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Calculations in New Sequence Spaces and Application to Statistical Convergence

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ABSTRACT

In this paper we recall recent results that are direct consequences of the fact that $(w_\infty(\lambda), w_\infty(\lambda))$ is a Banach algebra. Then we define the set $W_\tau = D_\tau w_\infty$ and characterize the sets $W_\tau(A)$ where A is either of the operators Δ, Σ, $\Delta(\lambda)$, or $C(\lambda)$. Afterwards we consider the sets $[A_1, A_2]_{W_\tau}$ of all sequences X such that $A_1(\lambda)(|A_2(\mu)X|) \in W_\tau$ where A_1 and A_2 are of the form $C(\xi)$, $C^+(\xi)$, $\Delta(\xi)$, or $\Delta^+(\xi)$ and it is given necessary conditions to get $[A_1(\lambda), A_2(\mu)]_{W_\tau}$ in the form W_ξ. Finally we apply the previous results to statistical convergence. So we have conditions to have $x_k \to L(S(A))$ where A is either of the infinite matrices $D_{1/\tau}C(\lambda)C(\mu)$, $D_{1/\tau}\Delta(\lambda)\Delta(\mu)$, $D_{1/\tau}\Delta(\lambda)C(\mu)$. We also give conditions to have $x_k \to 0(S(A))$ where A is either of the operators $D_{1/\tau}C^+(\lambda)\Delta(\mu)$, $D_{1/\tau}C^+(\lambda)C(\mu)$, $D_{1/\tau}C^+(\lambda)C^+(\mu)$, or $D_{1/\tau}\Delta(\lambda)C^+(\mu)$.

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RESUMEN

Recordamos resultados recientes que son consecuencia directa del hecho de que $(w_\infty(\lambda), w_\infty(\lambda))$ es una algebra de Banach. Entonces nosotros definimos el conjunto $W_\tau = D_\tau w_\infty$ y caracterizamos los conjuntos $W_\tau(A)$ donde A es uno de los siguientes operadores Δ, Σ, $\Delta(\lambda)$, o $C(\lambda)$. Después consideramos los conjuntos $[A_1, A_2]_{W_\tau}$ de todas las sucesiones X tal que $A_1(\lambda)(|A_2(\mu)X|) \in W_\tau$ donde A_1 y A_2 son de la forma $C(\xi)$, $C^+(\xi)$, $\Delta(\xi)$, o $\Delta^+(\xi)$ y son dadas condiciones necesarias para obtener $[A_1(\lambda), A_2(\mu)]_{W_\tau}$ en la forma W_ξ. Finalmente, aplicamos los resultados previos para tener $x_k \to L(S(A))$ donde A es una de las matrices infinitas $D_{1/\tau}C(\lambda)C(\mu)$, $D_{1/\tau}\Delta(\lambda)\Delta(\mu)$, $D_{1/\tau}\Delta(\lambda)C(\mu)$. Nosotros también damos condiciones para tener $x_k \to 0(S(A))$ donde A es uno de los operadores $D_{1/\tau}C^+(\lambda)\Delta(\mu)$, $D_{1/\tau}C^+(\lambda)C(\mu)$, $D_{1/\tau}C^+(\lambda)C^+(\mu)$, o $D_{1/\tau}\Delta(\lambda)C^+(\mu)$.

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1 Introduction

In this paper we consider spaces generalizing the well-known sets w^0 and w_∞ introduced and studied by Maddox [12, 13]. Recall that w^0 and w_∞ are the sets of *strongly summable and strongly bounded sequences*. In [15] Malkowsky and Rakočević gave characterizations of matrix maps between w^0, w, or w_∞ and w_∞^p and between w^0, w, or w_∞ and l_1. In [2] de Malafosse defined the spaces $w_\alpha(\lambda)$, $w_\alpha^{(c)}(\lambda)$ and $w_\alpha^0(\lambda)$ of all sequences that are *$\alpha-$strongly bounded, summable and summable to zero* respectively. For instance recall that $w_\alpha(\lambda)$ is the set of all sequences $(x_n)_n$ such that $1/\lambda_n \sum_{m=1}^n |x_m| = \alpha_n O(1)$ as n tends to infinity. It was shown that these spaces can be written in the form s_ξ, $s_\xi^{(c)}$ and s_ξ^0 under some condition on α and λ.

More recently in [5] it was shown that if λ is a *sequence exponentially bounded* then $(w_\infty(\lambda), w_\infty(\lambda))$ is a Banach algebra. This result led to consider bijective operators mapping between $w_\infty(\lambda)$. Here we will use these results to study sets of the form $W_\tau = D_\tau w_\infty$, $W_\tau(\Delta(\lambda))$, $W_\tau(C(\lambda))$ and $W_\tau(C^+(\lambda))$ generalizing the well-known *set of strongly bounded sequences* $c_\infty = w_\infty(\Delta(\mu))$ where $\mu_n = n$ for all n. These results lead to the study of *statistical convergence* which was introduced by Steinhaus in 1949, see [16], and studied by several authors such as Fast [7], Fridy, Orhan [8-11] and Connor. Here we will deal with the notion of $A-$ *statistical convergence* which generalizes the notion of *statistical convergence,* see [6], where A belongs to a special class of operators.

The paper is organized as follows. In Section 2 among other things we recall a recent result on the operators Δ_ρ and Δ_ρ^T considered as map from $w_\infty(\lambda)$ to itself. In Sections 3 and 4 our aim is to give necessary conditions to have $W_\tau(A)$ in the form W_ξ when A is either one of the matrices $\Delta(\lambda)$, $C(\lambda)$ or $C^+(\lambda)$. Then we consider spaces generalizing the well-known set of all strongly bounded sequences $[C,\Delta] = c_\infty$ defined and studied by Maddox. Then we will define the sets $[A_1,A_2]_{W_\tau}$ of all sequences X with $A_1(\lambda)(|A_2(\mu)X|) \in W_\tau$ where A_1 and A_2 are of the form $C(\xi)$, $C^+(\xi)$, $\Delta(\xi)$, or $\Delta^+(\xi)$ and we will give necessary conditions to get $[A_1(\lambda),A_2(\mu)]$ in the form W_τ. In Section 5 we apply these results to $A-$ *statistical convergence*, where A is equal to $D_{1/\tau}A_1A_2$ and A_1, A_2 are of the form $C(\xi)$, $\Delta(\xi)$, $\Delta(\mu)$, or $C^+(\xi)$.

2 Well Known Results

For a given infinite matrix $A = (a_{nm})_{n,m\geq 1}$ we define the operators A_n for any integer $n \geq 1$, by

$$A_n(X) = \sum_{m=1}^{\infty} a_{nm}x_m \tag{1}$$

where $X = (x_n)_{n\geq 1}$, the series intervening in the second member being convergent. So we are led to the study of the infinite linear system

$$A_n(X) = b_n \quad n = 1,2,... \tag{2}$$

where $B = (b_n)_{n\geq 1}$ is a one-column matrix and X the unknown, see [2-5]. The equations (2) can be written in the form $AX = B$, where $AX = (A_n(X))_{n\geq 1}$. In this paper we shall also consider A as an operator from a sequence space into another sequence space.

We will write s for the set of all complex sequences and ℓ_∞ for the set of all *bounded sequences*.

Let E and F be any subsets of s. When A maps E into F we write that $A \in (E,F)$. So for every $X \in E$, $AX \in F$, ($AX \in F$ means that for each $n \geq 1$ the series defined by $y_n = \sum_{m=1}^{\infty} a_{nm}x_m$ is convergent and $(y_n)_{n\geq 1} \in F$).

Body Math For any subset E of s, we put

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$$AE = \{Y \in s \ : \ Y = AX \text{ for some } X \in E\}. \tag{3}$$

If F is a subset of s, we shall denote

$$F(A) = F_A = \{X \in s \ : \ Y = AX \in F\}. \tag{4}$$

In all what follows we will use the set

$$U^+ = \{(u_n)_{n\geq 1} \in s \ : \quad u_n > 0 \text{ for all } n\}$$

and the notation $e = (1,...,1,...)$. So for $\lambda = (\lambda_n)_{n\geq 1} \in U^+$ we will consider the sets of *strongly bounded and strongly summable sequences*, respectively, that is

$$\begin{aligned} w_\infty(\lambda) &= \left\{X = (x_n)_{n\geq 1} \in s \ : \ \sup_n \frac{1}{\lambda_n} \sum_{m=1}^{n} |x_m| < \infty\right\}, \\ w^0(\lambda) &= \left\{X = (x_n)_{n\geq 1} \in s \ : \ \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{m=1}^{n} |x_m| = 0\right\} \end{aligned}$$

and

$$w(\lambda) = \{X = (x_n)_{n\geq 1} \in s \ : \ X - le \in w^0(\lambda) \text{ for some } l \in \mathbb{C}\}$$

were studied by Malkowsky, with the concept of *exponentially bounded sequences*, see [3]. Recall that Maddox [12, 13], defined and studied the sets $w_\infty(\lambda) = w_\infty$, $w_0(\lambda) = w^0$ and $w(\lambda) = w$ where $\lambda_n = n$ for all n.

A Banach space E of complex sequences with the norm $\|\,\|_E$ *is a BK space if each projection* $P_n : X \mapsto P_n X = x_n$ *is continuous. A BK space E is said to have AK if every sequence* $X = (x_n)_{n\geq 1} \in E$ *has a unique representation* $X = \sum_{n=1}^{\infty} x_n e_n$ where e_n is the sequence with 1 in the n-th position and 0 otherwise.

Recall that a nondecreasing sequence $\lambda = (\lambda_n)_{n\geq 1} \in U^+$ is *exponentially bounded* if there is an integer $m \geq 2$ such that for all non-negative integers ν there is at least one term $\lambda_n \in I_m^{(\nu)} = [m^\nu, m^{\nu+1} - 1]$. It was shown (cf. [14, Lemma 1]) that a non-decreasing sequence $\lambda = (\lambda_n)_{n\geq 1}$ is *exponentially bounded* if and only if there are reals $s \leq t$ such that for some subsequence $(\lambda_{n_i})_{i\geq 1}$

$$0 < s \leq \frac{\lambda_{n_i}}{\lambda_{n_{i+1}}} \leq t < 1 \text{ for all } i = 1,2,...;$$

such a sequence is called an *associated subsequence*. Consider now the norm

$$\|X\|_\lambda = \sup_n \left(\frac{1}{\lambda_n} \sum_{m=1}^{n} |x_m|\right).$$

In [5] it was shown that if $\lambda = (\lambda_n)_{n\geq 1} \in U^+$ is *exponentially bounded* the class $(w_\infty(\lambda), w_\infty(\lambda))$ is a *Banach algebra* with the norm

$$\|A\|_{(w_\infty(\lambda),w_\infty(\lambda))} = \sup_{X\neq 0} \left(\frac{\|AX\|_\lambda}{\|X\|_\lambda}\right). \tag{5}$$

For $\rho = (\rho_n)_{n\geq 1}$ consider now the following matrices

$$\Delta_\rho^+ = \begin{pmatrix} 1 & -\rho_1 & & & \\ & . & . & & \\ & & 1 & -\rho_n & \\ & 0 & & . & . \\ & & & & \end{pmatrix} \text{ and } \Delta_\rho = \begin{pmatrix} 1 & & & & \\ -\rho_1 & 1 & & 0 & \\ & . & . & & \\ & & -\rho_{n-1} & 1 & . \\ & & & . & . \end{pmatrix}.$$

It can easily be shown that if $\rho = (\rho_n)_{n\geq 1}$ and $(\lambda_{n+1}/\lambda_n)_{n\geq 1} \in \ell_\infty$ then $\Delta_\rho^+ \in (w_\infty(\lambda), w_\infty(\lambda))$. We also see that $\Delta_\rho \in (w_\infty(\lambda), w_\infty(\lambda))$ for ρ, $(\lambda_{n-1}/\lambda_n)_{n\geq 2} \in \ell_\infty$. Recall the next result which is a direct consequence of [5, Theorem 5.1 and Theorem 5.12].

Lemma 2.1. *Let $\lambda \in U^+$ be a sequence exponentially bounded.*

(i) If

$$\overline{\lim_{n\to\infty}} \left(\frac{\lambda_{n+1}}{\lambda_n}\right) < \infty \text{ and } \overline{\lim_{n\to\infty}} |\rho_n| < \frac{1}{\overline{\lim_{n\to\infty}} \left(\frac{\lambda_{n+1}}{\lambda_n}\right)}, \tag{6}$$

for given $B \in w_\infty(\lambda)$ the equation $\Delta_\rho^+ X = B$ has a unique solution in $w_\infty(\lambda)$.

(ii) If

$$\overline{\lim_{n\to\infty}} |\rho_n| < \frac{1}{\overline{\lim_{n\to\infty}} \left(\frac{\lambda_{n-1}}{\lambda_n}\right)}, \tag{7}$$

then for any given $B \in w_\infty(\lambda)$ the equation $\Delta_\rho X = B$ has a unique solution in $w_\infty(\lambda)$.

When λ is a strictly increasing sequence tending to infinity we obtain similar results on the Banach algebra $(w^0(\lambda), w^0(\lambda))$ with the norm $\|A\|_{(w_\infty(\lambda), w_\infty(\lambda))}$.

3 On the Sets $W_\tau(A)$ Where A is Either $\Delta(\lambda)$, $C(\lambda)$ or $C^+(\lambda)$

In the following we will use the operators represented by $C(\lambda)$ and $\Delta(\lambda)$. Let U be the set of all sequences $(u_n)_{n\geq 1}$ with $u_n \neq 0$ for all n. We define $C(\lambda)$ for $\lambda = (\lambda_n)_{n\geq 1} \in U$, by

$$[C(\lambda)]_{nm} = \begin{cases} \dfrac{1}{\lambda_n} & \text{if } m \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

We will write $C(\lambda)^T = C^+(\lambda)$, $C(e) = \Sigma$, $\Sigma^+ = \Sigma^T$, and for $\lambda_n = n$, the matrix $C_1 = C((n)_n)$ is called the Cesaro operator. If It can be proved that the matrix $\Delta(\lambda)$ with

$$[\Delta(\lambda)]_{nm} = \begin{cases} \lambda_n & \text{if } m = n, \\ -\lambda_{n-1} & \text{if } m = n-1 \text{ and } n \geq 2, \\ 0 & \text{otherwise,} \end{cases}$$

is the inverse of $C(\lambda)$, see [2, 3]. We will use the following sets

$$\Gamma = \left\{X \in U^+ : \overline{\lim_{n\to\infty}} \left(\frac{x_{n-1}}{x_n}\right) < 1\right\},$$
$$\Gamma^+ = \left\{X \in U^+ : \overline{\lim_{n\to\infty}} \left(\frac{x_{n+1}}{x_n}\right) < 1\right\}.$$

Note that $X \in \Gamma^+$ if and only if $1/X \in \Gamma$.

For given sequence $\tau = (\tau_n)_{n\geq 1} \in U^+$, we write D_τ for the diagonal matrix defined by $[D_\tau]_{nn} = \tau_n$ for all n. For any subset E of s, we write

$$D_\tau E = \left\{X = (x_n)_{n\geq 1} \in s \ : \left(\frac{x_n}{\tau_n}\right)_n \in E\right\}.$$

We put $W_\tau = D_\tau w_\infty$ for $\tau \in U^+$, that is

$$W_\tau = \left\{X : \|X\|_{W_\tau} = \sup_n \left(\frac{1}{n}\sum_{m=1}^{\infty} \frac{|x_m|}{\tau_m}\right) < \infty\right\}.$$

It can easily be seen that $W_\tau = w_\infty(D_{1/\tau})$ is a BK space with norm $\|\|_{W_\tau}$, (cf. [17, Theorem 4.3.6, p. 52]). In all that follows we will use the convention that the entries with subscripts strictly less than 1 are equal to zero. Then we are interested in the study of the following sets where $\lambda, \tau \in U^+$.

$$W_\tau(\Delta(\lambda)) = \left\{X : \sup_n \left(\frac{1}{n}\sum_{m=1}^{n} \frac{1}{\tau_m} |\lambda_m x_m - \lambda_{m-1}x_{m-1}|\right) < \infty\right\},$$
$$W_\tau(C(\lambda)) = \left\{X : \sup_n \frac{1}{n}\sum_{m=1}^{n} \left(\frac{1}{\lambda_m\tau_m}\sum_{k=1}^{m} |x_k|\right) < \infty\right\},$$
$$W_\tau\left(C^+(\lambda)\right) = \left\{X : \sup_n \frac{1}{n}\sum_{m=1}^{n} \left(\frac{1}{\tau_m}\sum_{k=m}^{\infty} \frac{|x_k|}{\lambda_k}\right) < \infty\right\}.$$

Note that for $\lambda_n = n$ and $\tau = e$, $W_\tau(\Delta(\lambda))$ is the well known set of all strongly and bounded sequences c_∞. We obtain the following result that is a direct consequence of Lemma 2.1.

Proposition 3.1. *(i) If $\tau \in \Gamma$ then the operators Δ and Σ are bijective from W_τ into itself and*

$$W_\tau(\Delta) = W_\tau,\ W_\tau(\Sigma) = W_\tau.$$

(ii) a) If $\lambda\tau \in \Gamma$ then

$$W_\tau(C(\lambda)) = W_{\lambda\tau}.$$

b) If $\tau \in \Gamma$ then

$$W_\tau(\Delta(\lambda)) = W_{\tau/\lambda}.$$

(iii) Let $\tau \in \Gamma^+$. Then

a) the operators Δ^+ and Σ^+ are bijective from W_τ into itself and

$$W_\tau\left(\Sigma^+\right) = W_\tau.$$

b) the operator $C^+(\lambda)$ is bijective from $W_{\lambda\tau}$ into W_τ and

$$W_\tau\left(C^+(\lambda)\right) = W_{\lambda\tau}.$$

Proof. (i) By Lemma 2.1 where $\rho_n = \tau_{n-1}/\tau_n$ and $\lambda_n = n$ for all n, we easily see that if

$$\overline{\lim_{n\to\infty}} \frac{\tau_{n-1}}{\tau_n} < \frac{1}{\lim_{n\to\infty}\left(\frac{n-1}{n}\right)} = 1,$$

that is $\tau \in \Gamma$, then $D_{1/\tau}\Delta D_\tau$ is bijective from w_∞ to itself. This means that Δ is bijective from $D_\tau w_\infty$ to itself. Since Σ is also bijective from $D_\tau w_\infty$ to itself, this shows $W_\tau(\Delta) = W_\tau$ and $W_\tau(\Sigma) = W_\tau$.

(ii) We have $X \in W_\tau(C(\lambda))$ if and only if $\Sigma X \in D_{\lambda\tau} w_\infty = W_{\lambda\tau}$. This means that $X \in W_{\lambda\tau}(\Sigma)$ and by (i) the condition $\lambda\tau \in \Gamma$ implies $W_{\lambda\tau}(\Sigma) = W_{\lambda\tau}$. Then $W_\tau(C(\lambda)) = W_{\lambda\tau}$ and $C(\lambda)$ is bijective from $W_{\lambda\tau}$ to W_τ. Since $\Delta(\lambda) = C(\lambda)^{-1}$ we conclude $\Delta(\lambda)$ bijective from W_τ to $W_{\lambda\tau}$ and $W_{\lambda\tau}(\Delta(\lambda)) = W_\tau$. We deduce that for $\tau \in \Gamma$, $W_\tau(\Delta(\lambda)) = W_{\tau/\lambda}$.

(iii) a) By Lemma 2.1 with $\rho_n = \tau_{n+1}/\tau_n$ and $\lambda_n = n$ we have $\Delta_\rho^+ = D_{1/\tau}\Delta^+ D_\tau$ and Δ^+ is bijective from $D_\tau w_\infty = W_\tau$ into itself for $\tau \in \Gamma^+$ and it is the same for Σ^+. Now the equation $\Sigma^+ X = Y$ for $Y \in W_\tau$ is equivalent to

$$\sum_{m=n}^{\infty} x_m = y_n \text{ for all } n. \tag{8}$$

We deduce (8) has a unique solution $X = (y_n - y_{n+1})_{n\geq 1} = \Delta^+ Y \in W_\tau$ and $W_\tau\left(\Sigma^+\right) = W_\tau$.

b) We have

$$W_\tau\left(C^+(\lambda)\right) = \left\{X : \Sigma^+ D_{1/\lambda} X \in W_\tau\right\} = D_\lambda W_\tau\left(\Sigma^+\right).$$

Now as we have seen above since $\tau \in \Gamma^+$ we get $W_\tau\left(\Sigma^+\right) = W_\tau$ and

$$W_\tau\left(C^+(\lambda)\right) = D_\lambda W_\tau\left(\Sigma^+\right) = D_\lambda W_\tau = W_{\lambda\tau}.$$

This gives the conclusion. □

4 Calculations in New Sequence Spaces

4.1 The sets $[C,\Delta]_{W_\tau}$, $[C,C]_{W_\tau}$, $[C^+,\Delta]_{W_\tau}$, $[C^+,C]_{W_\tau}$ and $[C^+,C^+]_{W_\tau}$.

In [4], were defined and studied the sets

$$[A_1,A_2] = [A_1(\lambda), A_2(\mu)] = \{X \in s \ : \ A_1(\lambda)(|A_2(\mu)X|) \in D_\tau l_\infty\}$$

where $|X| = (|x_n|)_{n\geq 1}$, A_1 and A_2 of the form $C(\xi)$, $C^+(\xi)$, $\Delta(\xi)$, or $\Delta^+(\xi)$ for $\xi \in U^+$. It was given necessary conditions to get $[A_1(\lambda), A_2(\mu)]$ in the form s_γ.

Similarly in the following we will put

$$[A_1,A_2]_{W_\tau} = [A_1(\lambda), A_2(\mu)]_{W_\tau} = \{X \in s \ : \ A_1(\lambda)(|A_2(\mu)X|) \in W_\tau\}$$

for λ, μ, $\tau \in U^+$. We can explicitly write the previous sets $[A_1,A_2]_{W_\tau}$ as follows.

$$\begin{aligned}
[C,\Delta]_{W_\tau} &= \left\{X : \sup_n \left(\frac{1}{n}\sum_{m=1}^{n}\frac{1}{\lambda_m\tau_m}\sum_{k=1}^{m}|\mu_k x_k - \mu_{k-1}x_{k-1}|\right) < \infty\right\},\\
[C,C]_{W_\tau} &= \left\{X : \sup_n \left(\frac{1}{n}\sum_{m=1}^{n}\left(\frac{1}{\lambda_m\tau_m}\sum_{k=1}^{m}\frac{1}{\mu_k}\left|\sum_{i=1}^{k}x_i\right|\right)\right) < \infty\right\},\\
[C^+,\Delta]_{W_\tau} &= \left\{X : \sup_n \left(\frac{1}{n}\sum_{m=1}^{n}\left(\frac{1}{\tau_m}\sum_{k=m}^{\infty}\frac{1}{\lambda_k}|\mu_k x_k - \mu_{k-1}x_{k-1}|\right)\right) < \infty\right\},\\
[C^+,C]_{W_\tau} &= \left\{X : \sup_n \left(\frac{1}{n}\sum_{m=1}^{n}\left(\frac{1}{\tau_m}\sum_{k=m}^{\infty}\frac{1}{\lambda_k}\frac{1}{\mu_k}\left|\sum_{i=1}^{k}x_i\right|\right)\right) < \infty\right\},\\
[C^+,C^+]_{W_\tau} &= \left\{X : \sup_n \left(\frac{1}{n}\sum_{m=1}^{n}\left(\frac{1}{\tau_m}\sum_{k=m}^{\infty}\frac{1}{\lambda_k}\left|\sum_{i=k}^{\infty}\frac{x_i}{\mu_i}\right|\right)\right) < \infty\right\}.
\end{aligned}$$

Note that if $\lambda_n = \mu_n$ for all n we get the well known set of sequences that are strongly bounded $[C,\Delta]_{W_e} = c_\infty(\lambda)$. We can state the following.

Theorem 4.1. *Let λ, $\mu, \tau \in U^+$.*

(i) If $\lambda\tau \in \Gamma$ then

$$[C,\Delta]_{W_\tau} = W_{\lambda\tau/\mu};$$

(ii) if $\lambda\tau$, $\lambda\mu\tau \in \Gamma$ then

$$[C,C]_{W_\tau} = W_{\lambda\mu\tau};$$

(iii) if $\tau \in \Gamma^+$ and $\lambda\tau \in \Gamma$ then

$$[C^+,\Delta]_{W_\tau} = W_{\lambda\tau/\mu};$$

(iv) if $\tau \in \Gamma^+$ and $\lambda\mu\tau \in \Gamma$ then

$$[C^+, C]_{W_\tau} = W_{\lambda\mu\tau};$$

(v) if τ, $\lambda\tau \in \Gamma^+$ then

$$[C^+, C^+]_{W_\tau} = W_{\lambda\mu\tau}.$$

Proof. In the following we will use the fact that for any $\xi \in U^+$ we have $|X| \in W_\xi$ if and only if $X \in W_\xi$.

(i) We have $C(\lambda)(|\Delta(\mu)X|) \in W_\tau$ if and only if $|\Delta(\mu)X| \in W_\tau(C(\lambda))$ and by Proposition 3.1, since $\lambda\tau \in \Gamma$ we get $W_\tau(C(\lambda)) = W_{\lambda\tau}$. Then by Proposition 3.1 (ii) we have $W_{\lambda\tau}(\Delta(\mu)) = W_{\lambda\tau/\mu}$ and we conclude $\Delta(\mu)X \in W_{\lambda\tau}$ if and only if $X \in W_{\lambda\tau}(\Delta(\mu)) = W_{\lambda\tau/\mu}$, that is $[C, \Delta]_{W_\tau} = W_{\lambda\tau/\mu}$.

(ii) Here we have $C(\lambda)(|C(\mu)X|) \in W_\tau$ if and only if $|C(\mu)X| \in W_\tau(C(\lambda))$; and since $\lambda\tau \in \Gamma$ by Proposition 3.1 we have $W_\tau(C(\lambda)) = W_{\lambda\tau}$. So $X \in [C, C]_{W_\tau}$ if and only if $C(\mu)X \in W_{\lambda\tau}$, that is $X \in W_{\lambda\tau}(C(\mu))$. Then by Proposition 3.1 (ii) a) $\lambda\mu\tau \in \Gamma$ implies $W_{\lambda\tau}(C(\mu)) = W_{\lambda\mu\tau}$ and we have shown (ii).

(iii) For any given $X \in [C^+, \Delta]_{W_\tau}$ we have $\Delta(\mu)X \in W_\tau(C^+(\lambda))$ and for $\tau \in \Gamma^+$ we have $W_\tau(C^+(\lambda)) = W_{\lambda\tau}$. Now the condition $\lambda\tau \in \Gamma$ implies $X \in [C^+, \Delta]_{W_\tau}$ if and only if $X \in W_{\lambda\tau}(\Delta(\mu)) = W_{\lambda\tau/\mu}$ and we have shown (iii).

(iv) Let $X \in [C^+, C]_{W_\tau}$. We have $\tau \in \Gamma^+$ implies $W_\tau(C^+(\lambda)) = W_{\lambda\tau}$ and so $X \in [C^+, C]_{W_\tau}$ if and only if $C(\mu)X \in W_{\lambda\tau}$. Now since $\lambda\mu\tau \in \Gamma$ we have $W_{\lambda\tau}(C(\mu)) = W_{\lambda\mu\tau}$ and we conclude $[C^+, C]_{W_\tau} = W_{\lambda\mu\tau}$.

(v) As above $X \in [C^+, C^+]_{W_\tau}$ if and only if $C^+(\mu)X \in W_\tau(C^+(\lambda))$ and the condition $\tau \in \Gamma^+$ implies $W_\tau(C^+(\lambda)) = W_{\lambda\tau}$. Since $\lambda\tau \in \Gamma^+$ we conclude $W_{\lambda\tau}(C^+(\mu)) = W_{\lambda\mu\tau}$ that is $[C^+, C^+]_{W_\tau} = W_{\lambda\mu\tau}$. □

Now we are led to study sets of the form $[\Delta, A_2]_{W_\tau}$ for $A_2 \in \{\Delta, \Delta, C^+\}$.

4.2 The sets $[\Delta, \Delta]_{W_\tau}$, $[\Delta, C]_{W_\tau}$ and $[\Delta, C^+]_{W_\tau}$

Using the convention $\mu_0 = 0$, and the notation $\Delta(\mu)x_m = \mu_m x_m - \mu_{m-1}x_{m-1}$ for $m \geq 1$ we explicitly have

$$[\Delta, \Delta]_{W_\tau} = \left\{X : \sup_n \left(\frac{1}{n}\sum_{m=1}^{n}\frac{1}{\tau_m}\left|\lambda_m\left|\Delta(\mu)x_m\right| - \lambda_{m-1}\left|\Delta(\mu)x_{m-1}\right|\right|\right) < \infty\right\},$$

$$[\Delta, C]_{W_\tau} = \left\{X : \sup_n \left(\frac{1}{n}\sum_{m=1}^{n}\frac{1}{\tau_m}\left|\lambda_m\left|\frac{1}{\mu_m}\sum_{k=1}^{m}x_k\right| - \lambda_{m-1}\left|\frac{1}{\mu_{m-1}}\sum_{k=1}^{m-1}x_k\right|\right|\right) < \infty\right\},$$

$$[\Delta, C^+]_{W_\tau} = \left\{ X : \sup_n \left(\frac{1}{n} \sum_{m=1}^{n} \frac{1}{\tau_m} \left| \lambda_m \left| \sum_{k=m}^{\infty} \frac{x_k}{\mu_k} \right| - \lambda_{m-1} \left| \sum_{k=m-1}^{\infty} \frac{x_k}{\mu_k} \right| \right| \right) < \infty \right\}.$$

As a direct consequence of Proposition 3.1 we also obtain the following results.

Theorem 4.2. *Let λ, μ, $\tau \in U^+$. Then*

(i) If τ, $\tau/\lambda \in \Gamma$ then

$$[\Delta, \Delta]_{W_\tau} = W_{\tau/\lambda\mu}.$$

(ii) If τ, $\tau\mu/\lambda \in \Gamma$ then

$$[\Delta, C]_{W_\tau} = W_{\tau\mu/\lambda}.$$

(iii) If τ, $\tau/\lambda \in \Gamma^+$ then

$$[\Delta, C^+]_{W_\tau} = W_{\tau\mu/\lambda}.$$

Proof. (i) Let $X \in [\Delta, \Delta]_{W_\tau}$. Since $\tau \in \Gamma$ we have $W_\tau(\Delta(\lambda)) = W_{\tau/\lambda}$ and $\Delta(\lambda) \left| \Delta(\mu) X \right| \in W_\tau$ means $\Delta(\mu) X \in W_{\tau/\lambda}$. We conclude $W_{\tau/\lambda}(\Delta(\mu)) = W_{\tau/\lambda\mu}$ for $\tau/\lambda \in \Gamma$.

(ii) Reasoning as above since $\tau \in \Gamma$ we have $X \in [\Delta, C]_{W_\tau}$ if and only if $C(\mu) X \in W_{\tau/\lambda}$. We conclude since the condition $\tau\mu/\lambda \in \Gamma$ implies $W_{\tau/\lambda}(C(\mu)) = W_{\tau\mu/\lambda}$.

(iii) Here under the conditions τ, $\tau/\lambda \in \Gamma^+$, we have $X \in [\Delta, C^+]_{W_\tau}$ if and only if $X \in W_{\tau/\lambda}(C^+(\mu)) = W_{\tau\mu/\lambda}$. □

The previous results can be applied to the case when w_∞ is replaced by w^0.

4.3 The sets $[A_1, A_2]_{W_\tau^0}$

Using the Banach algebra $(w^0(\lambda), w^0(\lambda))$ we get similar results to those given above replacing $w_\infty(\lambda)$ by $w^0(\lambda)$ and W_τ by $W_\tau^0 = D_\tau w^0$. Note that $X \in W_\tau^0$ if and only if

$$\frac{1}{n} \sum_{m=1}^{n} \frac{|x_m|}{\tau_m} \to 0 \ (n \to \infty).$$

By [17, Theorem 4.3.6, p. 52] the set W_τ^0 is a BK space with AK normed by $\| \|_{W_\tau}$. So we can state the following.

Proposition 4.3. *Let λ, $\mu \in U^+$.*

(i) If $\lambda\tau \in \Gamma$ then $[C, \Delta]_{W_\tau^0} = W^0_{\lambda\tau/\mu}$;

(ii) if $\lambda\tau$, $\lambda\mu\tau \in \Gamma$ then $[C, C]_{W_\tau^0} = W^0_{\lambda\mu\tau}$;

(iii) if $\tau \in \Gamma^+$ and $\lambda\tau \in \Gamma$ then $[C^+, \Delta]_{W_\tau^0} = W^0_{\lambda\tau/\mu}$;

(iv) if $\tau \in \Gamma^+$ *and* $\lambda\mu\tau \in \Gamma$ *then* $\left[C^+, C\right]_{W_\tau^0} = W_{\lambda\mu\tau}^0$;

(v) if τ, $\lambda\tau \in \Gamma^+$ *then* $\left[C^+, C^+\right]_{W_\tau^0} = W_{\lambda\mu\tau}^0$;

(vi) if τ, $\tau/\lambda \in \Gamma$ *then* $[\Delta, \Delta]_{W_\tau^0} = W_{\tau/\lambda\mu}^0$;

(vii) if τ, $\tau\mu/\lambda \in \Gamma$ *then* $[\Delta, C]_{W_\tau^0} = W_{\tau\mu/\lambda}^0$;

(viii) if τ, $\tau/\lambda \in \Gamma^+$ *then* $\left[\Delta, C^+\right]_{W_\tau^0} = W_{\tau\mu/\lambda}^0$.

We immediatly get the next remark.

Remark 4.4. *It can easily be seen that in Proposition 4.3 each of the sets* $[A_1, A_2]_{W_\tau^0}$ *is equal to* $W_\tau^0(A_1A_2)$*. This result is a direct consequence of the previous proofs and of the fact that* W_τ^0 *is of absolute type, that is* $|X| \in W_\tau^0$ *if and only if* $X \in W_\tau^0$.

These results can be applied to statistical convergence.

5 Application to $A-$Statistical Convergence

In this section we will give conditions to have $x_k \to L(S(A))$ where A is either of the infinite matrices $D_{1/\tau}C(\lambda)C(\mu)$, $D_{1/\tau}\Delta(\lambda)\Delta(\mu)$, or $D_{1/\tau}\Delta(\lambda)C(\mu)$. Then we give conditions to have $x_k \to 0(S(A))$ where A is either of the operators $D_{1/\tau}C^+(\lambda)\Delta(\mu)$, $D_{1/\tau}C^+(\lambda)C(\mu)$, $D_{1/\tau}C^+(\lambda)$ $C^+(\mu)$ and $D_{1/\tau}\Delta(\lambda)C^+(\mu)$.

The sequence $X = (x_n)_{n\geq 1}$ is said to be *statiscally convergent to the number L* if

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0 \text{ for all } \varepsilon > 0,$$

where the vertical bars indicate the number of elements in the enclosed set. In this case we will write $x_k \to L(S)$ or $st - \lim X = L$.

Let $A \in (E, F)$ for given $L \in \mathbb{C}$ and for every $\varepsilon > 0$ we will use the notation

$$I_\varepsilon(A) = \{k \leq n : |[AX]_k - L| \geq \varepsilon\},$$

(where we assume that every series $[AX]_k = A_k(X) = \sum_{m=1}^{\infty} a_{km}x_m$ for $k \geq 1$ is convergent). We will say that $X = (x_n)_{n\geq 1}$ *is* $A-$ *statistically convergent to L if for every* $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |I_\varepsilon(A)| = 0.$$

Then we will write $x_k \to L(S(A))$ and for $A = I$, $x_k \to L(S(I))$ means that $st - \lim X = L$, (cf. [6]).

Now we require a lemma where we will put $T^{-1}e = \widetilde{l} = (l_n)_{n\geq 1}$ for given triangle T, that is $T = (t_{nm})_{n,m\geq 1}$ with $t_{nn} \neq 0$ and $t_{nm} = 0$ if $m > n$ for all n, m.

We can state the following.

Lemma 5.1. *If $X - L\widetilde{l} \in w^0(T)$ then x_k is $T-$ statistically convergent to L.*

Proof. The condition $X - L\widetilde{l} \in w^0(T)$ means that $T\left(X - L\widetilde{l}\right) \in w^0$. Since

$$TX - Le = T\left(X - LT^{-1}e\right) = T\left(X - L\widetilde{l}\right)$$

for any $\varepsilon > 0$ we have

$$\begin{aligned} y_n &= \frac{1}{n}\sum_{k=1}^{n} \left|[TX]_k - L\right| = \frac{1}{n}\sum_{k=1}^{n}\left|\left[T\left(X - L\widetilde{l}\right)\right]_k\right| \\ &\geq \frac{1}{n}\sum_{k\in I_\varepsilon(T)}\left|\left[T\left(X - L\widetilde{l}\right)\right]_k\right| \\ &\geq \frac{1}{n}\sum_{k\in I_\varepsilon(T)}\varepsilon \\ &\geq \frac{\varepsilon}{n}\left|\left\{k \leq n : \left|[TX]_k - L\right| \geq \varepsilon\right\}\right|. \end{aligned}$$

We conclude that $X - L\widetilde{l} \in w^0(T)$ implies $y_n \to 0$ $(n \to \infty)$ and $x_k \to L(S(T))$. □

We are led to state the next results.

Theorem 5.2. *(i) Let $\lambda\tau$, $\lambda\tau\mu \in \Gamma$. If*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{\left|x_k - L\left[\lambda_k\mu_k\tau_k + \left(\mu_{k-1}+\mu_k\right)\lambda_{k-1}\tau_{k-1} - \lambda_{k-2}\mu_{k-2}\tau_{k-2}\right]\right|}{\lambda_k\mu_k\tau_k} = 0 \tag{9}$$

then $x_k \to L\left(S\left(D_{1/\tau}C(\lambda)C\left(\mu\right)\right)\right)$, that is for every $\varepsilon > 0$

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k \leq n : \left|\frac{1}{\lambda_k\tau_k}\sum_{i=1}^{k}\frac{1}{\mu_i}\left(\sum_{j=1}^{i}x_j\right) - L\right| \geq \varepsilon\right\}\right| = 0.$$

(ii) Let τ, $\tau/\lambda \in \Gamma$. If

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{\lambda_k\mu_k}{\tau_k}\left|x_k - L\left(\frac{1}{\mu_k}\sum_{i=1}^{k}\frac{1}{\lambda_i}\sum_{j=1}^{i}\tau_j\right)\right| = 0$$

then $x_k \to L\left(S\left(D_{1/\tau}\Delta(\lambda)\Delta\left(\mu\right)\right)\right)$, that is for every $\varepsilon > 0$

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k \leq n : \left|\frac{1}{\tau_k}\left[\lambda_k\Delta\left(\mu\right)x_k - \lambda_{k-1}\Delta\left(\mu\right)x_{k-1}\right] - L\right| \geq \varepsilon\right\}\right| = 0.$$

(iii) Let τ, $\tau\mu/\lambda \in \Gamma$. If

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{\lambda_k}{\mu_k\tau_k}\left|x_k - L\left[\left(\frac{\mu_k}{\lambda_k} - \frac{\mu_{k-1}}{\lambda_{k-1}}\right)\sum_{i=1}^{k-1}\tau_i + \frac{\mu_k}{\lambda_k}\tau_k\right]\right| = 0$$

then $x_k \to L\left(S\left(D_{1/\tau}\Delta(\lambda)C(\mu)\right)\right)$, *that is for every* $\varepsilon > 0$

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k \le n : \left|\frac{1}{\tau_k}\left[\left(\frac{\lambda_k}{\mu_k}-\frac{\lambda_{k-1}}{\mu_{k-1}}\right)\sum_{i=1}^{k-1}x_i+\frac{\lambda_k}{\mu_k}x_k\right]-L\right|\ge\varepsilon\right\}\right|=0.$$

Proof. (i) First by Proposition 4.3 (ii) and Remark 4.4, we easily see that for $\lambda\tau$, $\lambda\tau\mu \in \Gamma$ we have $W_\tau^0\left(C(\lambda)C(\mu)\right) = W_{\lambda\mu\tau}^0$. Then putting $T = D_{1/\tau}C(\lambda)C(\mu)$ we get

$$w^0(T) = W_\tau^0\left(C(\lambda)C(\mu)\right) = W_{\lambda\mu\tau}^0. \tag{10}$$

Then $\widetilde{l} = T^{-1}e = \Delta(\mu)\Delta(\lambda)D_\tau e$ for each n with

$$l_n = \left[\Delta(\mu)\Delta(\lambda)D_\tau e\right]_n = \lambda_n\mu_n\tau_n + \left(\mu_{n-1}+\mu_n\right)\lambda_{n-1}\tau_{n-1} - \lambda_{n-2}\mu_{n-2}\tau_{n-2} \tag{11}$$

Using (10) and (11) we see that condition (9) is equivalent $X - L\widetilde{l} \in w^0(T)$. We conclude by Lemma 5.1 that $x_k \to L(S(T))$. This completes the proof of (i).

(ii) By Proposition 4.3 (vi) and Remark 4.4, since τ, $\tau/\lambda \in \Gamma$ we have $W_\tau^0\left(\Delta(\lambda)\Delta(\mu)\right) = W_{\tau/\lambda\mu}^0$. Then putting $T' = D_{1/\tau}\Delta(\lambda)\Delta(\mu)$ we get

$$w^0\left(T'\right) = W_\tau^0\left(\Delta(\lambda)\Delta(\mu)\right) = W_{\tau/\lambda\mu}^0. \tag{12}$$

Since $\widetilde{l'} = T'^{-1}e = C(\mu)C(\lambda)D_\tau e$ we have

$$l'_n = \left[C(\mu)C(\lambda)D_\tau e\right]_n = \frac{1}{\mu_n}\sum_{i=1}^{n}\frac{1}{\lambda_i}\left(\sum_{j=1}^{i}\tau_j\right) \text{ for all } n.$$

By Lemma 5.1 we conclude $x_k \to L\left(S\left(D_{1/\tau}\Delta(\lambda)\Delta(\mu)\right)\right)$ for all X with

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left|x_k - Ll'_k\right|\frac{\lambda_k\mu_k}{\tau_k} = 0$$

This shows (ii).

(iii) Again by Proposition 4.3 (vii) and Remark 4.4, since τ, $\tau\mu/\lambda \in \Gamma$ we have $W_\tau^0\left(\Delta(\lambda)C(\mu)\right) = W_{\tau\mu/\lambda}^0$. Then putting $T'' = D_{1/\tau}\Delta(\lambda)C(\mu)$ we get

$$w^0\left(T''\right) = W_\tau^0\left(\Delta(\lambda)C(\mu)\right) = W_{\tau\mu/\lambda}^0. \tag{13}$$

Writing $\widetilde{l''} = T''^{-1}e = \Delta(\mu)C(\lambda)D_\tau e$ we successively get

$$D_\tau e = (\tau_n)_{n\ge1}, C(\lambda)D_\tau e = \left(\left(\sum_{i=1}^{n}\tau_i\right)/\lambda_n\right)_{n\ge1}$$

and

$$\Delta(\mu)C(\lambda)D_\tau e = \left(\frac{\mu_n}{\lambda_n}\sum_{i=1}^{n}\tau_i - \frac{\mu_{n-1}}{\lambda_{n-1}}\sum_{i=1}^{n-1}\tau_i\right)_{n\ge1}.$$

So for each n we have

$$l''_n = \left[\Delta(\mu) C(\lambda) D_\tau e\right]_n = \left(\frac{\mu_n}{\lambda_n} - \frac{\mu_{n-1}}{\lambda_{n-1}}\right)\sum_{i=1}^{n-1}\tau_i + \frac{\mu_n}{\lambda_n}x_k.$$

We conclude that for every X with

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left|x_k - Ll''_k\right|\frac{\lambda_k}{\mu_k\tau_k} = 0$$

then $x_k \to L\left(S\left(T''\right)\right)$. Finally we easily get

$$\begin{aligned}\left[T''X\right]_n &= \frac{1}{\tau_n}\left(\frac{\lambda_n}{\mu_n}\sum_{i=1}^{n}x_i - \frac{\lambda_{n-1}}{\mu_{n-1}}\sum_{i=1}^{n-1}x_i\right)\\ &= \frac{1}{\tau_n}\left[\left(\frac{\lambda_n}{\mu_n} - \frac{\lambda_{n-1}}{\mu_{n-1}}\right)\sum_{i=1}^{n-1}x_i + \frac{\lambda_n}{\mu_n}x_n\right].\end{aligned}$$

This shows (iii). □

We are led to illustrate the previous results with some examples where we must have in mind that the condition $x_k/\tau_k \to 0$ $(k\to\infty)$ implies $X \in W_\tau^0$.

Example 5.3. *The condition*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left|\frac{x_k}{2^k} - \frac{7}{4}L\right| = 0$$

for given $L \in \mathbb{C}$ implies $x_k \to L\left(S\left(D_{(n/2^n)_n}C_1\Sigma\right)\right)$, that is, for each $\varepsilon > 0$

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k \le n : \left|\frac{1}{2^k}\sum_{i=1}^{k}\sum_{j=1}^{i}x_j - L\right| \ge \varepsilon\right\}\right| = 0. \tag{14}$$

Indeed it is enough to apply Theorem 5.2 (i) with $\lambda_k = k$, $\tau_k = 2^k/k$ and $\mu_k = 1$ for all k. Note that if $x_k/2^k \to 7L/4$ $(k\to\infty)$ then $x_k \to L\left(S\left(D_{(n/2^n)_n}C_1\Sigma\right)\right)$.

We can also state the next application.

Example 5.4. *If $\lim_{n\to\infty}(1/n)\sum_{k=1}^{n}|x_k|/k2^k = 0$ then $x_k \to L\left(S\left(D_{(2^{-n})_n}\Delta C_1\right)\right)$, that is for each $\varepsilon > 0$*

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k \le n : \left|\frac{1}{2^k}\left(\frac{1}{k} - \frac{1}{k-1}\right)\sum_{i=1}^{k-1}x_i + \frac{1}{k}x_k\right| \ge \varepsilon\right\}\right| = 0.$$

This result is a direct consequence of Theorem 5.2 (iii) with $\lambda_k = 1$, $\tau_k = 2^k$ and $\mu_k = k$ for all k. Again note that we have $x_k \to L\left(S\left(D_{(2^{-n})_n}\Delta C_1\right)\right)$ if $x_k/k2^k \to 0$ $(k\to\infty)$.

In the following we will use the previous Proposition 4.3 and the expressions of $W_\tau^0\left(C^+(\lambda)\Delta(\mu)\right)=\left[C^+,\Delta\right]_{W_\tau^0}$, $W_\tau^0\left(C^+(\lambda)C(\mu)\right)=\left[C^+,C\right]_{W_\tau^0}$, $W_\tau^0\left(C^+(\lambda)C^+(\mu)\right)=\left[C^+,C^+\right]_{W_\tau^0}$ and $W_\tau^0\left(\Delta(\lambda)C^+(\mu)\right)=\left[\Delta,C^+\right]_{W_\tau^0}$. We now require a lemma which is a direct consequence of Lemma 5.1.

Lemma 5.5. *Let A be an infinite matrix. If $X\in w^0(A)$ then*

$$x_k\to 0(S(A)).$$

we deduce the next results.

Theorem 5.6. *(i) Let $\tau\in\Gamma^+$ and $\lambda\tau\in\Gamma$. If*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{|x_k|}{\lambda_k\tau_k}\mu_k=0 \tag{15}$$

then $x_k\to 0\left(S\left(D_{1/\tau}C^+(\lambda)\Delta(\mu)\right)\right)$, that is for every $\varepsilon>0$

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k\le n:\left|\frac{1}{\tau_k}\sum_{i=k}^{\infty}\frac{\mu_i x_i-\mu_{i-1}x_{i-1}}{\lambda_i}\right|\ge\varepsilon\right\}\right|=0. \tag{16}$$

(ii) Let $\tau\in\Gamma^+$ and $\lambda\mu\tau\in\Gamma$. If

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{|x_k|}{\lambda_k\mu_k\tau_k}=0 \tag{17}$$

then $x_k\to 0\left(S\left(D_{1/\tau}C^+(\lambda)C(\mu)\right)\right)$, that is for every $\varepsilon>0$

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k\le n:\left|\frac{1}{\tau_k}\sum_{i=k}^{\infty}\frac{1}{\lambda_i}\left(\frac{1}{\mu_i}\sum_{j=1}^{i}x_j\right)\right|\ge\varepsilon\right\}\right|=0. \tag{18}$$

(iii) Let τ, $\lambda\tau\in\Gamma^+$. If

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{|x_k|}{\lambda_k\mu_k\tau_k}=0 \tag{19}$$

then $x_k\to 0\left(S\left(D_{1/\tau}C^+(\lambda)C^+(\mu)\right)\right)$, that is for every $\varepsilon>0$

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k\le n:\left|\frac{1}{\tau_k}\sum_{i=k}^{\infty}\frac{1}{\lambda_i}\left(\sum_{j=i}^{\infty}\frac{x_j}{\mu_j}\right)\right|\ge\varepsilon\right\}\right|=0. \tag{20}$$

(iv) Let τ, $\tau/\lambda\in\Gamma^+$. If

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{\lambda_k|x_k|}{\mu_k\tau_k}=0$$

then $x_k\to 0\left(S\left(D_{1/\tau}\Delta(\lambda)C^+(\mu)\right)\right)$, that is for every $\varepsilon>0$

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k\le n:\frac{1}{\tau_k}\left|(\lambda_k-\lambda_{k-1})\sum_{i=k-1}^{\infty}\frac{x_i}{\mu_i}+\frac{\lambda_k}{\mu_k}x_k\right|\ge\varepsilon\right\}\right|=0. \tag{21}$$

Proof. (i) Condition (15) implies $X \in W^0_{\lambda\tau/\mu}$ and by Proposition 4.3 and Remark 4.4 since $\tau \in \Gamma^+$ and $\lambda\tau \in \Gamma$ we have $W^0_{\lambda\tau/\mu} = W^0_\tau\left(C^+(\lambda)\Delta(\mu)\right)$ and $X \in W^0_\tau\left(C^+(\lambda)\Delta(\mu)\right)$. Now it can be easily seen that

$$\left[D_{1/\tau}C^+(\lambda)\Delta(\mu)\right]_n = \frac{1}{\tau_n}\sum_{i=n}^{\infty}\frac{\mu_i x_i - \mu_{i-1}x_{i-1}}{\lambda_i},$$

so by Lemma 5.5 with $A = D_{1/\tau}C^+(\lambda)\Delta(\mu)$ we conclude $x_k \to 0\left(S\left(D_{1/\tau}C^+(\lambda)\Delta(\mu)\right)\right)$. This shows (i).

(ii) Here condition (17) means $X \in W^0_{\lambda\mu\tau}$ and by Proposition 4.3 and Remark 4.4 since $\tau \in \Gamma^+$ and $\lambda\mu\tau \in \Gamma$ we have $W^0_{\lambda\mu\tau} = W^0_\tau\left(C^+(\lambda)C(\mu)\right)$ and $X \in W^0_\tau\left(C^+(\lambda)C(\mu)\right)$. Now since

$$\left[D_{1/\tau}C^+(\lambda)C(\mu)\right]_n = \frac{1}{\tau_n}\sum_{i=n}^{\infty}\frac{1}{\lambda_i}\left(\frac{1}{\mu_i}\sum_{j=1}^{i}x_j\right),$$

by Lemma 5.5 where $A' = D_{1/\tau}C^+(\lambda)C(\mu)$, we conclude $x_k \to 0\left(S\left(D_{1/\tau}C^+(\lambda)C(\mu)\right)\right)$. So we have shown (ii).

(iii) can be obtained reasoning as above with $A'' = D_{1/\tau}C^+(\lambda)C^+(\mu)$ and so $x_k \to 0$ $\left(S\left(D_{1/\tau}C^+(\lambda)C^+(\mu)\right)\right)$.

(iv) can also be obtained similarly. It is enough to put $A''' = D_{1/\tau}\Delta(\lambda)C^+(\mu)$. An elementary calculation gives

$$\left[A'''X\right]_k = \frac{1}{\tau_k}\left[(\lambda_k - \lambda_{k-1})\sum_{i=k-1}^{\infty}\frac{x_i}{\mu_i} + \frac{\lambda_k}{\mu_k}x_k\right]$$

and we conclude that $x_k \to 0\left(S\left(D_{1/\tau}\Delta(\lambda)C^+(\mu)\right)\right)$, that is (21). □

We can state the next example

Example 5.7. *for each $\varepsilon > 0$ and for every $X \in W^0_{3/2}$ we have $x_k \to 0\left(S\left(D_{(2^n)_n}\Sigma^+C((3^n)_n)\right)\right)$, that is*

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k \le n : \left|2^k\sum_{i=1}^{\infty}\frac{1}{3^i}\left(\sum_{j=1}^{i}x_j\right)\right| \ge \varepsilon\right\}\right| = 0. \tag{22}$$

It is enough to apply Theorem 5.6 (ii) with $\tau_k = 2^{-k}$, $\mu_k = 3^k$ and $\lambda_k = 1$ for all k. So if $(2/3)^k x_k \to 0$ $(k \to \infty)$ then (22) holds.

We also have the next example.

Example 5.8. *From Theorem 5.6 (iii) with $\lambda_k = \mu_k = k$ and $\tau_k = 2^{-k}$ the condition*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}2^k\frac{|x_k|}{k^2} = 0$$

implies $x_k \to 0\left(S\left(D_{(2^n)_n} C_1 C_1^+\right)\right)$ *that is, for each* $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \le n : \left| 2^k \sum_{i=k}^{\infty} \frac{1}{i} \left(\sum_{j=i}^{\infty} \frac{x_j}{j} \right) \right| \ge \varepsilon \right\} \right| = 0. \tag{23}$$

As in the previous cases (23) holds if $2^k x_k / k^2 \to 0$ $(k \to \infty)$.

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