } 1 \leq i \leq p, \\ \mathcal{C}\ell_{p,q-1} & \text{if } p+1 \leq i \leq n. \end{cases} \tag{2.66}$$

The collection $\{R_i\}_{1\leq i\leq n}$ then induces the following sequence of monomorphisms:

$$\mathcal{C}\ell_{0,0} \xrightarrow{R_1} \mathcal{C}\ell_{1,0} \xrightarrow{R_2} \cdots \xrightarrow{R_{p+1}} \mathcal{C}\ell_{p,1} \xrightarrow{R_{p+2}} \cdots \xrightarrow{R_n} \mathcal{C}\ell_{p,q}. \tag{2.67}$$

2.4 Cohomology and the canonical raising operator $\mathcal{R}$

Let p and q be fixed nonnegative integers. Let $\mathcal{R}$ be the canonical raising operator defined on the Clifford algebra $\mathcal{C}\ell_{p,q}$ by

$$\mathcal{R} = \bigoplus_{i=1}^{n} R_i. \tag{2.68}$$

Let the basis blades of $\mathcal{C}\ell_{p,q}$ be canonically ordered by $\prec$ as defined in (1.17). Given this canonical ordering, define the matrix representation of
$\mathcal{R} : \mathcal{C}\ell_{p,q} \to \mathcal{C}\ell_{p,q}$ as the $2^n \times 2^n$ matrix

$$\mathcal{R}^{(n)}_{\underline{i},\underline{j}} = \begin{cases} 1 & \text{if } |\underline{j}| = |\underline{i}| + 1 \ \text{ and } \mathbf{e}_\ell\, \mathbf{e}_{\underline{i}} = \mathbf{e}_{\underline{j}} \text{ for some } 1 \leq \ell \leq n, \\ -1 & \text{if } |\underline{j}| = |\underline{i}| + 1 \ \text{ and } \mathbf{e}_\ell\, \mathbf{e}_{\underline{i}} = -\mathbf{e}_{\underline{j}} \text{ for some } 1 \leq \ell \leq n, \\ 0 & \text{otherwise.} \end{cases} \tag{2.69}$$

Here the rows and columns of $\mathcal{R}^{(n)}$ have been labeled in one-to-one fashion by multi-indices $\underline{i}, \underline{j} \in 2^{[n]}$.

Remark 2.3. *It is apparent from the definition that unlike the canonical lowering operator, $\mathcal{R}$ is signature independent because the operator involves no squaring of basis vectors. This is illustrated in Figure 2.*

Again using the canonical vector space isomorphism (2.24), one has

$$\mathcal{R}\, u \simeq \vec{u}\mathcal{R}^{(n)}. \tag{2.70}$$

$$\mathtt{R}_{0,2} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathtt{R}_{1,1} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathtt{R}_{2,0} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure 2: The canonical raising operator in the case $n = 2$.

Lemma 2.5. *Let $n = p + q \geq 1$. Given $\mathcal{R}^{(n)}$,*

$$\mathcal{R}^{(n+1)} = \begin{pmatrix} \mathcal{R}^{(n)} & \Xi \\ 0 & \mathcal{R}^{(n)} \end{pmatrix}, \tag{2.71}$$

where 0 represents the $2^n \times 2^n$ zero matrix and I denotes the $2^n \times 2^n$ identity matrix.

Proof. As in the proof of Lemma 2.2, $\mathcal{R}^{(n+1)}$ has the form

$$\mathcal{R}^{(n+1)} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{2.72}$$

Entries in A correspond to the action of the raising operator restricted to $\mathcal{C}\ell_{p,q}$. Hence, $A = \mathcal{R}^{(n)}$. Similarly, entries in D correspond to the action of the raising operator on multi-vectors containing $\mathbf{e}_{n+1}$ that leave $\mathbf{e}_{n+1}$ in place. Thus, $D = \mathcal{R}^{(n)}$.

Rows $2^n + 1$ through 2^{n+1} correspond to pre-images containing $\mathbf{e}_{n+1}$. Columns 1 through 2^n correspond to raising these to elements of $\mathcal{C}\ell_{p,q}$. Hence, $C = \mathbf{0}$.

Finally, because (i) $\mathcal{R}^{(n+1)}$ represents a raising operator, (ii) $\mathbf{e}_{n+1}$ is not found in any of the first 2^n multi-vectors in the ordered basis, and (iii) columns $2^n + 1$ through 2^{n+1} correspond to images containing $\mathbf{e}_{n+1}$, it follows that B corresponds to the raising operator R_{n+1} with domain restricted to $\mathcal{C}\ell_{p,q}$. Hence, each blade indexed by $\underline{i} \in [n]$ is mapped to the corresponding blade indexed by $\underline{i}^+ = \underline{i} \cup \{n+1\}$. Since the action is left multiplication by $\mathbf{e}_{n+1}$, the number of transpositions required is $|\underline{i}|$, contributing $(-1)^{|\underline{i}|}$ to the sign. In particular, canonical ordering of the basis gives $B_{\underline{i},\underline{i}^+} = (-1)^{|\underline{i}|} = \Xi_{\underline{i},\underline{i}}$.

$\square$

Lemma 2.6.

$$\mathrm{Im}\mathcal{R} = \mathrm{Ker}\mathcal{R}. \tag{2.73}$$

Proof. Because $\mathcal{R}^2 = 0$, it is clear that $\mathrm{Im}\mathcal{R} \subseteq \mathrm{Ker}\mathcal{R}$. To establish the reverse inclusion, let $n \geq 1$ and consider the matrix representation $\mathcal{R}^{(n)} \simeq \mathcal{R}$.

Let $\vec{x} = (\vec{x_1}, \vec{x_2}) \in \mathrm{Ker}\mathcal{R}^{(n)}$. Then,

$$\begin{aligned}(\vec{x_1}, \vec{x_2})\mathcal{R}^{(n)} = (\vec{x_1}, \vec{x_2}) \begin{pmatrix} \mathcal{R}^{(n-1)} & \Xi \\ 0 & \mathcal{R}^{(n-1)} \end{pmatrix} &= (\vec{x_1}\mathcal{R}^{(n-1)}, \vec{x_1}\Xi + \vec{x_2}\mathcal{R}^{(n-1)}) \\ &= (\vec{0}, \vec{0}). \end{aligned} \tag{2.74}$$

This implies $\vec{x_1} \in \mathrm{Ker}\mathcal{R}^{(n-1)}$ and $\vec{x_1} = -\vec{x_2}\mathcal{R}^{(n-1)}\Xi$. In other words,

$$(\vec{x_1}, \vec{x_2}) = (-\vec{x_2}\Upsilon, \vec{x_2}),$$

where $\Upsilon = \mathcal{R}^{(n-1)}\Xi$.

To see that $(\vec{x_2}\Upsilon, \vec{x_2}) \in \mathrm{Im}\mathcal{R}^{(n)}$, solve the equation

$$(\vec{s}, \vec{t})\mathcal{R}^{(n)} = (-\vec{x_2}\Upsilon, \vec{x_2}) \tag{2.75}$$

for $\vec{s}$ and $\vec{t}$. One solution is to let $\vec{s} = \vec{x_2}\Xi$ and choose any $\vec{t} \in \mathrm{Ker}\mathcal{R}^{(n-1)}$. Then,

$$\begin{aligned}(\vec{s}, \vec{t})\mathcal{R}^{(n)} = (\vec{x_2}\Xi, \vec{t})\mathcal{R}^{(n)} = (\vec{x_2}\Xi\mathcal{R}^{(n-1)}, \vec{x_2} - \vec{t}\mathcal{R}^{(n-1)}) &= (\vec{x_2}\Xi\mathcal{R}^{(n-1)}, \vec{x_2}) \\ &= (-\vec{x_2}\mathcal{R}^{(n-1)}\Xi, \vec{x_2}) = (-\vec{x_2}\Upsilon, \vec{x_2}). \end{aligned} \tag{2.76}$$

Hence, $\mathrm{Im}\mathcal{R}^{(p,q)} = \mathrm{Ker}\mathcal{R}^{(p,q)}$ for any signature (p, q), and therefore $\mathrm{Im}\mathcal{R} = \mathrm{Ker}\mathcal{R}$ for the canonical raising operator on a Clifford algebra of arbitrary signature. □

Turning now to the chain complex

$$\cdots \xrightarrow{\mathcal{R}} \mathcal{C}\ell_{p,q} \xrightarrow{\mathcal{R}} \mathcal{C}\ell_{p,q} \xrightarrow{\mathcal{R}} \cdots, \tag{2.77}$$

the homology group $\mathrm{Ker}\mathcal{R} \diagup \mathrm{Im}\mathcal{R}$ is trivial at each stage.

The canonical raising operator $\mathcal{R}$ induces the following exact sequence:

$$\mathbb{R} \xrightarrow{\mathcal{R}} \mathcal{C}\ell_{p,q}^{\lfloor 1 \rfloor} \xrightarrow{\mathcal{R}} \mathcal{C}\ell_{p,q}^{\lfloor 2 \rfloor} \xrightarrow{\mathcal{R}} \cdots \xrightarrow{\mathcal{R}} \mathcal{C}\ell_{p,q}^{\lfloor n-1 \rfloor} \xrightarrow{\mathcal{R}} \mathcal{C}\ell_{p,q}. \tag{2.78}$$

Theorem 2.2. *At the k^{th} step of the exact sequence* (2.46)*, where $0 \leq k \leq n-1$, the vector space homomorphism*

$$\mathcal{R} : \mathcal{C}\ell_{p,q}^{\lfloor k \rfloor} \to \mathcal{C}\ell_{p,q}^{\lfloor k+1 \rfloor}$$

satisfies the following condition:

$$\mathcal{C}\ell_{p,q}^{\lfloor k \rfloor} \diagup \mathrm{Ker}\mathcal{R} \cong \mathbb{R}^m, \tag{2.79}$$

where

$$m = \sum_{\ell=0}^{k} \binom{n-1}{\ell}. \tag{2.80}$$

Proof. The proof is by induction on k. Observe that $\mathcal{C}\ell_{p,q}^{\lfloor k \rfloor}$ is of dimension $\displaystyle\sum_{\ell=0}^{k} \binom{n}{k}$. It is clear that when $k = 0$, $\mathrm{Ker}\,\mathcal{R} = \{0\}$, which is of dimension 0. Rank-nullity then implies $\mathrm{Im}\,\mathcal{R}$ is of dimension 1.

Assume that $\mathcal{C}\ell_{p,q}^{\lfloor k \rfloor} \diagup \mathrm{Ker}\mathcal{R} \cong \mathbb{R}^m$, where $m = \displaystyle\sum_{\ell=0}^{k} \binom{n-1}{\ell}$ for some $1 \leq k \leq n-2$. Consider the homomorphism $\mathcal{R} : \mathcal{C}\ell_{p,q}^{\lfloor k+1 \rfloor} \to \mathcal{C}\ell_{p,q}^{\lfloor k+2 \rfloor}$.

Then,

$$\begin{aligned}
\mathrm{Dim}\left(\mathcal{C}\ell_{p,q}^{\lfloor k+1\rfloor}\big/\mathrm{Ker}\,\mathcal{R}\right) &= \mathrm{Dim}\left(\mathcal{C}\ell_{p,q}^{\lfloor k+1\rfloor}\right) - \mathrm{Dim}\left(\mathrm{Ker}\,\mathcal{R}\right)\\
&= \mathrm{Dim}\left(\mathcal{C}\ell_{p,q}^{\lfloor k+1\rfloor}\right) - \mathrm{Dim}\left(\mathcal{C}\ell_{p,q}^{\lfloor k\rfloor}\big/\mathrm{Ker}\,\mathcal{R}\right)\\
&= \sum_{\ell=0}^{k+1}\binom{n}{\ell} - \sum_{\ell=0}^{k}\binom{n-1}{\ell}\\
&= \sum_{\ell=0}^{k+1}\binom{n}{\ell} - \sum_{\ell=1}^{k+1}\binom{n-1}{\ell-1}\\
&= \binom{n}{0} + \sum_{\ell=1}^{k+1}\left(\binom{n}{\ell} - \binom{n-1}{\ell-1}\right)\\
&= \binom{n}{0} + \sum_{\ell=1}^{k+1}\binom{n-1}{\ell} = \sum_{\ell=0}^{k+1}\binom{n-1}{\ell}. \qquad (2.81)
\end{aligned}$$

□

Example 2.4. *For the Clifford algebra $\mathcal{C}\ell_{4,0}$, one has the following exact sequence:*

$$\underset{1}{\mathcal{C}\ell_{4,0}^{\lfloor 0\rfloor}} \xrightarrow{\mathcal{R}} \underset{5}{\mathcal{C}\ell_{4,0}^{\lfloor 1\rfloor}} \xrightarrow{\mathcal{R}} \underset{11}{\mathcal{C}\ell_{4,0}^{\lfloor 2\rfloor}} \xrightarrow{\mathcal{R}} \underset{15}{\mathcal{C}\ell_{4,0}^{\lfloor 3\rfloor}} \xrightarrow{\mathcal{R}} \underset{16}{\mathcal{C}\ell_{4,0}^{\lfloor 4\rfloor}}. \qquad (2.82)$$

Here the number beneath each vector space indicates the dimension of the space. The cohomology sequence is

$$\mathcal{C}\ell_{4,0}^{\lfloor 0\rfloor}\big/\mathrm{Ker}\mathcal{R} \cong \mathbb{R} \qquad (2.83)$$
$$\mathcal{C}\ell_{4,0}^{\lfloor 1\rfloor}\big/\mathrm{Ker}\mathcal{R} \cong \mathbb{R}^4 \qquad (2.84)$$
$$\mathcal{C}\ell_{4,0}^{\lfloor 2\rfloor}\big/\mathrm{Ker}\mathcal{R} \cong \mathbb{R}^7 \qquad (2.85)$$
$$\mathcal{C}\ell_{4,0}^{\lfloor 3\rfloor}\big/\mathrm{Ker}\mathcal{R} \cong \mathbb{R}^8. \qquad (2.86)$$

Example 2.5. *Consider the Clifford algebra $\mathcal{C}\ell_{0,2}$, which is canonically isomorphic to the algebra of quaternions.*

The linear spaces of the exact sequences are given by

$$\mathcal{C}\ell_{0,2} = \mathcal{C}\ell_{0,2}^{\lfloor 2\rfloor} = \{u : u = u_0 + u_1\,\mathbf{e}_1 + u_2\,\mathbf{e}_2 + u_{12}\,\mathbf{e}_{12}\}, \qquad (2.87)$$
$$\mathcal{C}\ell_{0,2}^{\lfloor 1\rfloor} = \{u : u = u_0 + u_1\,\mathbf{e}_1 + u_2\,\mathbf{e}_2\}, \qquad (2.88)$$
$$\mathcal{C}\ell_{0,2}^{\lfloor 0\rfloor} = \{u : u = u_0\} = \mathbb{R}. \qquad (2.89)$$

3 Generalized lowering and raising operators

Given any nonzero vector $\vec{\lambda} \in \mathbb{R}^n$, the following define lowering and raising operators on $\mathcal{C}\ell_{p,q}$, respectively:

$$\nabla_{\vec{\lambda}} = \sum_{i=1}^{n} \lambda_i\, D_i, \qquad (3.1)$$

and

$$\mathcal{R}_{\vec{\lambda}} = \sum_{i=1}^{n} \lambda_i R_i. \tag{3.2}$$

The structure of the matrix representations of these operators is identical to that of the canonical lowering and raising operators as established in Lemmas 2.2 and 2.5. Thus, all properties established for ∇ and $\mathcal{R}$ hold also for $\nabla_{\vec{\lambda}}$ and $\mathcal{R}_{\vec{\lambda}}$. Note that in particular, $\nabla_{\vec{\lambda}}$ and $\mathcal{R}_{\vec{\lambda}}$ are nilpotent of index 2.

Example 3.1. *The matrix representation and kernel of $\nabla_{\vec{\lambda}}$ in $\mathcal{C}\ell_{1,2}$ are computed with Mathematica:*

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ -\lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_3 & 0 & 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & -\lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_3 & 0 & \lambda_2 & \lambda_1 & 0 \end{pmatrix}$$

In $\text{Cl}_{1,2}$, Ker(L) is spanned by :

$$e_{\{2,3\}} + \frac{e_{\{1,3\}}\lambda_2}{\lambda_1} - \frac{e_{\{1,2\}}\lambda_3}{\lambda_1}$$

$$e_{\{3\}} + \frac{e_{\{1\}}\lambda_3}{\lambda_1}$$

$$e_{\{2\}} + \frac{e_{\{1\}}\lambda_2}{\lambda_1}$$

1

Example 3.2. *Consider $\mathcal{C}\ell_{1,3}$, which is canonically isomorphic to the space-time algebra. Mathematica computations reveal the kernels of the generalized lowering and raising operators:*

In $\text{Cl}_{1,3}$, Ker(L) is spanned by :

$$e_{\{2,3,4\}} + \frac{e_{\{1,3,4\}}\lambda_2}{\lambda_1} - \frac{e_{\{1,2,4\}}\lambda_3}{\lambda_1} + \frac{e_{\{1,2,3\}}\lambda_4}{\lambda_1}$$

$$e_{\{3,4\}} + \frac{e_{\{1,4\}}\lambda_3}{\lambda_1} - \frac{e_{\{1,3\}}\lambda_4}{\lambda_1}$$

$$e_{\{2,4\}} + \frac{e_{\{1,4\}}\lambda_2}{\lambda_1} - \frac{e_{\{1,2\}}\lambda_4}{\lambda_1}$$

$$e_{\{4\}} + \frac{e_{\{1\}}\lambda_4}{\lambda_1}$$

$$e_{\{2,3\}} + \frac{e_{\{1,3\}}\lambda_2}{\lambda_1} - \frac{e_{\{1,2\}}\lambda_3}{\lambda_1}$$

$$e_{\{3\}} + \frac{e_{\{1\}}\lambda_3}{\lambda_1}$$

$$e_{\{2\}} + \frac{e_{\{1\}}\lambda_2}{\lambda_1}$$

1

In $\text{Cl}_{1,3}$, Ker(R) is spanned by :

$$e_{\{1,2,3,4\}}$$

$$e_{\{2,3,4\}} + \frac{e_{\{1,2,3\}}\lambda_1}{\lambda_4}$$

$$e_{\{1,3,4\}} - \frac{e_{\{1,2,3\}}\lambda_2}{\lambda_4}$$

$$e_{\{3,4\}} - \frac{e_{\{1,3\}}\lambda_1}{\lambda_4} - \frac{e_{\{2,3\}}\lambda_2}{\lambda_4}$$

$$e_{\{1,2,4\}} + \frac{e_{\{1,2,3\}}\lambda_3}{\lambda_4}$$

$$e_{\{2,4\}} - \frac{e_{\{1,2\}}\lambda_1}{\lambda_4} + \frac{e_{\{2,3\}}\lambda_3}{\lambda_4}$$

$$e_{\{1,4\}} + \frac{e_{\{1,2\}}\lambda_2}{\lambda_4} + \frac{e_{\{1,3\}}\lambda_3}{\lambda_4}$$

$$e_{\{4\}} + \frac{e_{\{1\}}\lambda_1}{\lambda_4} + \frac{e_{\{2\}}\lambda_2}{\lambda_4} + \frac{e_{\{3\}}\lambda_3}{\lambda_4}$$

3.1 Graphs associated with raising and lowering operators

The action of the canonical lowering and raising operators on blades in $\mathcal{C}\ell_{p,q}$ can be depicted graphically by treating the matrix representations as graph adjacency matrices.

Vertices represent blades, and two vertices are adjacent if and only if the difference of their respective grades is exactly one. I.e., one vertex is the image of the other under the action of the canonical raising/lowering operator being considered.

When a graph represents the canonical lowering operator, edges run from blades of higher grade to lower grade. Edges run the opposite direction in graph representations of the canonical raising operator.

Two graphs are associated with each operator to make clear the sign changes induced by multiplication within the algebra. Figures 3 and 4 depict thes action of the canonical raising and lowering operators, respectively, in $\mathcal{C}\ell_{1,3}$.

3.2 Quantum Probability

While a proper treatment of quantum probability is beyond the scope of the current work, some connections should be mentioned.

Given a Hilbert space $\mathcal{H}$ of dimension $n < \infty$, *observables* are Hermitian operators on $\mathcal{H}$. Observables are the quantum probability analogues of random variables in classical probability. Any observable X has the spectral resolution $X = \sum_i x_i E_i^X$, where the x_is are the distinct eigenvalues and E_i^X is the *event that* X *takes the value* x_i.

Proposition 3.1. *Fix $n > 0$, and let $\mathcal{L}_{\vec{\lambda}}$ and $\mathcal{R}_{\vec{\lambda}}$ denote the matrix representations of the generalized lowering and raising operators in $\mathcal{C}\ell_{p,q}$. Then,*

$$\left(i\,\mathcal{L}_{\vec{\lambda}}\right)^\dagger = i\,\mathcal{R}_{\vec{\lambda}}, \tag{3.3}$$

and hence

$$\left(i\mathcal{L}_{\vec{\lambda}} + i\mathcal{R}_{\vec{\lambda}}\right)^\dagger = i\mathcal{L}_{\vec{\lambda}} + i\mathcal{R}_{\vec{\lambda}} \tag{3.4}$$

is a traceless, bounded Hermitian linear operator; i.e., $i\mathcal{L}_{\vec{\lambda}} + i\mathcal{R}_{\vec{\lambda}}$ is a quantum observable if and only if $\lambda_1 = \cdots = \lambda_p = 0$.

Proof. Let $\{L_j\}_{1\le j\le n}$ and $\{R_j\}_{1\le j\le n}$ denote the collections of lowering operators and raising operators in $\mathcal{C}\ell_{p,q}$. The proposition is proved by showing $(iL_j)^\dagger = iR_j$ holds only for $p+1 \le j \le n$.

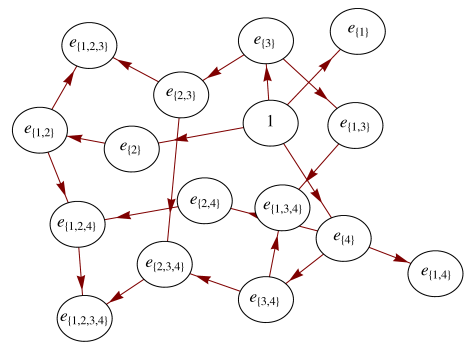
Fix $1 \le j \le n$, denote iL_j by $(\ell_{\underline{k},\underline{m}})_{\underline{k},\underline{m}\in 2^{[n]}}$, and denote iR_j by $(r_{\underline{k},\underline{m}})_{\underline{k},\underline{m}\in 2^{[n]}}$. By construction of the lowering operators, iL_j is the lower triangular matrix defined by

$$\ell_{\underline{k},\underline{m}} = \begin{cases} i & \text{if } \mathbf{e}_s\,\mathbf{e}_{\underline{k}} = \mathbf{e}_{\underline{m}} \text{ for some } 1 \le s \le n \\ -i & \text{if } \mathbf{e}_s\,\mathbf{e}_{\underline{k}} = -\mathbf{e}_{\underline{m}} \text{ for some } 1 \le s \le n \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

```
Canonical Raising Operator in Cl_{1,3}

Operator action not resulting in change of sign...
```

Out[83]=

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Operator action resulting in change of sign...
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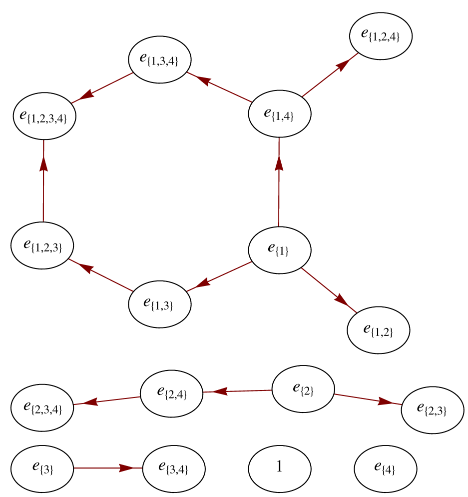

Figure 3: Action of canonical raising operator on $\mathcal{C}\ell_{1,3}$.

```
Canonical Lowering Operator in Cl₁,₃
```

```
Operator action not resulting in change of sign...
```

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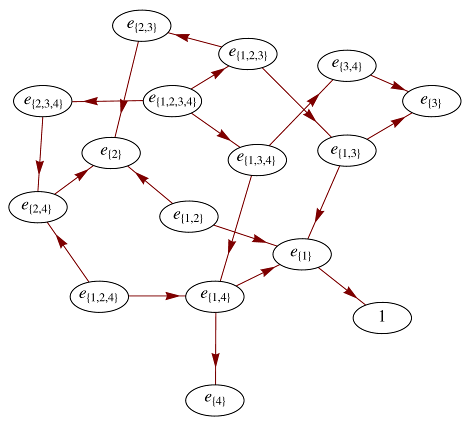


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Operator action resulting in change of sign...
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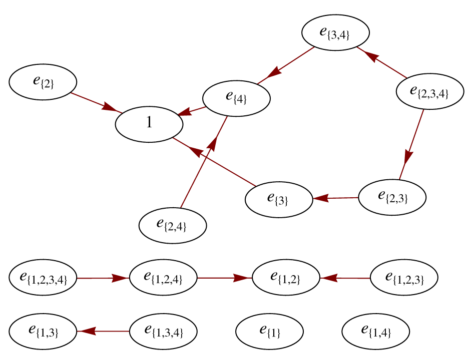

Figure 4: Action of canonical lowering operator on $\mathcal{C}\ell_{1,3}$.

By construction of the raising operators, iR_j is the upper triangular matrix defined by

$$r_{\underline{k},\underline{m}} = \begin{cases} i & \text{if } \mathbf{e}_s\, \mathbf{e}_{\underline{k}} = \mathbf{e}_{\underline{m}} \text{ for some } 1 \leq s \leq n \\ -i & \text{if } \mathbf{e}_s\, \mathbf{e}_{\underline{k}} = -\mathbf{e}_{\underline{m}} \text{ for some } 1 \leq s \leq n \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

For fixed $\underline{m}$ and $\underline{k}$, $\mathbf{e}_s\, \mathbf{e}_{\underline{m}} = \mathbf{e}_{\underline{k}}$ if and only if $\mathbf{e}_s\, \mathbf{e}_{\underline{k}} = \pm\mathbf{e}_{\underline{m}}$. Since the same number of transpositions are involved in canonically ordering the multi-indices of the products in either case, the only case in which $\ell_{\underline{k},\underline{m}} = -r_{\underline{m},\underline{k}}$ occurs is when ${\mathbf{e}_s}^2 = -1$. Hence, $iL_j = (iL_j)^\dagger$ if and only if $p+1 \leq j \leq n$. □

Note that an immediate consequence of Proposition 3.1 is that $i\mathcal{L}_{\vec{\lambda}} + i\mathcal{R}_{\vec{\lambda}}$ is a quantum observable in $\mathcal{C}\ell_{0,n}$ for any positive integer n.

Example 3.3. *The result of Proposition 3.1 is illustrated by generating the matrix* $(i\,\mathcal{L}_{\vec{\lambda}} + i\,\mathcal{R}_{\vec{\lambda}}) - (i\,\mathcal{L}_{\vec{\lambda}} + i\,\mathcal{R}_{\vec{\lambda}})^\dagger$ *over the 4-dimensional Clifford algebras:*

$\mathrm{Cl}_{0,2}$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\mathrm{Cl}_{1,1}$

$$\begin{pmatrix} 0 & 2\,i\,\lambda_1 & 0 & 0 \\ 2\,i\,\lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\,i\,\lambda_1 \\ 0 & 0 & 2\,i\,\lambda_1 & 0 \end{pmatrix}$$

$\mathrm{Cl}_{2,0}$

$$\begin{pmatrix} 0 & 2\,i\,\lambda_1 & 2\,i\,\lambda_2 & 0 \\ 2\,i\,\lambda_1 & 0 & 0 & -2\,i\,\lambda_2 \\ 2\,i\,\lambda_2 & 0 & 0 & 2\,i\,\lambda_1 \\ 0 & -2\,i\,\lambda_2 & 2\,i\,\lambda_1 & 0 \end{pmatrix}$$

Example 3.4. *In* $\mathcal{C}\ell_{0,3}$, $\dfrac{i\mathcal{L} + i\mathcal{R}}{\sqrt{3}}$ *is a quantum observable taking values* ± 1 *with equal probability. This is indicated by the spectrum of the matrix representation.*

$$\frac{\mathrm{i}L+\mathrm{i}R}{\sqrt{3}}=\begin{pmatrix}0&\frac{\mathrm{i}}{\sqrt{3}}&\frac{\mathrm{i}}{\sqrt{3}}&0&\frac{\mathrm{i}}{\sqrt{3}}&0&0&0\\-\frac{\mathrm{i}}{\sqrt{3}}&0&0&-\frac{\mathrm{i}}{\sqrt{3}}&0&-\frac{\mathrm{i}}{\sqrt{3}}&0&0\\-\frac{\mathrm{i}}{\sqrt{3}}&0&0&\frac{\mathrm{i}}{\sqrt{3}}&0&0&-\frac{\mathrm{i}}{\sqrt{3}}&0\\0&\frac{\mathrm{i}}{\sqrt{3}}&-\frac{\mathrm{i}}{\sqrt{3}}&0&0&0&0&\frac{\mathrm{i}}{\sqrt{3}}\\-\frac{\mathrm{i}}{\sqrt{3}}&0&0&0&0&\frac{\mathrm{i}}{\sqrt{3}}&\frac{\mathrm{i}}{\sqrt{3}}&0\\0&\frac{\mathrm{i}}{\sqrt{3}}&0&0&-\frac{\mathrm{i}}{\sqrt{3}}&0&0&-\frac{\mathrm{i}}{\sqrt{3}}\\0&0&\frac{\mathrm{i}}{\sqrt{3}}&0&-\frac{\mathrm{i}}{\sqrt{3}}&0&0&\frac{\mathrm{i}}{\sqrt{3}}\\0&0&0&-\frac{\mathrm{i}}{\sqrt{3}}&0&\frac{\mathrm{i}}{\sqrt{3}}&-\frac{\mathrm{i}}{\sqrt{3}}&0\end{pmatrix}$$

```
Eigenvalues: {-1, -1, -1, -1, 1, 1, 1, 1}
```

4 Mathematica

The Mathematica package "CliffMath08" underlying the examples is available online at `http://www.siue.edu/~sstaple` and provides basic machinery for performing computations using Clifford algebras. This section details the operator calculus procedures used to generate the examples.

```
(* ⊙ denotes the Clifford product *)

(* k^th lowering operator *)
Lower_k_[u_] :=
  Module[{j}, Sum[DegreeKPart[ClExpand[e_{k} ⊙ DegreeKPart[u, j]], j - 1], {j, 1, MaxIndex}]];

(* k^th raising operator *)

Raise_k_[u_] :=
  Module[{j}, Sum[DegreeKPart[ClExpand[e_{k} ⊙ DegreeKPart[u, j]], j + 1], {j, 0, MaxIndex - 1}]];

(* λ-Lowering *)
LambdaLowering[u_, λ_] := Module[{j}, Sum[λ[[j]] Lower_j[u], {j, 1, Length[λ]}]];

(* λ-Raising *)
LambdaRaising[u_, λ_] := Module[{j}, Sum[λ[[j]] Raise_j[u], {j, 1, Length[λ]}]];

(* Canonical lowering *)
Lower[u_] := Module[{j1}, Sum[Lower_j1[u], {j1, 1, MaxIndex}]];

(* Canonical raising *)
Raise[u_] := Module[{j1}, Sum[Raise_j1[u], {j1, 1, MaxIndex}]];

LambdaLower[u_, Λ_] := Module[{j1}, Sum[Λ[[j1]] Lower_j1[u], {j1, 1, MaxIndex}]];

LambdaRaise[u_, Λ_] := Module[{j1}, Sum[Λ[[j1]] Raise_j1[u], {j1, 1, MaxIndex}]];
```

```
(* Matrix rep of ith lowering *)
ithLoweringRep[ith_] := Module[{m, B, rw, co, mulvr, mpr, mprs, A},
   B = ClBasis; A = Table[0, {rw, 1, 2^MaxIndex}, {co, 1, 2^MaxIndex}];
   For[rw = 1, rw ≤ 2^MaxIndex, rw++, For[co = 1, co ≤ rw, co++, mulvr = ClExpand[B⟦rw⟧ ⊙ B⟦co⟧];
     If[mulvr == e{ith} || mulvr == -e{ith}, mpr = ClExpand[ e{ith} ⊙ B⟦rw⟧];
      mprs = mpr /. {e_ → 1}; If[mprs == 1, A⟦rw, co⟧ = 1, A⟦rw, co⟧ = -1];]]]; Return[A];];

(* Matrix rep of ith raising *)
ithRaisingRep[ith_] := Module[{m, B, rw, co, mulvr, mpr, mprs, A},
   B = ClBasis; A = Table[0, {rw, 1, 2^MaxIndex}, {co, 1, 2^MaxIndex}];
   For[rw = 1, rw ≤ 2^MaxIndex, rw++, For[co = rw, co ≤ 2^MaxIndex, co++, mulvr = ClExpand[B⟦rw⟧ ⊙ B⟦co⟧];
     If[mulvr == e{ith} || mulvr == -e{ith}, mpr = ClExpand[e{ith} ⊙ B⟦rw⟧ ];
      mprs = mpr /. {e_ → 1}; If[mprs == 1, A⟦rw, co⟧ = 1, A⟦rw, co⟧ = -1];]]]; Return[A];];

(* Matrix rep of canonical lowering *)
ClLoweringRep := Module[{i}, Return[Sum[ithLoweringRep[i], {i, 1, MaxIndex}]];];

(* Matrix rep of canonical raising *)
ClRaisingRep := Module[{i}, Return[Sum[ithRaisingRep[i], {i, 1, MaxIndex}]];];
```

```
(* Matrix rep for λ-raising *)
LambdaRaisingRep[λ_] := Module[{j}, If[Length[λ] ≠ MaxIndex, Message[lambda::"length",
     MaxIndex]; Abort[];, Sum[λ⟦j⟧ ithRaisingRep[j], {j, 1, Length[λ]}]]]];

(* Matrix rep for λ-lowering *)
LambdaLoweringRep[Λ_] := Module[{j}, If[Length[Λ] ≠ MaxIndex, Message[lambda::"length",
     MaxIndex]; Abort[];, Sum[Λ⟦j⟧ ithLoweringRep[j], {j, 1, Length[Λ]}]]]];

(* Return canonical lowering kernel as vector *)
GetLoweringKernel := Module[{CB, K, lp}, CB = ClBasis;
   K = NullSpace[Transpose[ClLoweringRep], Method → OneStepRowReduction];
   Return[Table[CB.K⟦lp⟧, {lp, 1, Length[K]}]];];

(* Return λ-lowering kernel as vector *)
GetLambdaLoweringKernel[lambda_] := Module[{lp, K, CB}, CB = ClBasis;
  K = NullSpace[Transpose[LambdaLoweringRep[lambda]], Method → OneStepRowReduction];
  Return[Table[CB.K⟦lp⟧, {lp, 1, Length[K]}]];]

(* Return canonical raising kernel as vector *)
GetRaisingKernel := Module[{CB, K, lp}, CB = ClBasis;
   K = NullSpace[Transpose[ClRaisingRep], Method → OneStepRowReduction];
   Return[Table[CB.K⟦lp⟧, {lp, 1, Length[K]}]];];

(* Return λ-raising kernel as vector *)
GetLambdaRaisingKernel[lambda_] := Module[{lp, K, CB}, CB = ClBasis;
  K = NullSpace[Transpose[LambdaRaisingRep[lambda]], Method → OneStepRowReduction];
  Return[Table[CB.K⟦lp⟧, {lp, 1, Length[K]}]];]
```

The following Mathematica code is useful for displaying kernels of raising and lowering operators.

```
(* Print canonical lowering kernel *)
PrintLoweringKernel := Module[{K, CB, lp}, CB = ClBasis;
   K = NullSpace[Transpose[ClLoweringRep], Method → OneStepRowReduction];
   Print["In ", Cl_{pPart,qPart}, ", Ker(L) is spanned by :"]; For[lp = 1, lp ≤ Length[K],
     lp++, Print[Simplify[Expand[CB.K[[lp]]]]];] Print["-----------"];];

(* Print λ-lowering kernel *)
PrintLambdaLoweringKernel[λ_] := Module[{K, lp},
   Print["In ", Cl_{pPart,qPart}, ", Ker(L) is spanned by :"]; K = GetLambdaLoweringKernel[λ];
   For[lp = 1, lp ≤ Length[K], lp++, Print[K[[lp]]];] Print["-----------"];];

(* Print canonical raising kernel *)
PrintRaisingKernel := Module[{CB, K, lp}, CB = ClBasis;
   K = NullSpace[Transpose[ClRaisingRep], Method → OneStepRowReduction];
   Print["In ", Cl_{pPart,qPart}, ",  Ker(R) is spanned by :"];
   For[lp = 1, lp ≤ Length[K], lp++, Print[CB.K[[lp]]];] Print["-----------"];];

(* Print λ-raising kernel *)
PrintLambdaRaisingKernel[λ_] := Module[{CB, K, lp},
   Print["In ", Cl_{pPart,qPart}, ", Ker(R) is spanned by :"]; K = GetLambdaRaisingKernel[λ];
   For[lp = 1, lp ≤ Length[K], lp++, Print[K[[lp]]];] Print["-----------"];];
```

Figures 3 and 4 were generated with the Mathematica code below.

```
(* Canonical raising in Cl_{1,3} *)

SetSignature[1, 3];
Print["Canonical Raising Operator in Cl_{1,3}"]
A = Floor[(1 + ClRaisingRep) / 2];
Print["Operator action not resulting in change of sign..."]
G = GraphPlot[A, {DirectedEdges → True, VertexRenderingFunction →
     ({White, EdgeForm[Black], Disk[#, {0.24, .2}], Black, Text[ClBasis[[#2]], #1]} &),
   Method → {"SpringElectricalEmbedding", "RepulsiveForcePower" → -4}}]
A2 = Floor[(1 - ClRaisingRep) / 2];
Print["Operator action resulting in change of sign..."]
G = GraphPlot[A2, {DirectedEdges → True, VertexRenderingFunction →
     ({White, EdgeForm[Black], Disk[#, {0.24, .18}], Black, Text[ClBasis[[#2]], #1]} &),
   Method → {"SpringElectricalEmbedding", "RepulsiveForcePower" → -4}}]



(* Canonical lowering in Cl_{1,3} *)

SetSignature[1, 3];
Print["Canonical Lowering Operator in Cl_{1,3}"]
A = (Abs[ClLoweringRep] + ClLoweringRep) / 2;
Print["Operator action not resulting in change of sign..."]
G = GraphPlot[A, {DirectedEdges → True, VertexRenderingFunction →
     ({White, EdgeForm[Black], Disk[#, {0.26, .14}], Black, Text[ClBasis[[#2]], #1]} &),
   Method → {"SpringElectricalEmbedding", "RepulsiveForcePower" → -3}}]
A2 = (Abs[ClLoweringRep] - ClLoweringRep) / 2;
Print["Operator action resulting in change of sign..."]
G = GraphPlot[A2, {DirectedEdges → True, VertexRenderingFunction →
     ({White, EdgeForm[Black], Disk[#, {0.28, .15}], Black, Text[ClBasis[[#2]], #1]} &),
   Method → {"SpringElectricalEmbedding", "RepulsiveForcePower" → -3}}]
```

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